Mechanics of Uncertainty: Managing Uncertainty in Mechanics

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Uncertainty is ubiquitous in the natural, engineered, and social environments. Devising rationales for explaining it, strategies for its integration into scientific determinism and mitigating its consequences has been an active arena of rational endeavor where many scientific concepts have taken turn at fame and infamy. Far from being a static concept, uncertainty is the complement of knowledge, and as thus, continually adapts itself to knowledge, feeding on its evolution to redefine its claim over science.

Mechanics is a framework for applying deductive and mathematical reasoning to enhance our understanding of the physical world. Thus, far from being accidental, the interaction of mechanics and uncertainty is rather by design, as they both mold the physical world in a complementary fashion. The substance of this interaction is attested to by the simultaneous evolution of mechanics and rational models of uncertainty as embodied, for example, in the contributions of Gauss, Euler, Legendre, Laplace, Einstein, Feynman, and vonMises.

Two driving forces behind a significant portion of current scientific research can be associated with technological developments in the areas of computing and sensing. Indeed, it has only recently become possible to resolve, numerically, very complex models of physical phenomena, as well as to probe these phenomena over length-scales that span orders of magnitude. This facility for doing science significantly changes the realm over which uncertainty can claim a hold, and merits a reconsideration of the scientific questions enabled through uncertainty modeling.

The paper focuses on a particular class of recent developments related to the quantification, propagation, and management of uncertainty, using a probabilistic framework. An attempt is made at presenting a formalism that facilitates the adaptive quantification of uncertainty and of its effect on mechanics-based predictions. In addition to the more traditional quest for estimating the probability of extreme events such as failure, attention is given to estimating the confidence in model predictions and to adaptive schemes for improving this confidence through model refinement (mechanistic and numerical) as well as data refinement. The possibility of performing such an adaptation can play a significant role in shaping performance-based design practice in science and engineering by quantifying the information, and its associated worth, required to

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achieve a target confidence in the predicted behavior of some contemplated design. The concept of combined stochastic-deterministic error and its estimation is introduced that permits the development of optimal numerical optimization strategies such as adaptive mesh refinement that are consistent with the level of accuracy justified by available data. Concepts are presented that can guide the simultaneous refinement of mesh and data. The uncertainty quantification, propagation and management framework to be presented is based on Hilbert space representations and projections of random functions. This permits the framework to be integrated with other Hilbert space representations used in computational mechanics and signal analysis.

Figure (1) shows a diagram of the various entities relevant to defining the problem. A mapping, $\mathcal{M}$, can be conceptually defined between events corresponding to random parameters and events corresponding to measurable outputs. Referring to Figure (1), this mapping takes subsets $\mathcal{T}$ into subsets $\mathcal{Q} = \mathcal{M}(\mathcal{T})$. A measure can be associated to each subset $\mathcal{T}$ reflecting a subjective assessment of the likelihood of that event occurring. The corresponding measure on $\mathcal{Q}$ is uniquely computable by the mapping $\mathcal{M}(.)$. In order to facilitate the analysis, and given that most measurable events of interest refer to numerical measurements, it is usually convenient to work on measures defined on the real line. Random variables provide a mechanism for effecting that, as they are defined as mappings from the set of basic events onto the real line. Loosely speaking, each of the events $\mathcal{T}$ and $\mathcal{Q}$ are mapped, through appropriate random variables, into subsets of the real line, the measure of which are identified with the measure of the corresponding event. The mapping $\mathcal{M}(.)$ can now be replaced by a new mapping $\mathcal{N}(.)$ between the random variables. It is usually this mapping that is dealt with in the context of mechanistic modeling. It can be shown that, as mappings go, second order random variables have a number of very interesting properties, so much so that they collectively form a Hilbert space when endowed with the inner product defined through the operation of statistical correlation. This Hilbert space structure is very convenient as it forms much of the foundation of deterministic numerical analysis: projections on subspaces as well as convergent approximations can now be meaningfully defined and implemented. It is clear, from Figure (1) that the space of random variables η contains that of the variables γ as a proper subset.

Contributions to formulating and solving the problem can be made at a number of key steps. Firstly, events $\mathcal{Q}$ of interest to a given problem must be identified. These consist, for example, of failure events, of levels of exceedance, or of some death/birth events. A mathematical description of the set to which these events belong must be adopted that is rich enough to distinguish between the various events of interest. Moreover, a corresponding, consistent, topology over the set to which events $\mathcal{T}$ belong must be identified that is commensurate with available instrumentation technology. Secondly, the measure on the various events $\mathcal{T}$ in the data space must be estimated. In a probabilistic framework such as the present one, this amounts to estimating the probability measure of the various events. The outcome of this task, and hence the confidence in the overall uncertainty quantification and propagation process, depends in no small measure on the choice of model and data used to calibrate the associated parameters. The ability to adaptively update the probabilistic characterization of the associated random variables is shown to be a significant feature of the framework to be presented in the paper. Thirdly, an algorithm for propagating the uncertainty in the parameters into uncertainty in the predictions must be developed. This algorithm should clearly take into consideration the mechanistic model adopted for the problem, the accuracy with which numerical predictions of this model can be resolved, as well as the practical questions that the uncertainty analysis is trying to address. Obviously, this uncertainty propagation task can be performed, at least conceptually, by relying on the Monte Carlo simulation (MCS) paradigm. It is shown that methods can be developed that generalize MCS, increasing its efficiency while providing it with improved error estimation capabilities, that are as much based on the mechanics of the problem as on statistics of the data. Based on the above, a solution of the following form is sought,

$$u = \hat{u} + \epsilon_h + \epsilon_p + \epsilon_d$$

where $\hat{u}$ is some computed prediction of the solution, ϵ_h, ϵ_p, and ϵ_d are error estimators that can be controlled by refining the numerical approximations, refining the probabilistic approach, and refining the data, respectively. Adaptive techniques for controlling ϵ_h are well established in deterministic computational mechanics. The formalism presented in the paper demonstrate the development of similar adaptation schemes for ϵ_p and ϵ_d. It is clear that the certification of the predicted solution $\hat{u}$ to be within a specified tolerance requires the rational control of all three error terms. Selection between competing models can then be made by identifying that model which achieves its target error reduction within specified limits on computational and data collection resources.

The problem can then be described as follows: The model parameters, as estimated from data and modeled as random variables or processes, live in the Hilbert space $\mathcal{H}_{\mathcal{G}}$. Assuming the data to be well defined in a probabilistic sense provides a full characterization of this space, in which a set of basis functions, $\boldsymbol{\xi}$, will have to be identified. This is accomplished using the Karhunen-Loeve expansion. The state of the system, again modeled as a random variable or process, resides in the Hilbert space $\mathcal{H}_{\mathcal{L}}$. A set of basis functions, $\boldsymbol{\psi}$, are also identified in this space, which in general are different from the basis $\boldsymbol{\xi}$ since this latter one spans only a subset of the space of second order random variables, namely those that characterize the data. Identifying a basis for the space $\mathcal{H}_{\mathcal{L}}$ is accomplished using the Chaos theory of Wiener. The solution of the problem can then be characterized by its projection on this basis. A Galerkin scheme is used to formulate this projection on a finite dimensional subspace, and equations governing the evolution of the associated coefficients are obtained. This form of characterizing the solution of a probabilistic problem, permits the propagation of the uncertainty with minimal loss of information. Indeed, the solution is being represented as a stochastic process, and not as an integrated norm of that process. Moreover, the representation used for that process, namely the

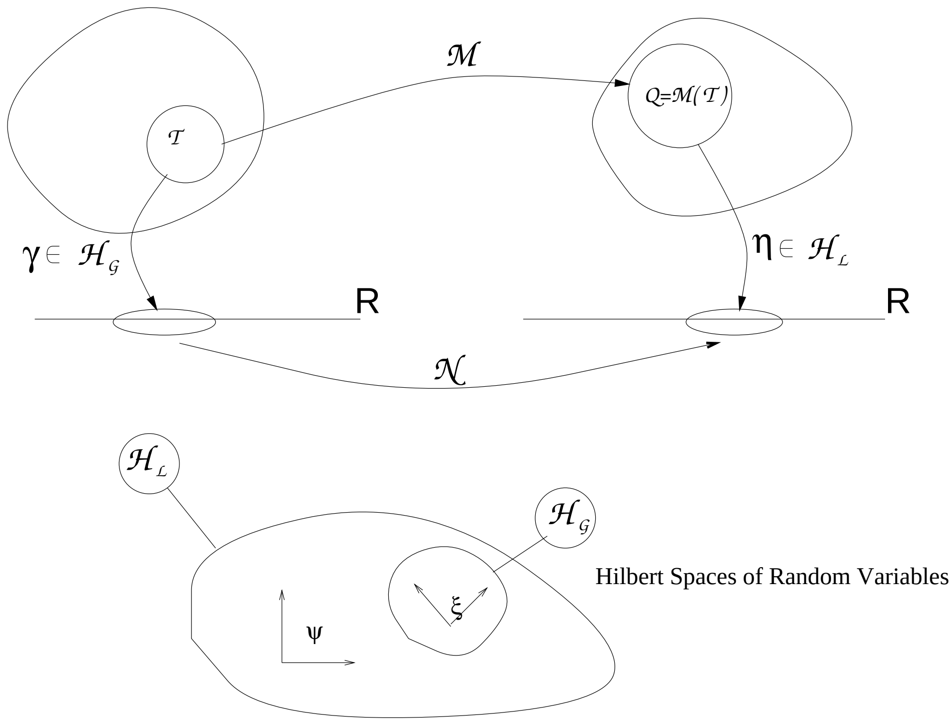

Figure 1: Functional Dependencies in a Probabilistic Characterization of the Problem.

basis set $\{\psi\}$ is appropriate for representing both the input and output of a mechanistic model, thus eliminating the need for significant post-processing and the associated loss of information.

Examples are shown from across a wide spectrum of applications that are relevant to engineering science. In particular, application to the problem of mechanical joints, shock-loaded structures, flows in random porous media, and bifurcation problems will be discussed in detail, highlighting the generality of the proposed methodology.

References

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