Penalty-Based Solution for the Interval Finite Element Methods

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Outline

- Interval Finite Elements
- Element-By-Element
- Penalty Approach
- Examples
- Conclusions
Center for Reliable Engineering Computing (REC)
Outline

- Interval Finite Elements
- Element-by-Element
- Penalty Approach
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Interval Finite Elements

- Follows conventional FEM
- Loads, nodal geometry and element materials are expressed as interval quantities
- Element-by-element method to avoid element stiffness coupling
- Lagrange Multiplier and Penalty function to impose compatibility
- Iterative approach to get enclosure
- Non-iterative approach to get exact hull for statically determinate structure
Interval Finite Elements

Uncertain Data

Geometry

Materials

Interval Stiffness Matrix

\[ K = \int B^T CB \, dV \]

Interval Load Vector

\[ F_i = \int N_i \, t \, dA \]

Element Level

\[ KU = F \]
Interval Finite Elements

\[ K \mathbf{u} = \mathbf{f} \]

- \( K = \int B^T C B \, dV \) = Interval element stiffness matrix
- \( B \) = Interval strain-displacement matrix
- \( C \) = Interval elasticity matrix
- \( \mathbf{f} = [F_1, \ldots, F_i, \ldots, F_n] \) = Interval element load vector (traction)
- \( F_i = \int N_i \, t \, dA \)
- \( N_i \) = Shape function corresponding to the \( i \)-th DOF
- \( t \) = Surface traction
Finite Element

1. Load Dependency
2. Stiffness Dependency
1. Load Dependency

\[ P_b = \sum L^T \int \! N^T b(x) \, dx \]

The global load vector \( P_b \) can be written as

\[ P_b = M \, q \]

where \( q \) is the vector of interval coefficients of the load approximating polynomial.
Finite Element – Load Dependency

Sharp solution for the interval displacement can be written as:

$$U = (K^{-1} \; M) \; q$$

Thus all non-interval values are multiplied first, the last multiplication involves the interval quantities

➢ If this order is not maintained, the resulting interval solution will not be sharp
Outline

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Finite Element – Element-by-Element Approach

- Stiffness Dependency

Coupling (assemblage process)
Finite Element – Element-by-Element Approach

Coupling

\[ k = \begin{pmatrix}
  k_1 + k_2 & -k_2 \\
  -k_2 & k_2
\end{pmatrix}, \quad k^{-1} = \begin{pmatrix}
  \frac{1}{k_1} & \frac{1}{k_1 + k_2} \\
  \frac{1}{k_1} & \frac{k_1}{k_1 k_2}
\end{pmatrix}, \quad p = \begin{pmatrix}
  0 \\
  1
\end{pmatrix} \]

\[ u_2 = \frac{1}{k_1}, \quad u_3 = \frac{k_1 + k_2}{k_1 k_2} \quad \text{(over estimation in } u_3, r_3 = 3r_{3\text{-exact}} \text{)} \]

\[ u_2 = \frac{1}{k_1}, \quad u_3 = \frac{1}{k_1} + \frac{1}{k_2} \quad \text{(exact solution)} \]
Finite Element – Element-by-Element Approach

Element by Element to construct global stiffness

Element level

\[ K_1 = \begin{pmatrix}
\frac{E_1 A_1}{L_1} & -\frac{E_1 A_1}{L_1} \\
-\frac{E_1 A_1}{L_1} & \frac{E_1 A_1}{L_1}
\end{pmatrix} = \begin{pmatrix}
E_1 & 0 \\
0 & E_1
\end{pmatrix} \begin{pmatrix}
\frac{A_1}{L_1} & -\frac{A_1}{L_1} \\
-\frac{A_1}{L_1} & \frac{A_1}{L_1}
\end{pmatrix} = D_1 S_1 = S_1 D_1 \]
Finite Element – Element-by-Element Approach

- $K$: block-diagonal matrix

\[
K = \begin{pmatrix}
  K_1 & & \\
  & K_2 & \\
  & & \ddots \\
  & & & K_n
\end{pmatrix} = \begin{pmatrix}
  D_1S_1 & & \\
  & D_2S_2 & \\
  & & \ddots \\
  & & & D_nS_n
\end{pmatrix}
\]

\[
K = \begin{pmatrix}
  D_1 & & \\
  & D_2 & \\
  & & \ddots \\
  & & & D_n
\end{pmatrix} \begin{pmatrix}
  S_1 & & \\
  & S_2 & \\
  & & \ddots \\
  & & & S_n
\end{pmatrix} = \begin{pmatrix}
  S_1 & & \\
  & S_2 & \\
  & & \ddots \\
  & & & S_n
\end{pmatrix} \begin{pmatrix}
  D_1 & & \\
  & D_2 & \\
  & & \ddots \\
  & & & D_n
\end{pmatrix}
\]
Finite Element – Element-by-Element Approach

Element-by-Element

\[ D = \begin{pmatrix} E_1 & 0 & 0 & 0 \\ 0 & E_1 & 0 & 0 \\ 0 & 0 & E_2 & 0 \\ 0 & 0 & 0 & E_2 \end{pmatrix} \]

\[ S = \begin{pmatrix} \frac{A_1}{L_1} & -\frac{A_1}{L_1} & 0 & 0 \\ -\frac{A_1}{L_1} & \frac{A_1}{L_1} & 0 & 0 \\ 0 & 0 & \frac{A_2}{L_2} & -\frac{A_2}{L_2} \\ 0 & 0 & -\frac{A_2}{L_2} & \frac{A_2}{L_2} \end{pmatrix} \]

\[ K = DS = SD \]
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- Interval Finite Elements
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Finite Element – Present Formulation

- In steady-state analysis-variational formulation

\[ \Pi = \frac{1}{2} U^T KU - U^T P \]

- With the constraints \( t = C U = 0 \)

\[ C = \begin{pmatrix} 0 & 1 & -1 & 0 \end{pmatrix} \text{ and } U^T = \begin{pmatrix} U_1 & U_2 & U_3 & U_4 \end{pmatrix} \]

- Adding the penalty function \( \frac{1}{2} t^T \alpha t \)

\( \alpha \) : a diagonal matrix of penalty numbers

\[ \Pi^* = \frac{1}{2} U^T KU - U^T P + \frac{1}{2} t^T \alpha t \]
Invoking the stationarity of $\Pi^*$, that is $\delta \Pi^* = 0$,

$$(K + C^T \alpha C)U = P$$

$$(K + Q)U = P$$

$$C^T = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \quad Q = C^T \alpha C = \alpha \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
The physical meaning of \( Q \) is an addition of a large spring stiffness

\[
Q = \alpha \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 \\
0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
K + Q = \begin{pmatrix}
\frac{E_1 A_1}{L_1} & -\frac{E_1 A_1}{L_1} & 0 & 0 \\
-\frac{E_1 A_1}{L_1} & \frac{E_1 A_1}{L_1} + \alpha & -\alpha & 0 \\
0 & -\alpha & \frac{E_2 A_2}{L_2} + \alpha & -\frac{E_2 A_2}{L_2} \\
0 & 0 & -\frac{E_2 A_2}{L_2} & \frac{E_2 A_2}{L_2}
\end{pmatrix}
\]
Finite Element – Penalty Approach

- Interval system of equations
  \[(K + Q)U = P\] or \[AU = P\]

- where
  \[A = \{\widetilde{A} \in R^{n \times n} \mid \widetilde{A}_{ik} \in A_{ik} \text{ for } i, k = 1, \ldots, n\}\]
  \[P = \{\widetilde{P} \in R^{n \times 1} \mid \widetilde{P}_i \in P_i \text{ for } i = 1, \ldots, n\}\]

- and
  \[D = \{\widetilde{D} \in R^{n \times n} \mid \widetilde{D}_{ii} \in D_{ii} \text{ for } i = 1, \ldots, n\}\]
  \[K = DS = SD\]
Finite Element – Penalty Approach

➤ The solution will have the following form

\[ RP - (I - RA)U \subseteq \text{int}(U) \]

➤ where \( R = \) inverse mid \((A)\) and \( U = U^* + U_0 \)

➤ or

\[ RP - RAU_0 + (I - RA)U^* \subseteq \text{int}(U^*) \]

\[ z + CU^* \subseteq \text{int}(U^*) \]
Finite Element – Penalty Approach

\[ z = RP - RAU_0 = RP - R(K + Q)U_0 \]
\[ z = RP - RQU_0 - RSDU_0 = RP - RQU_0 - RSM\delta \]

- \[ R = (S + Q)^{-1} \] and \[ U_0 = RP \]

\[ C = I - RA = I - RK - RQ = I - RQ - RSD \]

- Algorithm converges if and only if \[ \rho (|C|) < 1 \]
Finite Element – Penalty Approach

Rewrite \( DU_0 = M \delta \)

\[
\begin{pmatrix}
E_1 & 0 & 0 & 0 \\
0 & E_1 & 0 & 0 \\
0 & 0 & E_2 & 0 \\
0 & 0 & 0 & E_2
\end{pmatrix}
\begin{pmatrix}
U_{01} \\
U_{02} \\
U_{03} \\
U_{04}
\end{pmatrix}
= 
\begin{pmatrix}
U_{01} & 0 \\
U_{02} & 0 \\
0 & U_{03} \\
0 & U_{04}
\end{pmatrix}
\begin{pmatrix}
E_1 \\
E_2
\end{pmatrix}
= 
\begin{pmatrix}
E_1 U_{01} \\
E_1 U_{02} \\
E_2 U_{03} \\
E_2 U_{04}
\end{pmatrix}
\]
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- Interval Finite Elements
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Examples

- Statically indeterminate (general case)
  - Two-bay truss
  - Three-bay truss
  - Four-bay truss
  - Statically indeterminate beam

- Statically determinate
  - Three-step bar
Examples – Stiffness Uncertainty

- Two-bay truss
  - $A = 0.01 \, m^2$
  - $E$ (nominal) = 200 $GPa$

- Three-bay truss

![Two-bay truss diagram]

![Three-bay truss diagram]
Examples – **Stiffness Uncertainty**

- **Four-bay truss**
Examples – Stiffness Uncertainty 1%

- **Two-bay truss**

  Two bay truss (11 elements) with 1% uncertainty in Modulus of Elasticity, $E = [199, 201]$ GPa

<table>
<thead>
<tr>
<th></th>
<th>V2(LB)(m)</th>
<th>V2(UB)(m)</th>
<th>U4(LB)(m)</th>
<th>U4(UB)(m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Comb $\times 10^{-4}$</td>
<td>$-2.00326$</td>
<td>$-1.98333$</td>
<td>$0.38978$</td>
<td>$0.40041$</td>
</tr>
<tr>
<td>Present $\times 10^{-4}$</td>
<td>$-2.00338$</td>
<td>$-1.98302$</td>
<td>$0.38965$</td>
<td>$0.40050$</td>
</tr>
<tr>
<td>error</td>
<td>$-0.006%$</td>
<td>$0.015%$</td>
<td>$0.033%$</td>
<td>$-0.023%$</td>
</tr>
</tbody>
</table>
### Examples – Stiffness Uncertainty 1%

#### Three-bay truss

Three bay truss (16 elements) with 1% uncertainty in Modulus of Elasticity, \( E = [199, 201] \) GPa

<table>
<thead>
<tr>
<th></th>
<th>V2(LB)(m)</th>
<th>V2(UB)(m)</th>
<th>U5(LB)(m)</th>
<th>U5(UB)(m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Comb ( \times 10^{-4} )</td>
<td>− 5.84628</td>
<td>− 5.78663</td>
<td>1.54129</td>
<td>1.56726</td>
</tr>
<tr>
<td>Present ( \times 10^{-4} )</td>
<td>− 5.84694</td>
<td>− 5.78542</td>
<td>1.5409</td>
<td>1.5675</td>
</tr>
<tr>
<td>error</td>
<td>− 0.011%</td>
<td>0.021%</td>
<td>0.025%</td>
<td>− 0.015%</td>
</tr>
</tbody>
</table>
**Examples – Stiffness Uncertainty 1%**

### Four-bay truss

Four-bay truss (21 elements) with 1% uncertainty in Modulus of Elasticity, $E = [199, 201]$ GPa

<table>
<thead>
<tr>
<th>Comb $\times 10^{-4}$</th>
<th>Present $\times 10^{-4}$</th>
<th>error</th>
</tr>
</thead>
<tbody>
<tr>
<td>V2(LB)(m)</td>
<td>V2(UB)(m)</td>
<td>U6(LB)(m)</td>
</tr>
<tr>
<td>− 17.7729</td>
<td>− 17.5942</td>
<td>3.83417</td>
</tr>
<tr>
<td>− 17.7752</td>
<td>− 17.5902</td>
<td>3.83268</td>
</tr>
<tr>
<td>− 0.013%</td>
<td>0.023%</td>
<td>0.039%</td>
</tr>
</tbody>
</table>
Examples – **Stiffness Uncertainty 5%**

➢ **Two-bay truss**

Two bay truss (11 elements) with 5% uncertainty in Modulus of Elasticity, \( E = [195, 205] \) GPa

<table>
<thead>
<tr>
<th></th>
<th>V2(LB)(m)</th>
<th>V2(UB)(m)</th>
<th>U4(LB)(m)</th>
<th>U4(UB)(m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Comb × 10^{-4}</td>
<td>– 2.04435</td>
<td>– 1.94463</td>
<td>0.36866</td>
<td>0.42188</td>
</tr>
<tr>
<td>Present × 10^{-4}</td>
<td>– 2.04761</td>
<td>– 1.93640</td>
<td>0.36520</td>
<td>0.42448</td>
</tr>
<tr>
<td>error</td>
<td>– 0.159%</td>
<td>0.423%</td>
<td>0.939%</td>
<td>– 0.616%</td>
</tr>
</tbody>
</table>
Examples – Stiffness Uncertainty 5%

Three-bay truss

Three bay truss (16 elements) with 5% uncertainty in Modulus of Elasticity, $E = [195, 205]$ GPa

<table>
<thead>
<tr>
<th></th>
<th>V2(LB)(m)</th>
<th>V2(UB)(m)</th>
<th>U5(LB)(m)</th>
<th>U5(UB)(m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Comb $\times 10^{-4}$</td>
<td>$-5.9692233$</td>
<td>$-5.6708065$</td>
<td>$1.4906613$</td>
<td>$1.6195115$</td>
</tr>
<tr>
<td>Present $\times 10^{-4}$</td>
<td>$-5.98838$</td>
<td>$-5.63699$</td>
<td>$1.47675$</td>
<td>$1.62978$</td>
</tr>
<tr>
<td>error</td>
<td>$-0.321%$</td>
<td>$0.596%$</td>
<td>$0.933%$</td>
<td>$-0.634%$</td>
</tr>
</tbody>
</table>
Examples – Stiffness Uncertainty 10%

➢ Two-bay truss

Two bay truss (11 elements) with 10% uncertainty in Modulus of Elasticity, \( E = [190, 210] \) GPa

<table>
<thead>
<tr>
<th></th>
<th>V2(LB)(m)</th>
<th>V2(UB)(m)</th>
<th>U4(LB)(m)</th>
<th>U4(UB)(m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Comb ( \times 10^{-4} )</td>
<td>– 2.09815</td>
<td>– 1.89833</td>
<td>0.34248</td>
<td>0.44917</td>
</tr>
<tr>
<td>Present ( \times 10^{-4} )</td>
<td>– 2.11418</td>
<td>– 1.86233</td>
<td>0.32704</td>
<td>0.46116</td>
</tr>
<tr>
<td>error</td>
<td>– 0.764%</td>
<td>1.896%</td>
<td>4.508%</td>
<td>– 2.669%</td>
</tr>
</tbody>
</table>
Examples – **Stiffness Uncertainty 10%**

➢ **Three-bay truss**

Three bay truss (16 elements) with 10% uncertainty in Modulus of Elasticity, $E = [190, 210]$ GPa

<table>
<thead>
<tr>
<th></th>
<th>V2(LB)(m)</th>
<th>V2(UB)(m)</th>
<th>U5(LB)(m)</th>
<th>U5(UB)(m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Comb $\times 10^{-4}$</td>
<td>– 6.13014</td>
<td>– 5.53218</td>
<td>1.42856</td>
<td>1.68687</td>
</tr>
<tr>
<td>Present $\times 10^{-4}$</td>
<td>– 6.22965</td>
<td>– 5.37385</td>
<td>1.36236</td>
<td>1.7383</td>
</tr>
<tr>
<td>error</td>
<td>– 1.623%</td>
<td>2.862%</td>
<td>4.634%</td>
<td>– 3.049%</td>
</tr>
</tbody>
</table>
Examples – Stiffness and Load Uncertainty

- Statically indeterminate beam

\[ A = 0.086 \, m^2 \quad I = 10^{-4} \, m^4 \quad E \text{ (nominal)} = 200 \, GPa \]

\[ P \]

10 m
Examples – **Stiffness and Load Uncertainty**

➤ **Statically indeterminate beam**

Statically indeterminate beam (2 elements) with 1% uncertainty in Modulus of Elasticity, $E = [199, 201]$ GPa, 10% uncertainty in Load, $P = [9.5, 10.5]$ kN

<table>
<thead>
<tr>
<th></th>
<th>V2(LB)(m)</th>
<th>V2(UB)(m)</th>
<th>$\theta$ 2(LB)(rad)</th>
<th>$\theta$ 2(UB)(rad)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Comb $\times 10^{-3}$</td>
<td>– 4.80902</td>
<td>– 4.307888</td>
<td>1.47699</td>
<td>1.648869</td>
</tr>
<tr>
<td>Present $\times 10^{-3}$</td>
<td>– 4.80949</td>
<td>– 4.30487</td>
<td>1.47565</td>
<td>1.64928</td>
</tr>
<tr>
<td>error</td>
<td>– 0.00977%</td>
<td>0.07006%</td>
<td>0.09073%</td>
<td>– 0.02493%</td>
</tr>
</tbody>
</table>
Examples – Stiffness and Load Uncertainty

Statically indeterminate beam

Statically indeterminate beam (2 elements) with 1% uncertainty in Modulus of Elasticity, \( E = [199, 201] \) GPa, 20% uncertainty in Load, \( P=[9, 11] \) kN

<table>
<thead>
<tr>
<th></th>
<th>( V_2(LB)(m) )</th>
<th>( V_2(UB)(m) )</th>
<th>( \theta_2(LB)(\text{rad}) )</th>
<th>( \theta_2(UB)(\text{rad}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Comb ( \times 10^{-3} )</td>
<td>− 5.03821</td>
<td>− 4.081157</td>
<td>1.399254</td>
<td>1.727387</td>
</tr>
<tr>
<td>Present ( \times 10^{-3} )</td>
<td>− 5.03884</td>
<td>− 4.07552</td>
<td>1.39672</td>
<td>1.7282</td>
</tr>
<tr>
<td>error</td>
<td>− 0.01250%</td>
<td>0.13812%</td>
<td>0.18110%</td>
<td>− 0.04707%</td>
</tr>
</tbody>
</table>
Examples – Stiffness and Load Uncertainty

Statically indeterminate beam

Statically indeterminate beam (2 elements) with 1% uncertainty in Modulus of Elasticity, \( E = [199, 201] \) GPa, 40% uncertainty in Load, \( P = [8, 12] \) kN

<table>
<thead>
<tr>
<th></th>
<th>( V_2(\text{LB})(m) )</th>
<th>( V_2(\text{UB})(m) )</th>
<th>( \theta_2(\text{LB})(\text{rad}) )</th>
<th>( \theta_2(\text{UB})(\text{rad}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Comb ( \times 10^{-3} )</td>
<td>– 5.49623</td>
<td>– 3.62769</td>
<td>1.234378</td>
<td>1.8844221</td>
</tr>
<tr>
<td>Present ( \times 10^{-3} )</td>
<td>– 5.49751</td>
<td>– 3.61684</td>
<td>1.23888</td>
<td>1.88604</td>
</tr>
<tr>
<td>error</td>
<td>– 0.02329%</td>
<td>0.29909%</td>
<td>– 0.36472%</td>
<td>– 0.08586%</td>
</tr>
</tbody>
</table>
Examples – **Stiffness and Load Uncertainty**

➢ Three-bay truss

Three bay truss (16 elements) with 1% uncertainty in Modulus of Elasticity, $E = [199, 201]$ GPa, 5% uncertainty in Load, $P = [19.5, 20.5]$ kN

<table>
<thead>
<tr>
<th></th>
<th>$V2(LB)(m)$</th>
<th>$V2(UB)(m)$</th>
<th>$U4(LB)(m)$</th>
<th>$U4(UB)(m)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Comb $\times 10^{-4}$</td>
<td>–2.05334</td>
<td>–1.93374</td>
<td>0.38003</td>
<td>0.41042</td>
</tr>
<tr>
<td>Present $\times 10^{-4}$</td>
<td>–2.05381</td>
<td>–1.93259</td>
<td>0.37953</td>
<td>0.41062</td>
</tr>
<tr>
<td>error</td>
<td>–0.023%</td>
<td>0.060%</td>
<td>0.132%</td>
<td>–0.050%</td>
</tr>
</tbody>
</table>
### Three-bay truss

Three bay truss (16 elements) with 1% uncertainty in Modulus of Elasticity, \( E = [199, 201] \) GPa, 10% uncertainty in Load, \( P = [19, 21] \) kN

<table>
<thead>
<tr>
<th></th>
<th>( V_2(\text{LB})(m) )</th>
<th>( V_2(\text{UB})(m) )</th>
<th>( U_4(\text{LB})(m) )</th>
<th>( U_4(\text{UB})(m) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Comb ( \times 10^{-4} )</td>
<td>-2.10342</td>
<td>-1.88416</td>
<td>0.37029</td>
<td>0.42043</td>
</tr>
<tr>
<td>Present ( \times 10^{-4} )</td>
<td>-2.10425</td>
<td>-1.88215</td>
<td>0.36941</td>
<td>0.42074</td>
</tr>
<tr>
<td>error</td>
<td>-0.039%</td>
<td>0.107%</td>
<td>0.237%</td>
<td>-0.075%</td>
</tr>
</tbody>
</table>
Examples – Stiffness and Load Uncertainty

➢ Three-bay truss

Three bay truss (16 elements) with 1% uncertainty in Modulus of Elasticity,  $E = [199, 201]$ GPa, 20% uncertainty in Load, $P = [18, 22]$ kN

<table>
<thead>
<tr>
<th>Comb $\times 10^{-4}$</th>
<th>V2(LB)(m)</th>
<th>V2(UB)(m)</th>
<th>U4(LB)(m)</th>
<th>U4(UB)(m)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>− 2.20359</td>
<td>− 1.78499</td>
<td>0.35080</td>
<td>0.44045</td>
</tr>
<tr>
<td>Present $\times 10^{-4}$</td>
<td>− 2.20511</td>
<td>− 1.78129</td>
<td>0.34917</td>
<td>0.44098</td>
</tr>
<tr>
<td>error</td>
<td>− 0.069%</td>
<td>0.207%</td>
<td>0.465%</td>
<td>− 0.121%</td>
</tr>
</tbody>
</table>
Examples – Statically determinate

Three-step bar

E1 = [18.5, 21.5]MPa (15% uncertainty)
E2 = [21.875, 28.125]MPa (25% uncertainty)
E3 = [24, 36]MPa (40% uncertainty)
P1 = [-9, 9]kN P2 = [-15, 15]kN P3 = [2, 18]kN
## Examples – Statically determinate

### Statically determinate 3-step bar

<table>
<thead>
<tr>
<th></th>
<th>U1(LB)(m)</th>
<th>U1(UB)(m)</th>
<th>U2(LB)(m)</th>
<th>U2(UB)(m)</th>
<th>U3(LB)(m)</th>
<th>U3(UB)(m)</th>
</tr>
</thead>
</table>
Conclusions

- Formulation of interval finite element methods (IFEM) is introduced
- EBE approach was used to avoid overestimation
- Penalty approach for IFEM
- Enclosure was obtained with few iterations
- Problem size does not affect results accuracy
- For small stiffness uncertainty, the accuracy does not deteriorate with the increase of load uncertainty
- In statically determinate case, exact hull was obtained by non-iterative approach