Interval Finite Element Methods for Uncertainty Treatment in Structural Engineering Mechanics

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Outline

- Introduction
- Interval Finite Elements
- Element-By-Element
- Examples
- Conclusions

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- Professor Bob Mullen
- Dr. Hao Zhang

Center for Reliable Engineering Computing (REC)

We handle computations with care

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Introduction- Uncertainty

- Uncertainty is unavoidable in engineering system
  - structural mechanics entails uncertainties in material, geometry and load parameters
- Probabilistic approach is the traditional approach
  - requires sufficient information to validate the probabilistic model
  - criticism of the credibility of probabilistic approach when data is insufficient (Elishakoff, 1995; Ferson and Ginzburg, 1996)

Introduction- Interval Approach

- Nonprobabilistic approach for uncertainty modeling when only range information (tolerance) is available

$$t = t_0 \pm \delta$$

- Represents an uncertain quantity by giving a range of possible values

$$t = [t_0 - \delta,\ t_0 + \delta]$$

- How to define bounds on the possible ranges of uncertainty?
  - experimental data, measurements, statistical analysis, expert knowledge

Introduction- Why Interval?

- Simple and elegant
- Conforms to practical tolerance concept
- Describes the uncertainty that can not be appropriately modeled by probabilistic approach
- Computational basis for other uncertainty approaches (e.g., fuzzy set, random set)

- **Provides guaranteed enclosures**

Introduction- Finite Element Method

Finite Element Method (FEM) is a numerical method that provides approximate solutions to partial differential equations

Introduction- Uncertainty & Errors

- Mathematical model (validation)
- Discretization of the mathematical model into a computational framework
- Parameter uncertainty (loading, material properties)
- Rounding errors

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Interval Finite Elements

Uncertain Data

Geometry | **Materials** | **Loads**

Interval Stiffness Matrix

$$K = \int B^T C B \, dV$$

Interval Load Vector

$$F_i = \int N_i \, t \, dA$$

$$K\,U = F$$

Element Level

Interval Finite Elements

$\boldsymbol{K\,U = F}$

$\boldsymbol{K = \int B^T\, C\, B\, dV}$ = Interval element stiffness matrix

$\boldsymbol{B}$ = Interval strain-displacement matrix

$\boldsymbol{C}$ = Interval elasticity matrix

$\boldsymbol{F}$ = $[F_1, \ldots F_i, \ldots F_n]$ = Interval element load vector (traction)

$\boldsymbol{F_i = \int N_i\, t\, dA}$

N_i = Shape function corresponding to the *i-th* DOF

$\boldsymbol{t}$ = Surface traction

Interval Finite Elements (IFEM)

- Follows conventional FEM
- Loads, geometry and material property are expressed as interval quantities
- System response is a function of the interval variables and therefore varies in an interval
- Computing the exact response range is proven NP-hard
- The problem is to estimate the bounds on the unknown exact response range based on the bounds of the parameters

IFEM- Inner-Bound Methods

- Combinatorial method (Muhanna and Mullen 1995, Rao and Berke 1997)
- Sensitivity analysis method (Pownuk 2004)
- Perturbation (Mc William 2000)
- Monte Carlo sampling method
- Need for alternative methods that achieve
    - Rigorousness – guaranteed enclosure
    - Accuracy – sharp enclosure
    - Scalability – large scale problem
    - Efficiency

IFEM- Enclosure

- Linear static finite element
  - Muhanna, Mullen, 1995, 1999, 2001,and Zhang 2004
  - Popova 2003, and Kramer 2004
  - Neumaier and Pownuk 2004
  - Corliss, Foley, and Kearfott 2004
- Dynamic
  - Dessombz, 2000
- Free vibration-Buckling
  - Modares, Mullen 2004, and Billini and Muhanna 2005

Interval arithmetic

- Interval number: $\boldsymbol{x} = [\underline{x}, \overline{x}]$

  midpoint: $\breve{\boldsymbol{x}} = (\underline{x} + \overline{x})/2$, width: $\mathrm{wid}(\boldsymbol{x}) = \overline{x} - \underline{x}$,

  absolute value: $|\boldsymbol{x}| = \max\{|\underline{x}|, |\overline{x}|\}$.

- Interval vector and interval matrix, e.g.,

  $\boldsymbol{x} = (\boldsymbol{x}_1, \boldsymbol{x}_2)^T = ([0,1],\ [-2,1])^T$

  midpoint, width, absolute value: defined componentwise

- Notations

  intervals: boldface, e.g., $\boldsymbol{x}$, $\boldsymbol{b}$, $\boldsymbol{A}$

  real: non-boldface, $x \in \boldsymbol{x}$, $A \in \boldsymbol{A}$

Linear interval equation

- Linear interval equation

  $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ ($A \in \boldsymbol{A}$, $b \in \boldsymbol{b}$)

- Solution set

  $\Sigma(\boldsymbol{A}, \boldsymbol{b}) = \{x \in \mathrm{R} \mid \exists A \in \boldsymbol{A}\ \exists b \in \boldsymbol{b}: Ax = b\}$

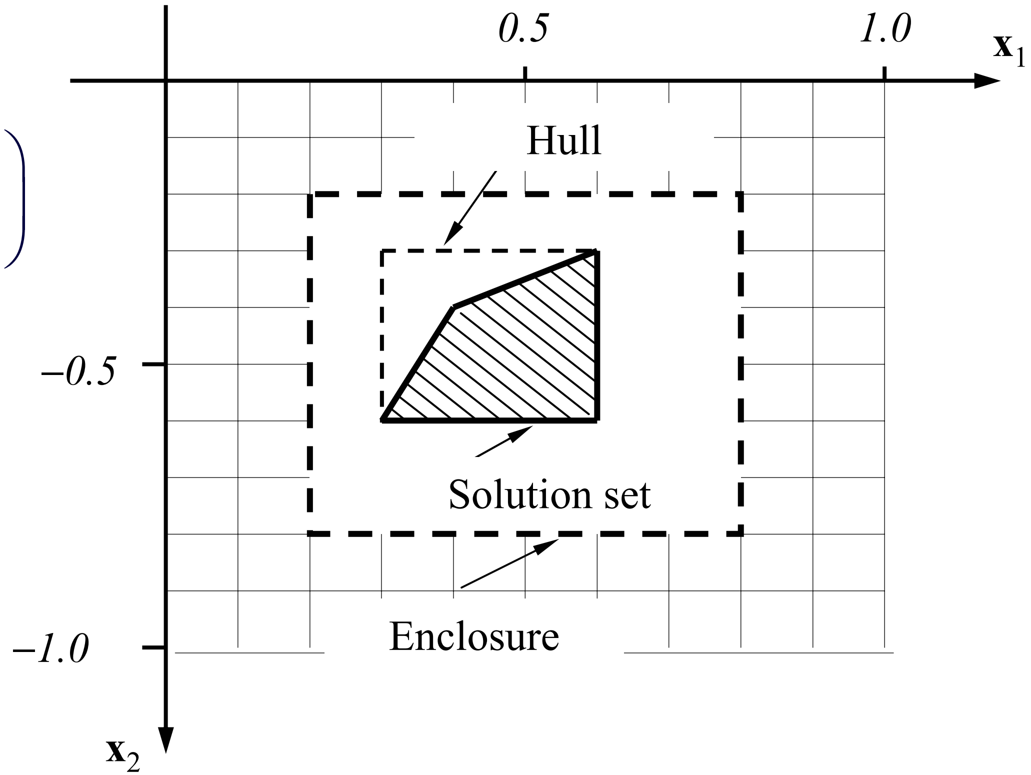
- Hull of the solution set $\Sigma(\boldsymbol{A}, \boldsymbol{b})$

  $\boldsymbol{A}^H\boldsymbol{b} := \Diamond\, \Sigma(\boldsymbol{A}, \boldsymbol{b})$

Linear interval equation

- Example

$$\begin{pmatrix} 2 & [-1,\ 0] \\ [-1,\ 0] & 2 \end{pmatrix} \begin{pmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{pmatrix} = \begin{pmatrix} 1.2 \\ -1.2 \end{pmatrix}$$



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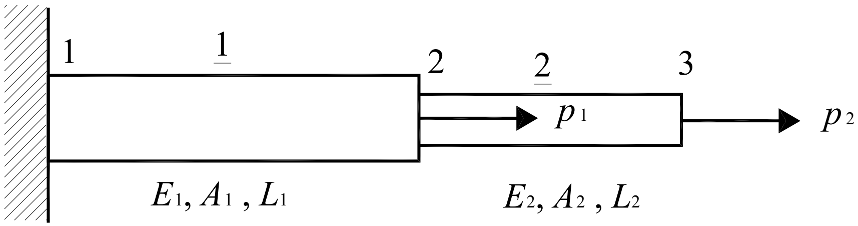
Naïve interval FEA



$E_1A_1/L_1 = \boldsymbol{k}_1 = [0.95,\ 1.05],$
$E_2A_2/L_2 = \boldsymbol{k}_2 = [1.9,\ 2.1],$
$p_1 = 0.5, \quad p_2 = 1$

$$\begin{pmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \Rightarrow \begin{pmatrix} [2.85,\ 3.15] & [-2.1,\ -1.9] \\ [-2.1,\ -1.9] & [1.9,\ 2.1] \end{pmatrix}\begin{pmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 1 \end{pmatrix}$$

- exact solution: $\boldsymbol{u}_1 = [1.429, 1.579]$, $\boldsymbol{u}_2 = [1.905, 2.105]$
- naïve solution: $\boldsymbol{u}_1 = [-0.052, 3.052]$, $\boldsymbol{u}_2 = [0.098, 3.902]$
- interval arithmetic assumes that all coefficients are independent
- uncertainty in the response is severely overestimated

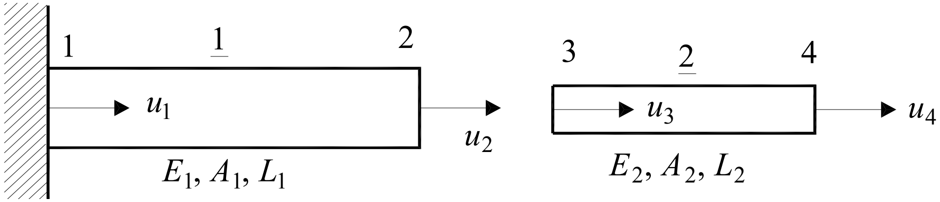
Element-By-Element

Element-By-Element (EBE) technique

- elements are detached – no element coupling
- structure stiffness matrix is block-diagonal (k_1 ,…, k_{Ne})
- the size of the system is increased

$$u = (u_1, \ldots, u_{Ne})^T$$

- need to impose necessary constraints for compatibility and equilibrium



Element-By-Element model

Element-By-Element

Suppose the modulus of elasticity is interval:

$$\boldsymbol{E} = \breve{E}(1+\boldsymbol{\delta})$$

$\boldsymbol{\delta}$: zero-midpoint interval

The element stiffness matrix can be split into two parts,

$$\boldsymbol{k} = \breve{k}(I+\boldsymbol{d}) = \breve{k} + \breve{k}\boldsymbol{d}$$

$\breve{k}$: deterministic part, element stiffness matrix evalued using $\breve{E}$,

$\breve{k}\boldsymbol{d}$: interval part

$\boldsymbol{d}$: interval diagonal matrix, diag($\boldsymbol{\delta},\ldots,\boldsymbol{\delta}$).

Element-By-Element

- Element stiffness matrix: $\boldsymbol{k} = \breve{k}(I + \boldsymbol{d})$

- Structure stiffness matrix:

$$\boldsymbol{K} = \breve{K}(I + \boldsymbol{D}) = \breve{K} + \breve{K}\boldsymbol{D}$$

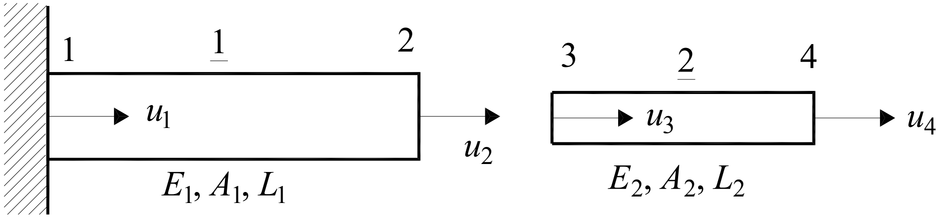
or

$$\boldsymbol{K} = \begin{pmatrix} \boldsymbol{k}_1 & & \\ & \ddots & \\ & & \boldsymbol{k}_{N_e} \end{pmatrix} = \begin{pmatrix} \breve{k}_1 & & \\ & \ddots & \\ & & \breve{k}_{N_e} \end{pmatrix} \left(I + \begin{pmatrix} \boldsymbol{d}_1 & & \\ & \ddots & \\ & & \boldsymbol{d}_{N_e} \end{pmatrix} \right)$$

Constraints

Impose necessary constraints for compatibility and equilibrium

- Penalty method
- Lagrange multiplier method



Element-By-Element model

Constraints – penalty method

Constraint conditions: $c\boldsymbol{u} = 0$

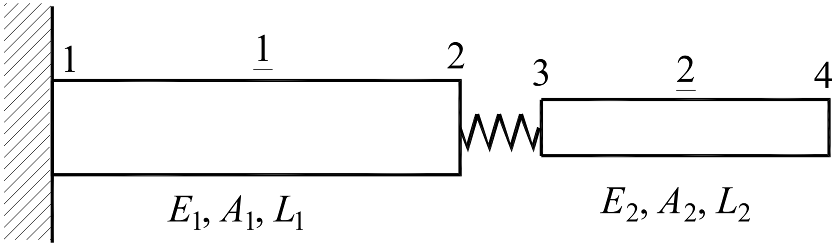
Using the penalty method:

$$(\boldsymbol{K} + Q)\boldsymbol{u} = \boldsymbol{p}$$

Q : penalty matrix, $Q = c^T \eta c$

η : diagonal matrix of penalty number η_i

Requires a careful choice of the penatly number



A spring of large stiffness is added to force node 2 and node 3 to have the same displacement.

Constraints – Lagrange multiplier

Constraint conditions: $c\boldsymbol{u} = 0$

Using the Lagrange multiplier method:

$$\begin{pmatrix} \boldsymbol{K} & c^T \\ c & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \boldsymbol{p} \\ 0 \end{pmatrix}$$

$\boldsymbol{\lambda}$: Lagrange multiplier vector, introdued as new unknowns

Load in EBE

Nodal load applied by elements $\boldsymbol{p}_b$

$$\boldsymbol{p}_b = (\boldsymbol{p}_1, \ldots, \boldsymbol{p}_{N_e})^T$$

where $\boldsymbol{p}_i = \int N^T \phi(x) dx$

Suppose the surface traction $\phi(x)$ is described by

an interval function: $\phi(x) = \sum_{j=0}^{m} \boldsymbol{a}_j x^j$.

$\boldsymbol{p}_b$ can be rewritten as

$$\boldsymbol{p}_b = W\boldsymbol{F}$$

W: deterministic matrix

$\boldsymbol{F}$: interval vector containing the interval coefficients of the surface tractiton

Fixed point iteration

- For the interval equation $\boldsymbol{Ax} = \boldsymbol{b}$,
  - preconditioning: $R\boldsymbol{Ax} = R\boldsymbol{b}$, R is the preconditioning matrix
  - transform it into $\boldsymbol{g}(\boldsymbol{x}^*) = \boldsymbol{x}^*$:

$$R\,\boldsymbol{b} - R\boldsymbol{A}\,x_0 + (I - R\boldsymbol{A})\,\boldsymbol{x}^* = \boldsymbol{x}^*, \qquad \boldsymbol{x} = \boldsymbol{x}^* + x_0$$

- **Theorem** (Rump, 1990): for some interval vector $\boldsymbol{x}^*$,

$$\text{if} \quad \boldsymbol{g}(\boldsymbol{x}^*) \subseteq \operatorname{int}(\boldsymbol{x}^*)$$

$$\text{then} \quad \boldsymbol{A}^H \boldsymbol{b} \subseteq \boldsymbol{x}^* + x_0$$

- Iteration algorithm:

iterate: $\boldsymbol{x}^{*(l+1)} = \boldsymbol{z} + \boldsymbol{G}(\boldsymbol{\varepsilon} \cdot \boldsymbol{x}^{*(l)})$

where $\boldsymbol{z} = R\boldsymbol{b} - R\boldsymbol{A}x_0$, $\boldsymbol{G} = I - R\boldsymbol{A}$, $R = \breve{A}^{-1}$, $\breve{A}x_0 = \breve{b}$

- No dependency handling

Fixed point iteration

- Interval FEA calls for a modified method which exploits the special form of the structure equations

$$(\boldsymbol{K}+Q)\boldsymbol{u}=\boldsymbol{p} \text{ with } \boldsymbol{K}=\breve{K}+\breve{K}\boldsymbol{D}$$

- Choose $R=(\breve{K}+Q)^{-1}$, construct iterations:

$$\boldsymbol{u}^{*(l+1)}=R\boldsymbol{p}-R(\boldsymbol{K}+Q)u_0+(I-R(\boldsymbol{K}+Q))(\boldsymbol{\varepsilon}\cdot\boldsymbol{u}^{*(l)})$$

$$=R\boldsymbol{p}-u_0-R\breve{K}\boldsymbol{D}(u_0+\boldsymbol{\varepsilon}\cdot\boldsymbol{u}^{*(l)})$$

$$=R\boldsymbol{p}-u_0-R\breve{K}\boldsymbol{M}^{(l)}\boldsymbol{\Delta}$$

if $\boldsymbol{u}^{*(l+1)}\subseteq\operatorname{int}(\boldsymbol{u}^{*(l)})$, then $\boldsymbol{u}=\boldsymbol{u}^{*(l+1)}+u_0=R\boldsymbol{p}-R\breve{K}\boldsymbol{M}^{(l)}\boldsymbol{\Delta}$

$\boldsymbol{\Delta}$: interval vector, $\boldsymbol{\Delta}=(\delta_1,\ldots,\delta_{N_e})^T$

The interval variables $\delta_1,\ldots,\delta_{N_e}$ appear only once in each iteration.

Most sources of dependence are eliminated.

Convergence of fixed point

- The algorithm converges if and only if

$$\rho(|\boldsymbol{G}|) < 1$$

  smaller $\rho(|\boldsymbol{G}|) \Rightarrow$ less iterations required, and less overestimation in results

- To minimize $\rho(|\boldsymbol{G}|)$:
  - choose $R = \breve{A}^{-1}$ so that $\boldsymbol{G} = I - R\boldsymbol{A}$ has a small spectral radius
  - reduce the overestimation in $\boldsymbol{G}$

    $$\boldsymbol{G} = I - R\boldsymbol{A} = \boldsymbol{I} - (\breve{K} + Q)^{-1}(\breve{K} + Q + \breve{K}\boldsymbol{D}) = -R\breve{K}\boldsymbol{D}$$

Stress calculation

- Conventional method:

$$\boldsymbol{\sigma} = \boldsymbol{C}B\boldsymbol{u}_e, \text{ (severe overestimation)}$$

$\boldsymbol{C}$: elasticity matrix, B: strain-displacement matrix

- Present method:

$$\boldsymbol{E} = (1+\boldsymbol{\delta})\breve{E}, \quad \boldsymbol{C} = (1+\boldsymbol{\delta})\breve{C}$$

$$\begin{aligned} \boldsymbol{\sigma} &= \boldsymbol{C}BL\boldsymbol{u} \\ &= \boldsymbol{C}BL(R\boldsymbol{p} - R\breve{K}\boldsymbol{M}^{(l)}\boldsymbol{\Delta}) \\ &= (1+\boldsymbol{\delta})(\breve{C}BLR\boldsymbol{p} - \breve{C}BLR\breve{K}\boldsymbol{M}^{(l)}\boldsymbol{\Delta}) \end{aligned}$$

L: Boolean matrix, $L\boldsymbol{u} = \boldsymbol{u}_e$

Element nodal force calculation

- Conventional method:

$$\boldsymbol{f} = T_e(\boldsymbol{k}\boldsymbol{u}_e - \boldsymbol{p}_e), \quad \text{(severe overestimation)}$$

- Present method:

in the EBE model, $T(\boldsymbol{K}\boldsymbol{u} - \boldsymbol{p}_b) = \begin{pmatrix} (\boldsymbol{T}_e)_1(\boldsymbol{k}_1(\boldsymbol{u}_e)_1 - (\boldsymbol{p}_e)_1) \\ \vdots \\ (\boldsymbol{T}_e)_{N_e}(\boldsymbol{k}_{N_e}(\boldsymbol{u}_e)_{N_e} - (\boldsymbol{p}_e)_{N_e}) \end{pmatrix}$

from $(\boldsymbol{K} + Q)\boldsymbol{u} = \boldsymbol{p}_c + \boldsymbol{p}_b \Rightarrow T(\boldsymbol{K}\boldsymbol{u} - \boldsymbol{p}_b) = T(\boldsymbol{p}_c - Q\boldsymbol{u})$

Calculate $T(\boldsymbol{p}_c - Q\boldsymbol{u})$ to obtain the element nodal forces for all elements.

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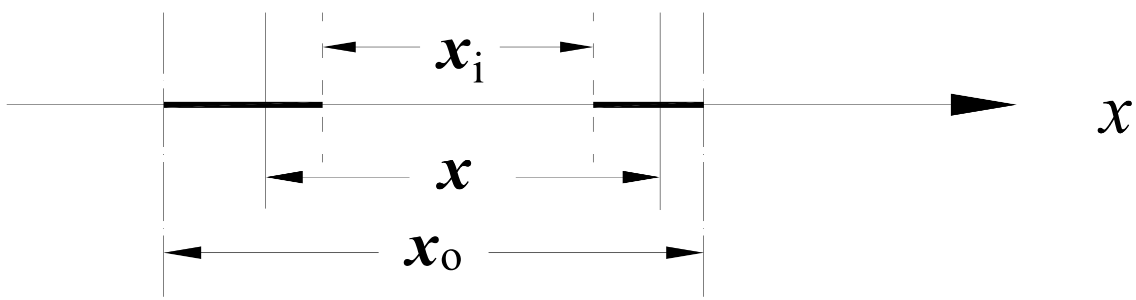
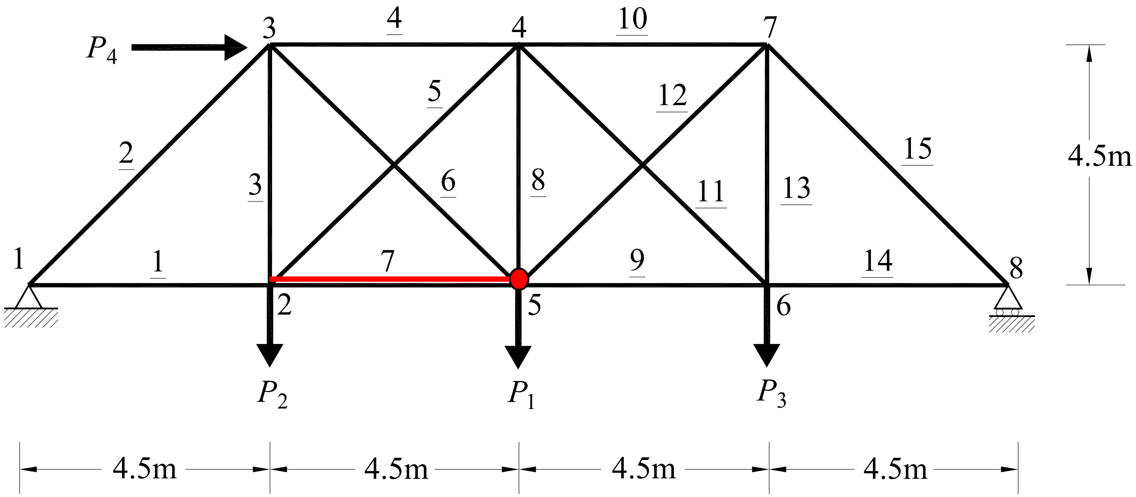
Numerical example

- Examine the rigorousness, accuracy, scalability, and efficiency of the present method
- Comparison with the alternative methods
  - the combinatorial method, sensitivity analysis method, and Monte Carlo sampling method
  - these alternative methods give inner estimation



$\boldsymbol{x}$: exact solution, $\boldsymbol{x}_\mathrm{i}$: inner bound, $\boldsymbol{x}_\mathrm{o}$: outer bound

Truss structure



$A_1, A_2, A_3, A_{13}, A_{14}, A_{15}$: $[9.95,\ 10.05]\ \mathrm{cm}^2$ (1% uncertainty)

cross-sectional area

of all other elements: $[5.97,\ 6.03]\ \mathrm{cm}^2$ (1% uncertainty)

modulus of elasticity of all elements: 200,000 MPa

$\boldsymbol{p}_1 = [190, 210]$ kN, $\boldsymbol{p}_2 = [95, 105]$ kN

$\boldsymbol{p}_3 = [95, 105]$ kN, $\boldsymbol{p}_4 = [85.5, 94.5]$ kN (10% uncertainty)

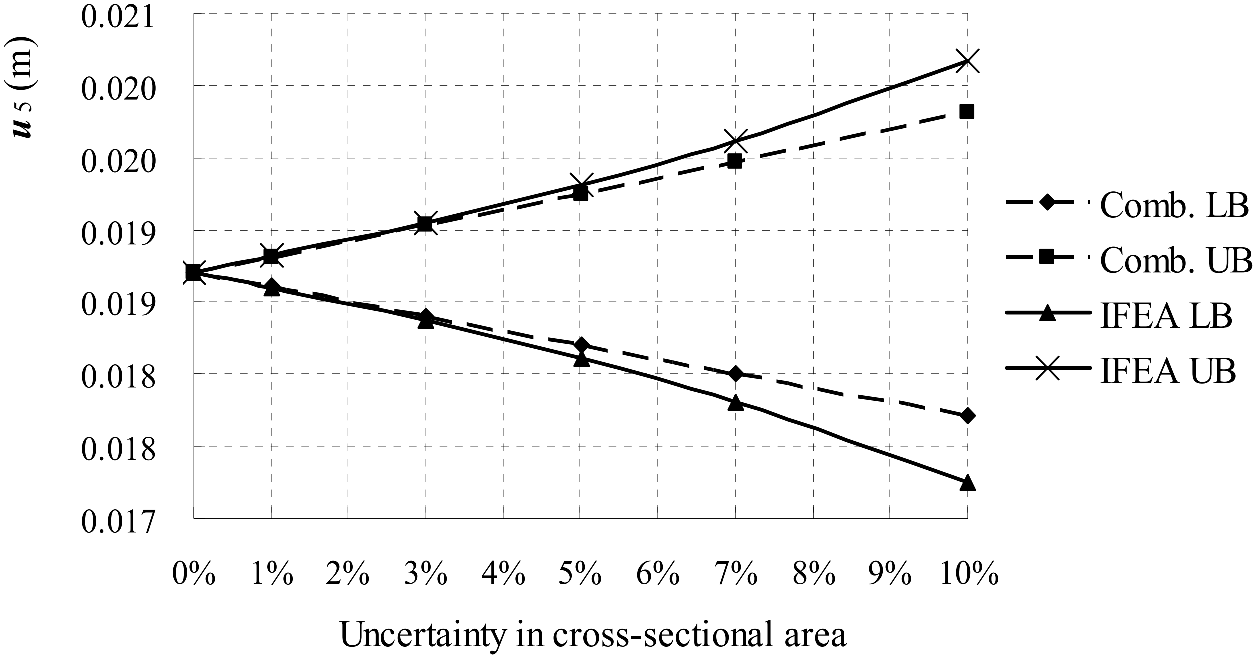
Truss structure - results

Table: results of selected responses

Method	$\boldsymbol{u}_5$(LB)	$\boldsymbol{u}_5$(UB)	$\boldsymbol{N}_7$(LB)	$\boldsymbol{N}_7$(UB)
Combinatorial	0.017676	0.019756	273.562	303.584
Naïve IFEA δ	– 0.011216 163.45%	0.048636 146.18%	– 717.152 362%	1297.124 327%
Present IFEA δ	0.017642 0.19%	0.019778 0.11%	273.049 0.19%	304.037 0.15%

unit: $\boldsymbol{u}_5$ (m), $\boldsymbol{N}_7$ (kN). LB: lower bound; UB: upper bound.

Truss structure – results



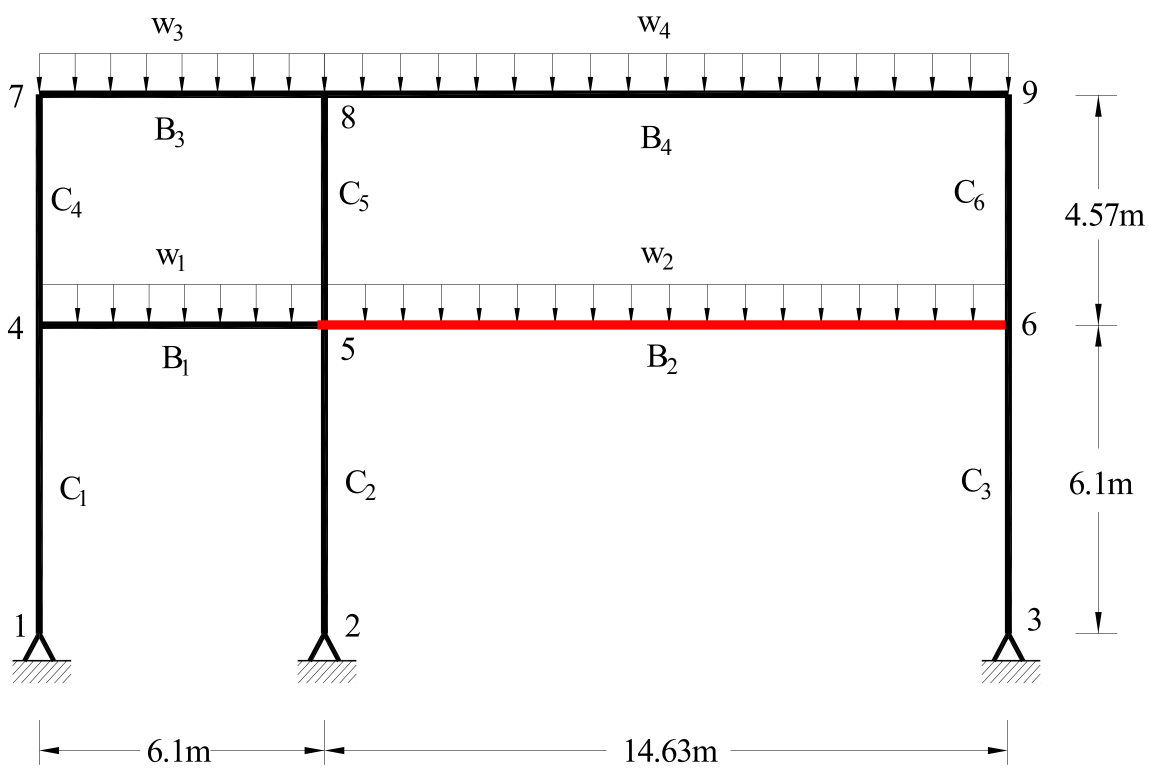
- for moderate uncertainty (≤ 5%), very sharp bounds are obtained
- for relatively large uncertainty, reasonable bounds are obtained

  in the case of 10% uncertainty:

  Comb.: $\boldsymbol{u}_5$= [0.017711,0.019811], IFEM: $\boldsymbol{u}_5$= [0.017252,0.020168]

  (relative difference: 2.59%, 1.80% for LB, UB, respectively)

Frame structure



Member	Shape
C_1	W12×19
C_2	W14×132
C_3	W14×109
C_4	W10×12
C_5	W14×109
C_6	W14×109
B_1	W27×84
B_2	W36×135
B_3	W18×40
B_4	W27×94

results listed: nodal forces at the left node of member B_2

Frame structure – case 1

Case 1: load uncertainty

$\mathbf{w}_1 = [105.8, 113.1]$ kN/m, $\mathbf{w}_2 = [105.8, 113.1]$ kN/m,

$\mathbf{w}_3 = [49.255, 52.905]$ kN/m, $\mathbf{w}_4 = [49.255, 52.905]$ kN/m,

Table: Nodal forces at the left node of member B_2

Nodal force	Combinatorial		Present IFEA	
	LB	UB	LB	UB
Axial (kN)	219.60	239.37	219.60	239.37
Shear (kN)	833.61	891.90	833.61	891.90
Moment (kN·m)	1847.21	1974.95	1847.21	1974.95

- exact solution is obtained in the case of load uncertainty

Frame structure – case 2

Case 2: stiffness uncertainty and load uncertainty

1% uncertainty introduced to $\boldsymbol{A}$, $\boldsymbol{I}$, and $\boldsymbol{E}$ of each element.

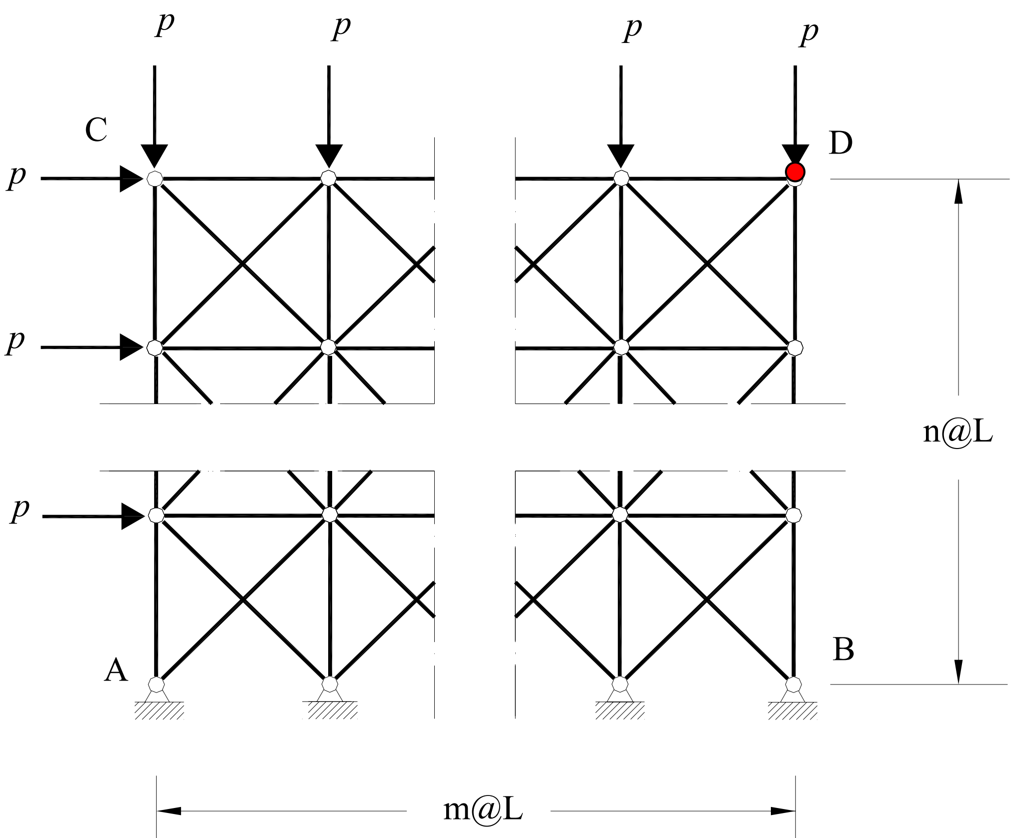
Number of interval variables: 34.

Table: Nodal forces at the left node of member B_2

Nodal force	Monte Carlo sampling*		Present IFEA	
	LB	UB	LB	UB
Axial (kN)	218.23	240.98	219.35	242.67
Shear (kN)	833.34	892.24	832.96	892.47
Moment (kN.m)	1842.86	1979.32	1839.01	1982.63

*10^6 samples are made.

Truss with a large number of interval variables



$$A_i = [0.995, 1.005] A_0,$$

$$E_i = [0.995, 1.005] E_0 \quad \text{for } i = 1, \ldots, N_e$$

story × bay	N_e	N_v
3×10	123	246
4×12	196	392
4×20	324	648
5×22	445	890
5×30	605	1210
6×30	726	1452
6×35	846	1692
6×40	966	1932
7×40	1127	2254
8×40	1288	2576

Scalability study

vertical displacement at right upper corner (node D): $v_D = a\dfrac{PL}{E_0 A_0}$

Table: displacement at node D

Story×bay	Sensitivity Analysis		Present IFEA				
	LB^*	UB^*	LB	UB	δ_{LB}	δ_{UB}	wid/d_0
3×10	2.5143	2.5756	2.5112	2.5782	0.12%	0.10%	2.64%
4×20	3.2592	3.3418	3.2532	3.3471	0.18%	0.16%	2.84%
5×30	4.0486	4.1532	4.0386	4.1624	0.25%	0.22%	3.02%
6×35	4.8482	4.9751	4.8326	4.9895	0.32%	0.29%	3.19%
7×40	5.6461	5.7954	5.6236	5.8166	0.40%	0.37%	3.37%
8×40	6.4570	6.6289	6.4259	6.6586	0.48%	0.45%	3.56%

$\delta_{LB} = |LB - LB^*|/ LB^*$, $\delta_{LB} = |UB - UB^*|/ UB^*$, $\delta_{LB} = (LB - LB^*)/ LB^*$

Efficiency study

Table: CPU time for the analyses with the present method (unit: seconds)

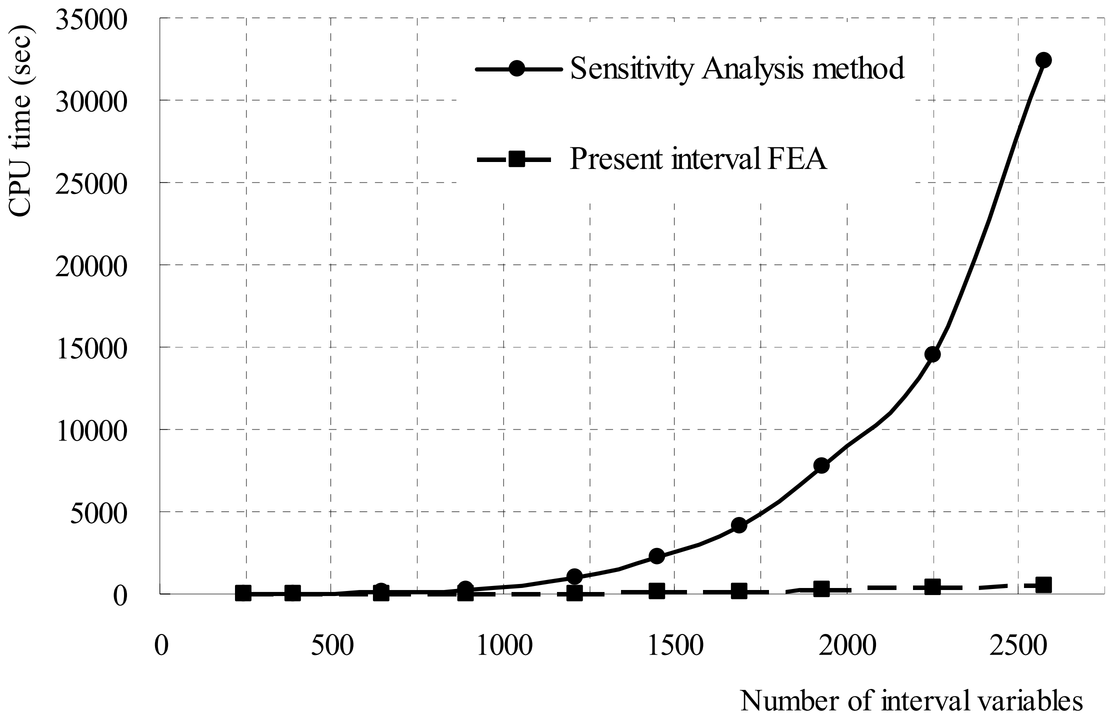
Story × bay	N_v	Iteration	t_i	t_r	t	t_i/t	t_r/t
3 × 10	246	4	0.14	0.56	0.72	19.5%	78.4%
4 × 20	648	5	1.27	8.80	10.17	12.4%	80.5%
5 × 30	1210	6	6.09	53.17	59.70	10.2%	89.1%
6 × 35	1692	6	15.11	140.23	156.27	9.7%	89.7%
7 × 40	2254	6	32.53	323.14	358.76	9.1%	90.1%
8 × 40	2576	7	48.454	475.72	528.45	9.2%	90.0%

t_i : iteration time, t_r : CPU time for matrix inversion, t : total comp. CPU time

- majority of time is spent on matrix inversion

Efficiency study

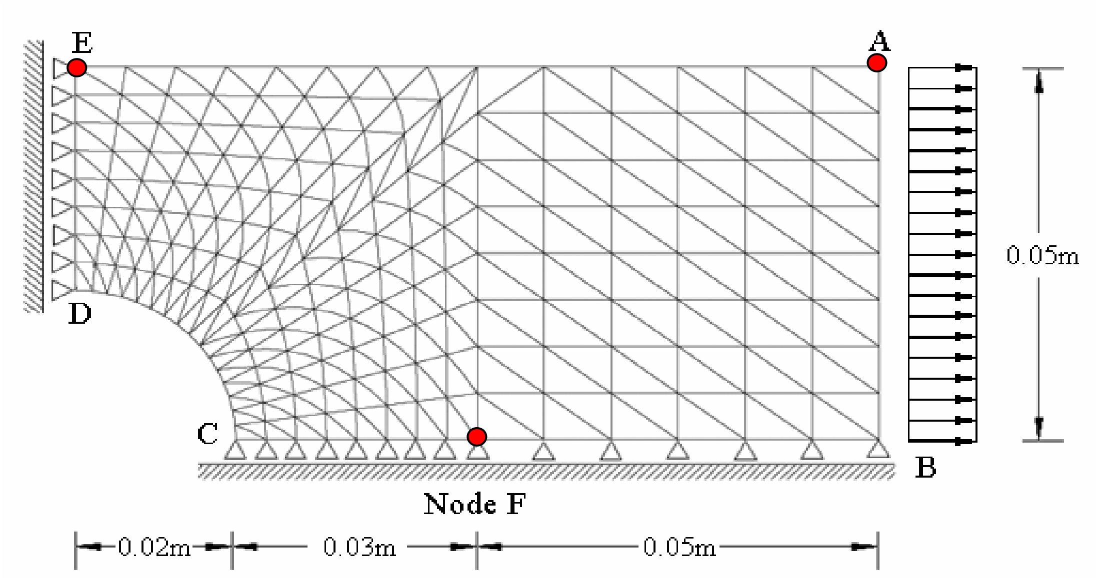
Computational time: a comparison of the senstitivity analysis method and the present method



Computational time (seconds)

N_v	Sens.	Present
246	1.06	0.72
648	64.05	10.17
1210	965.86	59.7
1692	4100	156.3
2254	14450	358.8
2576	32402	528.45

Plate with quarter-circle cutout



thickness: 0.005m

Possion ratio: 0.3

load: 100kN/m

modulus of elasticity:

$\boldsymbol{E}$ = [199, 201]GPa

number of element: 352

element type: six-node isoparametric quadratic triangle

results presented: $\boldsymbol{u}_A$, $\boldsymbol{v}_E$, $\boldsymbol{\sigma}_{xx}$ and $\boldsymbol{\sigma}_{yy}$ at node F

Plate – case 1

Case 1: the modulus of elasticity for each element varies independently in the interval [199, 201] GPa.

Table: results of selected responses

Response	Monte Carlo sampling*		Present IFEA	
	LB	UB	LB	UB
$\boldsymbol{u}_A$ (10^{-5} m)	1.19094	1.20081	1.18768	1.20387
$\boldsymbol{v}_E$ (10^{-5} m)	−0.42638	−0.42238	−0.42894	−0.41940
σ_{xx} (MPa)	13.164	13.223	12.699	13.690
σ_{yy} (MPa)	1.803	1.882	1.592	2.090

*10^6 samples are made.

Plate – case 2

Case 2: each subdomain has an independent modulus of elasticity.

$\boldsymbol{E}_i = [199,\ 201]$ GPa, for $i = 1, \ldots, 8$

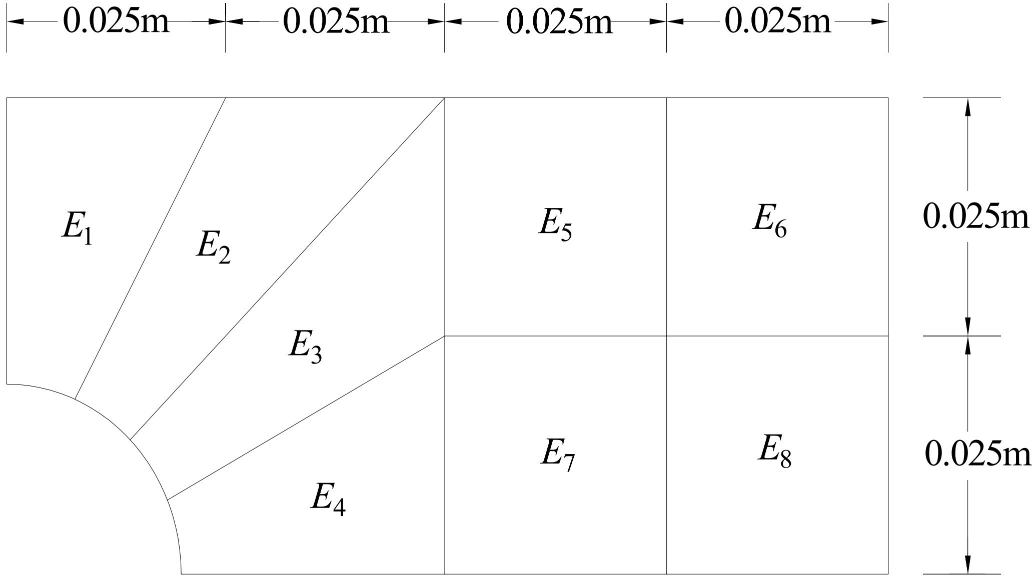


Plate – case 2

Table: results of selected responses

Response	Combinatorial		Present IFEA	
	LB	UB	LB	UB
$\boldsymbol{u}_A$ $(10^{-5}$ m)	1.19002	1.20197	1.18819	1.20368
$\boldsymbol{v}_E$ $(10^{-5}$ m)	−0.42689	−0.42183	−0.42824	−0.42040
σ_{xx} (MPa)	13.158	13.230	12.875	13. 513
σ_{yy} (MPa)	1.797	1.885	1.686	1.996

Outline

- Introduction
- Interval Finite Elements
- Element-by-Element
- Examples
- Conclusions

Conclusions

- Development and implementation of IFEM
  - uncertain material, geometry and load parameters are described by interval variables
  - interval arithmetic is used to guarantee an enclosure of response
- Enhanced dependence problem control
  - use Element-By-Element technique
  - use the penalty method or Lagrange multiplier method to impose constraints
  - modify and enhance fixed point iteration to take into account the dependence problem
  - develop special algorithms to calculate stress and element nodal force

Conclusions

- The method is generally applicable to linear static FEM, regardless of element type
- Evaluation of the present method
  - Rigorousness: in all the examples, the results obtained by the present method enclose those from the alternative methods
  - Accuracy: sharp results are obtained for moderate parameter uncertainty (no more than 5%); reasonable results are obtained for relatively large parameter uncertainty (5%~10%)

Conclusions

- Scalability: the accuracy of the method remains at the same level with increase of the problem scale
- Efficiency: the present method is significantly superior to the conventional methods such as the combinatorial, Monte Carlo sampling, and sensitivity analysis method

- The present IFEM represents an efficient method to handle uncertainty in engineering applications