Penalty-Based Solution for the Interval Finite Element Methods

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Outline

- Interval Finite Elements
- Element-By-Element
- Penalty Approach
- Examples
- Conclusions

Center for Reliable Engineering Computing (REC)

Interval Calculator

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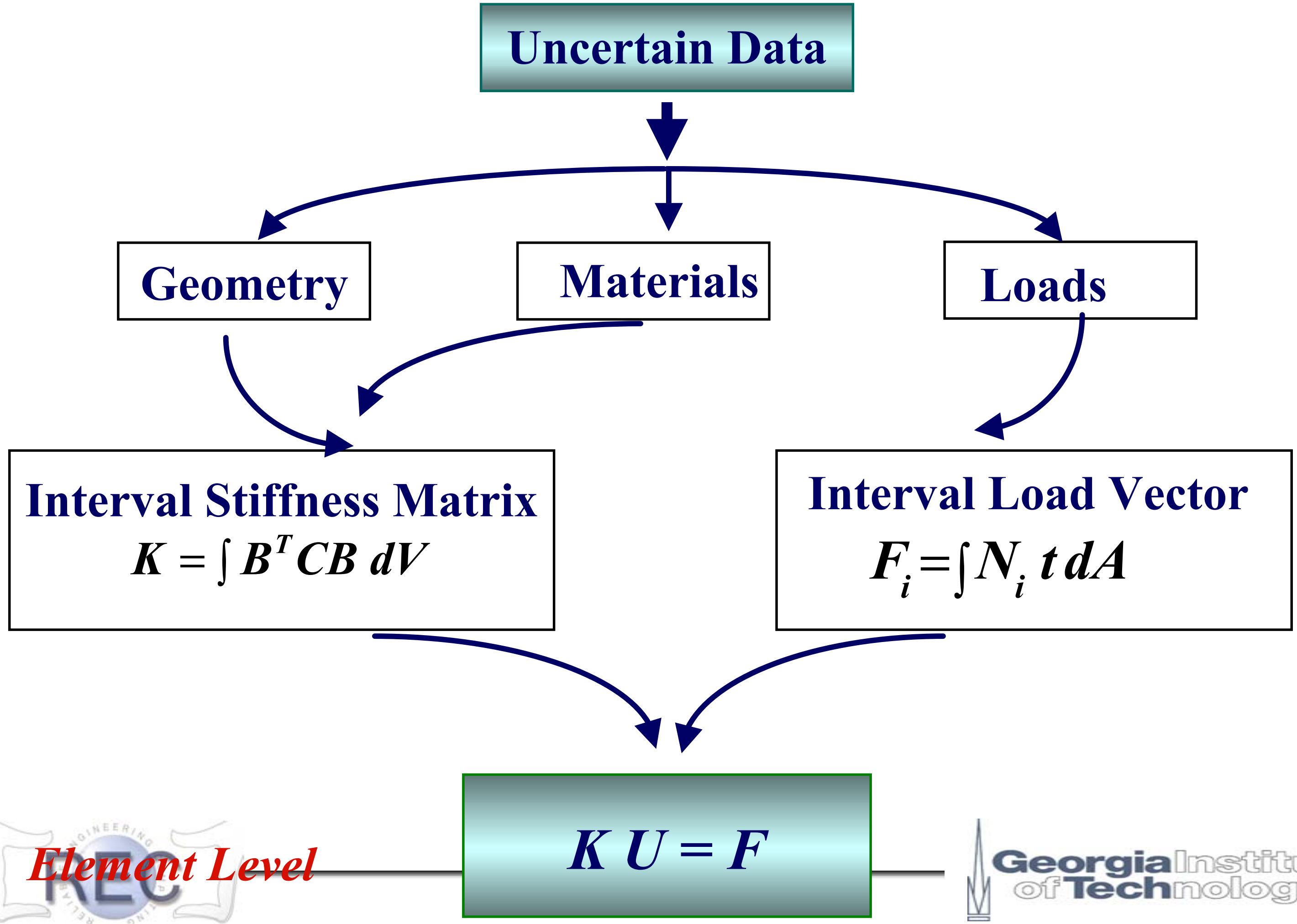
Interval Finite Elements

- Follows conventional FEM
- Loads, nodal geometry and element materials are expressed as interval quantities
- Element-by-element method to avoid element stiffness coupling
- Lagrange Multiplier and Penalty function to impose compatibility
- Iterative approach to get enclosure
- Non-iterative approach to get exact hull for statically determinate structure

Interval Finite Elements



Interval Finite Elements

$$\boldsymbol{K\,U = F}$$

$\boldsymbol{K} = \int \boldsymbol{B}^T \boldsymbol{C\,B}\, d\boldsymbol{V}$ = Interval element stiffness matrix

$\boldsymbol{B}$ = Interval strain-displacement matrix

$\boldsymbol{C}$ = Interval elasticity matrix

$\boldsymbol{F}$ = $[F_1, \ldots F_i, \ldots F_n]$ = Interval element load vector (traction)

$$\boldsymbol{F_i = \int N_i\, t\, dA}$$

$\boldsymbol{N_i}$ = Shape function corresponding to the *i-th* DOF

$\boldsymbol{t}$ = Surface traction

Finite Element

1. Load Dependency
2. Stiffness Dependency

Finite Element – Load Dependency

1. Load Dependency

$$P_b = \sum L^T \int_l N^T b(x)\, dx$$

The global load vector P_b can be written as

$$P_b = M\, q$$

where q is the vector of interval coefficients of the load approximating polynomial

Finite Element – Load Dependency

Sharp solution for the interval displacement can be written as:

$$U = (K^{-1} M)\, q$$

Thus all non-interval values are multiplied first, the last multiplication involves the interval quantities

- ***If this order is not maintained, the resulting interval solution will not be sharp***

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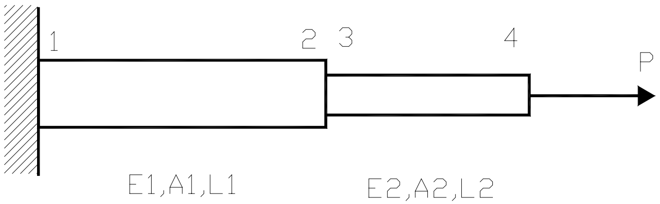
Finite Element – Element-by-Element Approach

- Stiffness Dependency

Coupling (assemblage process)

Finite Element – Element-by-Element Approach

- Coupling



$$k = \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix}, \quad k^{-1} = \begin{pmatrix} \frac{1}{k_1} & \frac{1}{k_1} \\ \frac{1}{k_1} & \frac{k_1 + k_2}{k_1 k_2} \end{pmatrix}, \qquad p = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$u_2 = \frac{1}{k_1}, \quad u_3 = \frac{k_1 + k_2}{k_1 k_2} \qquad (over\ estimation\ in\ u_3, r_3 = 3r_{3-exact})$$

$$u_2 = \frac{1}{k_1}, \quad u_3 = \frac{1}{k_1} + \frac{1}{k_2} \qquad (exact\ solution)$$

Finite Element – Element-by-Element Approach

- Element by Element to construct global stiffness
- Element level

$$K_1 = \begin{pmatrix} \frac{E_1 A_1}{L_1} & -\frac{E_1 A_1}{L_1} \\ -\frac{E_1 A_1}{L_1} & \frac{E_1 A_1}{L_1} \end{pmatrix} = \begin{pmatrix} E_1 & 0 \\ 0 & E_1 \end{pmatrix} \begin{pmatrix} \frac{A_1}{L_1} & -\frac{A_1}{L_1} \\ -\frac{A_1}{L_1} & \frac{A_1}{L_1} \end{pmatrix} = D_1 S_1 = S_1 D_1$$

Finite Element – Element-by-Element Approach

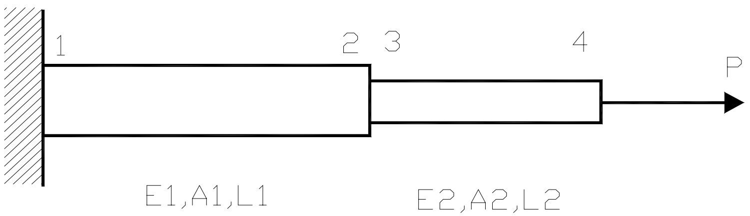
- K: block-diagonal matrix

$$K = \begin{pmatrix} K_1 & & & \\ & K_2 & & \\ & & \ddots & \\ & & & K_n \end{pmatrix} = \begin{pmatrix} D_1 S_1 & & & \\ & D_2 S_2 & & \\ & & \ddots & \\ & & & D_n S_n \end{pmatrix}$$

$$K = \begin{pmatrix} D_1 & & & \\ & D_2 & & \\ & & \ddots & \\ & & & D_n \end{pmatrix} \begin{pmatrix} S_1 & & & \\ & S_2 & & \\ & & \ddots & \\ & & & S_n \end{pmatrix} = \begin{pmatrix} S_1 & & & \\ & S_2 & & \\ & & \ddots & \\ & & & S_n \end{pmatrix} \begin{pmatrix} D_1 & & & \\ & D_2 & & \\ & & \ddots & \\ & & & D_n \end{pmatrix}$$

Finite Element – Element-by-Element Approach

- Element-by-Element



$$K = DS = SD$$

$$D = \begin{pmatrix} E_1 & 0 & 0 & 0 \\ 0 & E_1 & 0 & 0 \\ 0 & 0 & E_2 & 0 \\ 0 & 0 & 0 & E_2 \end{pmatrix}$$

$$S = \begin{pmatrix} \frac{A_1}{L_1} & -\frac{A_1}{L_1} & 0 & 0 \\ -\frac{A_1}{L_1} & \frac{A_1}{L_1} & 0 & 0 \\ 0 & 0 & \frac{A_2}{L_2} & -\frac{A_2}{L_2} \\ 0 & 0 & -\frac{A_2}{L_2} & \frac{A_2}{L_2} \end{pmatrix}$$

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Finite Element – Present Formulation

- In steady-state analysis-variational formulation

$$\Pi = \frac{1}{2}\boldsymbol{U}^{\boldsymbol{T}}\boldsymbol{K}\boldsymbol{U} - \boldsymbol{U}^{\boldsymbol{T}}\boldsymbol{P}$$

- With the constraints $t = CU = 0$

$$\boldsymbol{C} = \begin{pmatrix} 0 & 1 & -1 & 0 \end{pmatrix} \text{ and } \boldsymbol{U}^{\boldsymbol{T}} = \begin{pmatrix} \boldsymbol{U}_1 & \boldsymbol{U}_2 & \boldsymbol{U}_3 & \boldsymbol{U}_4 \end{pmatrix}$$

- Adding the penalty function $\frac{1}{2}t^T\alpha\, t$

α : a diagonal matrix of penalty numbers

$$\Pi^* = \frac{1}{2}U^T KU - U^T P + \frac{1}{2}t^T \alpha t$$

Finite Element – Present Formulation

- Invoking the stationarity of Π^*, that is $\delta\Pi^* = 0$

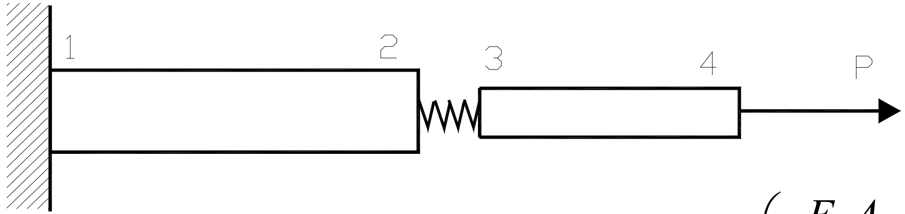
$$(K + C^T \alpha\, C)U = P$$

$$(K + Q)U = P$$

$$C^T = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \qquad Q = C^T \alpha\, C = \alpha \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Finite Element – Penalty Approach

- The physical meaning of Q is an addition of a large spring stiffness



$$Q = \alpha \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$K + Q = \begin{pmatrix} \frac{E_1 A_1}{L_1} & -\frac{E_1 A_1}{L_1} & 0 & 0 \\ -\frac{E_1 A_1}{L_1} & \frac{E_1 A_1}{L_1} + \alpha & -\alpha & 0 \\ 0 & -\alpha & \frac{E_2 A_2}{L_2} + \alpha & -\frac{E_2 A_2}{L_2} \\ 0 & 0 & -\frac{E_2 A_2}{L_2} & \frac{E_2 A_2}{L_2} \end{pmatrix}$$

Finite Element – Penalty Approach

- Interval system of equations

$$(K + Q)U = P \quad \text{or} \quad AU = P$$

- where

$$A = \{\tilde{A} \in R^{n \times n} \mid \tilde{A}_{ik} \in A_{ik} \quad \text{for } i,k = 1,\cdots,n\}$$

$$P = \{\tilde{P} \in R^{n \times 1} \mid \tilde{P}_i \in P_i \quad \text{for } i = 1,\cdots,n\}$$

- and

$$D = \{\tilde{D} \in R^{n \times n} \mid \tilde{D}_{ii} \in D_{ii} \quad \text{for } i = 1,\cdots,n\}$$

$$\boldsymbol{K} = \boldsymbol{DS} = \boldsymbol{SD}$$

Finite Element – Penalty Approach

- The solution will have the following form

$$RP - (I - RA)U \subseteq \text{int}(U)$$

- where R = inverse mid (A) and $U = U^* + U_0$
- or

$$RP - RAU_o + (I - RA)U^* \subseteq \text{int}(U^*)$$

$$z + CU^* \subseteq \text{int}(U^*)$$

Finite Element – Penalty Approach

$$z = RP - RAU_0 = RP - R(K+Q)U_0$$

$$z = RP - RQU_0 - RSDU_0 = RP - RQU_0 - RSM\delta$$

➢ $R = (S+Q)^{-1}$ and $U_0 = RP$

$$C = I - RA = I - RK - RQ = I - RQ - RSD$$

➢ Algorithm converges if and only if $\rho(|C|) < 1$

Finite Element – Penalty Approach

- Rewrite $DU_0 = M\delta$

$$\begin{pmatrix} E_1 & 0 & 0 & 0 \\ 0 & E_1 & 0 & 0 \\ 0 & 0 & E_2 & 0 \\ 0 & 0 & 0 & E_2 \end{pmatrix} \begin{pmatrix} U_{01} \\ U_{02} \\ U_{03} \\ U_{04} \end{pmatrix} = \begin{pmatrix} U_{01} & 0 \\ U_{02} & 0 \\ 0 & U_{03} \\ 0 & U_{04} \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \end{pmatrix} = \begin{pmatrix} E_1 U_{01} \\ E_1 U_{02} \\ E_2 U_{03} \\ E_2 U_{04} \end{pmatrix}$$

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Examples

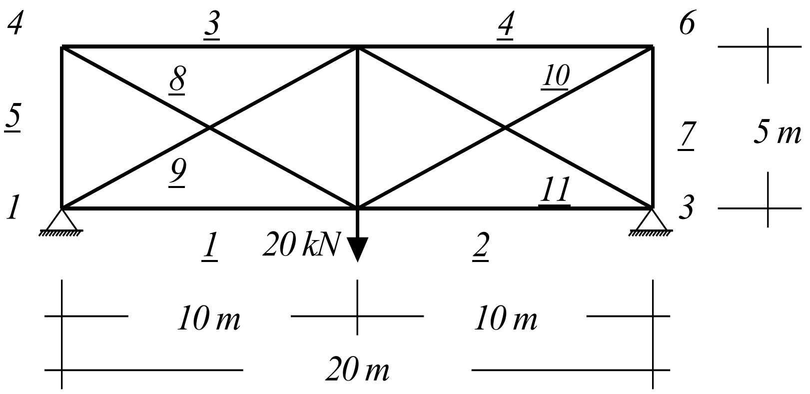
- Statically indeterminate (general case)
  - Two-bay truss
  - Three-bay truss
  - Four-bay truss
  - Statically indeterminate beam
- Statically determinate
  - Three-step bar

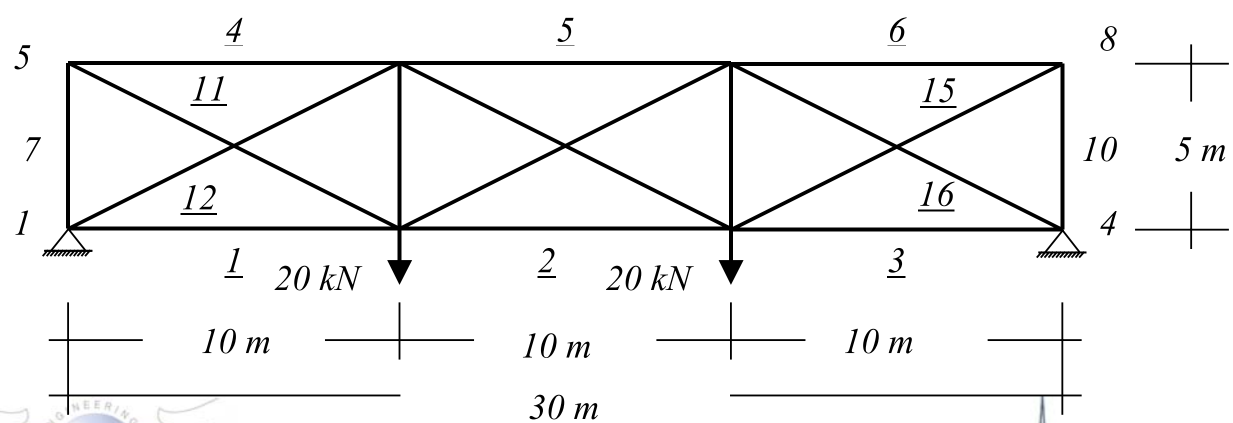
Examples – Stiffness Uncertainty

- Two-bay truss
- Three-bay truss

A = 0.01 m^2

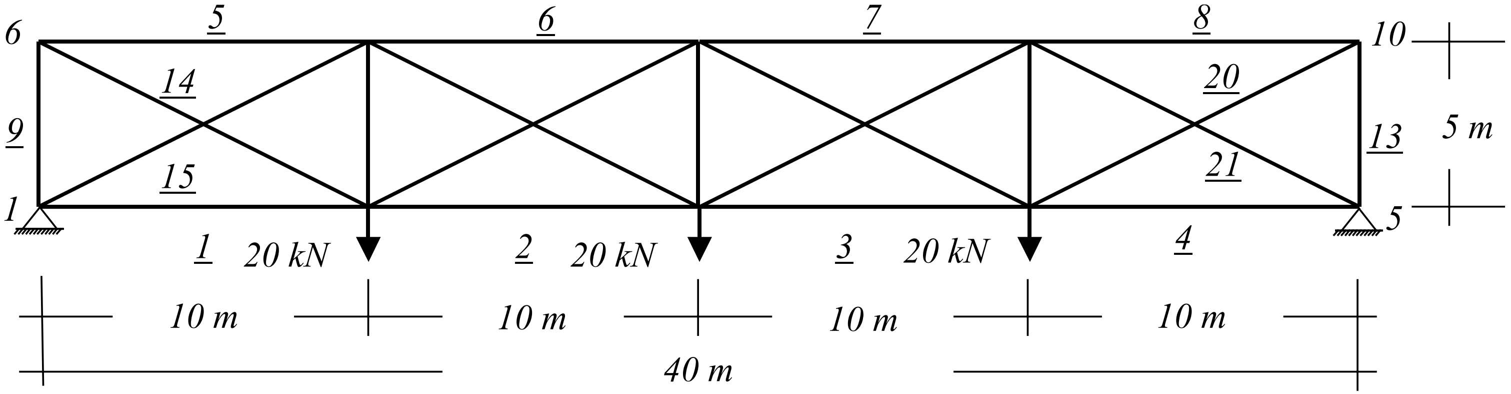
E (nominal) = 200 GPa





Examples – Stiffness Uncertainty

- Four-bay truss



Examples – Stiffness Uncertainty 1%

➢ Two-bay truss

Two bay truss (11 elements) with 1% uncertainty in Modulus of Elasticity, E = [199, 201] GPa

	V2(LB)(m)	V2(UB)(m)	U4(LB)(m)	U4(UB)(m)
Comb $\times 10^{-4}$	– 2.00326	– 1.98333	0.38978	0.40041
Present $\times 10^{-4}$	– 2.00338	– 1.98302	0.38965	0.40050
error	– 0.006%	0.015%	0.033%	– 0.023%

Examples – Stiffness Uncertainty 1%

➢ Three-bay truss

Three bay truss (16 elements) with 1% uncertainty in Modulus of Elasticity, E = [199, 201] GPa

	V2(LB)(m)	V2(UB)(m)	U5(LB)(m)	U5(UB)(m)
Comb $\times 10^{-4}$	– 5.84628	– 5.78663	1.54129	1.56726
Present $\times 10^{-4}$	– 5.84694	– 5.78542	1.5409	1.5675
error	– 0.011%	0.021%	0.025%	– 0.015%

Examples – Stiffness Uncertainty 1%

- **Four-bay truss**

Four-bay truss (21 elements) with 1% uncertainty in Modulus of Elasticity, E = [199, 201] GPa

	V2(LB)(m)	V2(UB)(m)	U6(LB)(m)	U6(UB)(m)	V6(LB)(m)	V6(UB)(m)
Comb $\times 10^{-4}$	– 17.7729	– 17.5942	3.83417	3.88972	– 0.226165	– 0.220082
Present $\times 10^{-4}$	– 17.7752	– 17.5902	3.83268	3.89085	– 0.226255	– 0.21995
error	– 0.013%	0.023%	0.039%	– 0.029%	– 0.040%	0.060%

Examples – Stiffness Uncertainty 5%

- **Two-bay truss**

Two bay truss (11 elements) with 5% uncertainty in Modulus of Elasticity, E = [195, 205] GPa

	V2(LB)(m)	V2(UB)(m)	U4(LB)(m)	U4(UB)(m)
Comb $\times 10^{-4}$	-2.04435	-1.94463	0.36866	0.42188
Present $\times 10^{-4}$	-2.04761	-1.93640	0.36520	0.42448
error	-0.159%	0.423%	0.939%	-0.616%

Examples – Stiffness Uncertainty 5%

➢ Three-bay truss

Three bay truss (16 elements) with 5% uncertainty in Modulus of Elasticity, E = [195, 205] GPa

	V2(LB)(m)	V2(UB)(m)	U5(LB)(m)	U5(UB)(m)
Comb $\times 10^{-4}$	– 5.9692233	– 5.6708065	1.4906613	1.6195115
Present $\times 10^{-4}$	– 5.98838	– 5.63699	1.47675	1.62978
error	– 0.321%	0.596%	0.933%	– 0.634%

Examples – Stiffness Uncertainty 10%

- **Two-bay truss**

Two bay truss (11 elements) with 10% uncertainty in Modulus of Elasticity, E = [190, 210] GPa

	V2(LB)(m)	V2(UB)(m)	U4(LB)(m)	U4(UB)(m)
Comb $\times 10^{-4}$	– 2.09815	– 1.89833	0.34248	0.44917
Present $\times 10^{-4}$	– 2.11418	– 1.86233	0.32704	0.46116
error	– 0.764%	1.896%	4.508%	– 2.669%

Examples – Stiffness Uncertainty 10%

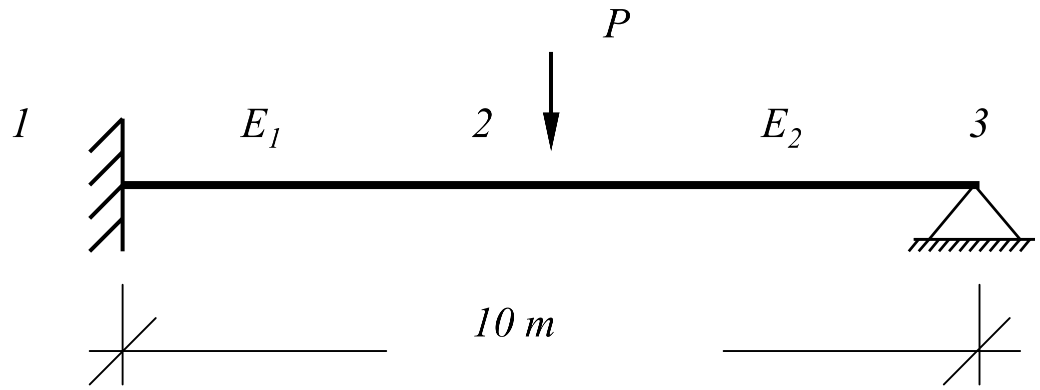
- ## Three-bay truss

Three bay truss (16 elements) with 10% uncertainty in Modulus of Elasticity, E = [190, 210] GPa

	V2(LB)(m)	V2(UB)(m)	U5(LB)(m)	U5(UB)(m)
Comb $\times 10^{-4}$	– 6.13014	– 5.53218	1.42856	1.68687
Present $\times 10^{-4}$	– 6.22965	– 5.37385	1.36236	1.7383
error	– 1.623%	2.862%	4.634%	– 3.049%

Examples – Stiffness and Load Uncertainty

- **Statically indeterminate beam**



A = 0.086 m^2 I = 10^{-4} m^4 E (nominal) = 200 *GPa*

Examples – Stiffness and Load Uncertainty

- **Statically indeterminate beam**

Statically indeterminate beam (2 elements) with 1% uncertainty in Modulus of Elasticity, E = [199, 201] GPa, 10% uncertainty in Load, P=[9.5, 10.5]kN

	V2(LB)(m)	V2(UB)(m)	θ 2(LB)(rad)	θ 2(UB)(rad)
Comb $\times 10^{-3}$	-4.80902	-4.307888	1.47699	1.648869
Present $\times 10^{-3}$	-4.80949	-4.30487	1.47565	1.64928
error	-0.00977%	0.07006%	0.09073%	-0.02493%

Examples – Stiffness and Load Uncertainty

➢ Statically indeterminate beam

Statically indeterminate beam (2 elements) with 1% uncertainty in Modulus of Elasticity, E = [199, 201] GPa, 20% uncertainty in Load, P=[9, 11]kN

	V2(LB)(m)	V2(UB)(m)	θ 2(LB)(rad)	θ 2(UB)(rad)
Comb $\times 10^{-3}$	– 5.03821	– 4.081157	1.399254	1.727387
Present $\times 10^{-3}$	– 5.03884	– 4.07552	1.39672	1.7282
error	– 0.01250%	0.13812%	0.18110%	– 0.04707%

Examples – Stiffness and Load Uncertainty

➢ Statically indeterminate beam

Statically indeterminate beam (2 elements) with 1% uncertainty in Modulus of Elasticity, E = [199, 201] GPa, 40% uncertainty in Load, P=[8, 12]kN

	V2(LB)(m)	V2(UB)(m)	θ 2(LB)(rad)	θ 2(UB)(rad)
Comb $\times 10^{-3}$	-5.49623	-3.62769	1.234378	1.8844221
Present $\times 10^{-3}$	-5.49751	-3.61684	1.23888	1.88604
error	-0.02329%	0.29909%	-0.36472%	-0.08586%

Examples – Stiffness and Load Uncertainty

➢ Three-bay truss

Three bay truss (16 elements) with 1% uncertainty in Modulus of Elasticity, E = [199, 201] GPa, 5% uncertainty in Load, P = [19.5,20.5]kN

	V2(LB)(m)	V2(UB)(m)	U4(LB)(m)	U4(UB)(m)
Comb $\times 10^{-4}$	-2.05334	-1.93374	0.38003	0.41042
Present $\times 10^{-4}$	-2.05381	-1.93259	0.37953	0.41062
error	-0.023%	0.060%	0.132%	-0.050%

Examples – Stiffness and Load Uncertainty

➢ Three-bay truss

Three bay truss (16 elements) with 1% uncertainty in Modulus of Elasticity, E = [199, 201] GPa, 10% uncertainty in Load, P = [19,21]kN

	V2(LB)(m)	V2(UB)(m)	U4(LB)(m)	U4(UB)(m)
Comb $\times 10^{-4}$	– 2.10342	– 1.88416	0.37029	0.42043
Present $\times 10^{-4}$	– 2.10425	– 1.88215	0.36941	0.42074
error	– 0.039%	0.107%	0.237%	– 0.075%

Examples – Stiffness and Load Uncertainty

- **Three-bay truss**

Three bay truss (16 elements) with 1% uncertainty in Modulus of Elasticity, E = [199, 201] GPa, 20% uncertainty in Load, P = [18,22]kN

	V2(LB)(m)	V2(UB)(m)	U4(LB)(m)	U4(UB)(m)
Comb $\times 10^{-4}$	-2.20359	-1.78499	0.35080	0.44045
Present $\times 10^{-4}$	-2.20511	-1.78129	0.34917	0.44098
error	-0.069%	0.207%	0.465%	-0.121%

Examples – Statically determinate

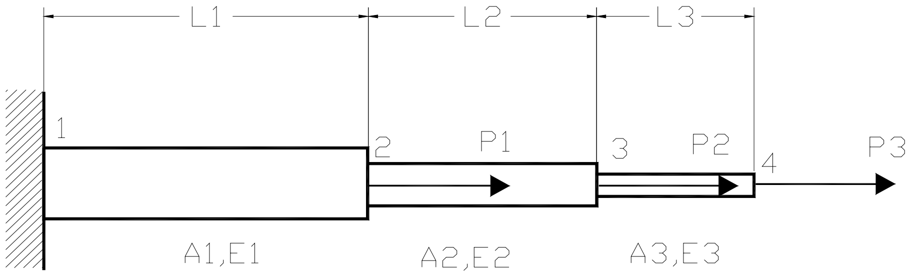
- ## Three-step bar

E1 = [18.5, 21.5]MPa (15% uncertainty)

E2 = [21.875,28.125]MPa (25% uncertainty)

E3 = [24, 36]MPa (40% uncertainty)

P1 = [-9, 9]kN P2 = [-15,15]kN P3 = [2, 18]kN



Examples – Statically determinate

- **Statically determinate 3-step bar**

	U1(LB)(m)	U1(UB)(m)	U2(LB)(m)	U2(UB)(m)	U3(LB)(m)	U3(UB)(m)
Comb $\times 10^{-3}$	-4.756756	9.081081	-7.72818	16.62393	-7.39485	21.1239
Present $\times 10^{-3}$	-4.756756	9.081081	-7.72818	16.62393	-7.39485	21.1239

Conclusions

- Formulation of interval finite element methods (IFEM) is introduced
- EBE approach was used to avoid overestimation
- Penalty approach for IFEM
- Enclosure was obtained with few iterations
- Problem size does not affect results accuracy
- For small *stiffness* uncertainty, the accuracy does not deteriorate with the increase of *load* uncertainty
- In statically determinate case, exact hull was obtained by non-iterative approach