Linear Relaxations – Validation in a Successful Non-Validated Technique

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- We first describe the technique, previously used successfully in non-validated contexts.
- We point out references for the technique and successful engineering applications.
- We explain how to validate the methods.
- We report some experimental results.
- We speculate on the type of problems for which it can be best applied.

General Problem

minimize $\varphi(x)$
subject to:

$$c_i(x) = 0,\ i = 1, \ldots, m_1,$$
$$g_i(x) \le 0,\ i = 1, \ldots, m_2,$$

where $\varphi : \boldsymbol{x} \to \mathbb{R}$ and $c_i, g_i : \boldsymbol{x} \to \mathbb{R}$, and where $\boldsymbol{x} \subset \mathbb{R}^n$ is the hyperrectangle (box) defined by

$$\underline{x}_{i_j} \le x_{i_j} \le \overline{x}_{i_j},\ 1 \le j \le m_3,$$

i_j between 1 and n, where the $\underline{x}_{i_j}$ and $\overline{x}_{i_j}$ are constant bounds.

If φ is constant or absent, this problem becomes a general constraint problem; if, in addition $m_2 = m_3 = 0$, this problem becomes a nonlinear system of equations.

Linear Relaxations

The basic idea

- If the objective φ is replaced by linear function $\varphi^{(\ell)}$ such that $\varphi^{(\ell)}(x) \le \varphi(x)$ for $x \in \boldsymbol{x}$, then the resulting problem has global optimum less than or equal to the global optimum of the original problem.
- If each inequality constraint g_i is replaced by a linear function $g_i^{(\ell)}$ such that $g_i^{(\ell)}(x) \le g_i(x)$ for $x \in \boldsymbol{x}$, then the resulting problem has optimum that is less than or equal to the optimum of the original problem.
- If there are equality constraints, then each equality constraint can be replaced by two linear inequality constraints, and these inequality constraints can be replaced as above by linear inequality constraints.
- The resulting linear program is termed a *linear relaxation.*

An Example

Take the constraint system

$$c_1(x) = x_1^2 - 2x_2, \quad c_2(x) = x_2^2 - 2x_1,$$
$$x_1 \in [-1, 1], \quad x_2 \in [-1, 1].$$

1. Solve for x_2 in c_1, to obtain $x_2 = x_1^2/2$, then plug $x_1 = [-1, 1]$ into $x_1^2/2$, to obtain $x_2 \in [0, 0.5]$.
2. Solve c_2 for x_1 to obtain $x_1 = x_2^2/2$, then plug the narrower value of x_2 into $x_2^2/2$, to obtain $x_1 \in [0, 0.125]$.
3. Use c_1 again to obtain an even narrower value for x_2.
4. This process can be continued to convergence to $x_1 = 0$, $x_2 = 0$.

Linear Relaxations

(applied to the example)

- Lower bounds of a convex function are tangent lines and upper bounds are secant lines.
- A corresponding linear program for computing an upper bound on x_2, using two underestimators for the convex function $x_2 = x_1^2$, is:

$$
\begin{array}{l}
\text{minimize } -x_2 \\
\text{subject to} \\
\quad x_2 \le x_1 \text{ (the overestimator)}, \\
\quad x_2 \ge .125 + .5(x_1 - .25), \\
\quad x_2 \le x_1 \text{ (the original constraint)}, \\
\quad x_1 \in [0,1],\ x_2 \in [0,1].
\end{array}
$$

Linear Relaxation Example

(continued)

- The exact minimum to this linear program is $\varphi = -.5$, corresponding to $x_2 \le 0.5$.
- Thus, we have narrowed x_2 to $x_2 \in [0, 0.5] \subset [0, 1]$.
- Basic constraint propagation now converges.
- Basic constraint propagation and interval Newton narrowing (applied to the Kuhn-Tucker system) alone are not successful on this example

Convex and Linear Relaxations

History and References

- Convex relaxations have been in use since the 1970's, starting perhaps with McCormick (G. P. McCormick, "Computability of global solutions to factorable nonconvex programs," *Math. Prog.* **10** (2), 1976.)
- Floudas et al have successfully used nonlinear but convex relaxations to solve chemical engineering and other problems. (C. A. Floudas, *Deterministic Global Optimization: Theory, Algorithms, and Applications*, Kluwer, 2000.
- Sahinidis and Tawarmalani have used linear relaxations successfully in similar applications (M. Tawarmalani and N. V. Sahinidis, *Convexification and Global Optimization in Continuous and Mixed-Integer Nonlinear Programming*, and have incorporated it into the BARON commercial software (available through GAMS).

Our Focus on Linear Relaxations

We focus on linear relaxations because:

- We can use linear programming technology, rather than convex programming technology.
- It is clearer how to compute machine-representable coefficients in linear relaxations in such a way that the resulting machine-representable linear program is a true relaxation.
- Convex terms can be approximated arbitrarily closely with multiple linear underestimators.

Rigor in Linear Relaxations

1. Typical procedures have been to compute the coefficients of the linear relaxation with floating point arithmetic, then to solve the relaxation with a state-of-the-art LP solver.
2. With carefully considered directed rounding and interval arithmetic, we can form a machine-representable LP that is an actual relaxation of the original problem.
3. Neumaier and Shcherbina, as well as Jansson, have presented a simple technique to utilize the duality gap to obtain a rigorous lower bound on the solution to an LP, given approximate values of the dual variables.
4. Combining (2) and (3) gives a procedure for rigorous computations of lower bounds on the solution to the original problem.

Linear Relaxations

A Computational Example

Nonlinear minimax problems have previously challenged our GlobSol software system. A particular such example, formulated as a continuous constrained problem with Lemaréchal's conditions, is:

$\min_{x\in\mathbb{R}^n} x_5$

such that:

$$x_4 - \left(x_1 t_i^2 + x_2 t_i + x_3\right)^2 - \sqrt{t_i} \quad - x_5 \le 0,$$

$$-\{x_4 - \left(x_1 t_i^2 + x_2 t_i + x_3\right)^2 - \sqrt{t_i}\} - x_5 \le 0,$$

$t_i = 0.25 + 0.75(i-1)/(m-1)$, $1 \le i \le m$, $m = 21$.

- GlobSol without linear relaxations has been unable to obtain bounds on the solution sets, using starting intervals $x_i \in [-5, 5]$, $i = 1, 2, 3, 4$, and $x_5 \in [-100, 100]$, for $m = 21$.

Computational Example

Results with Linear Relaxations

We tried this problem with linear relaxations within GlobSol.

- There are various parameters affecting the linear relaxations, dealing with the accuracy with which convex functions are underestimated with linear functions, etc. Depending on how we set these parameters, GlobSol completed the problem with roughly 12,000 boxes searched.
- We also tried GlobSol with linear relaxations on a "Standard" test set (the "tiny" Library 1 problems from the AMPL test set, as recommended by Neumaier et al).

Linear Relaxations in GlobSol

The Test Set Results

- For most problems in that test set, there was a reduction in the total number of boxes searched, but an increase in processor time, due to time spent solving the LP problems.
- For several problems, there was a large improvement in both number of boxes searched and total processor time.

On the Implementation in GlobSol

Summary

- We have implemented linear relaxations in GlobSol.
- Initial experiments indicate the technique makes possible solution of problems that were previously intractable within GlobSol.
- A preprint of experimental results is available.
- GlobSol still is not fully competitive with other packages using relaxations in a non-validated way (e.g. BARON).
- One possibility for improvement: Use a better LP solver. (GlobSol presently is using a free one from the SLATEC library.)

Possible Appropriate Applications

Structural Analysis

- Uncertain finite element computations in structural analysis are reduced to the form $Ax = b$, where A is a square matrix and A and b are uncertain. (That is, in general, $\underline{a}_{i,j} \le a_{i,j} \le \overline{a}_{i,j}$ and $\underline{b}_i \le b_i \le \overline{b}_i$.)
- One possibility is to formulate the problem as:

$$\begin{array}{ll} \min x_i & (\text{ or } \max\ -x_i) \\ \text{subject to } Ax = b, & \\ \underline{a}_{i,j} \le a_{i,j} \le \overline{a}_{i,j}, \quad \underline{b}_i \le b_i \le \overline{b}_i, & \end{array} \qquad (1)$$

  with a_{i_j} and $b_{i,j}$ unknowns.
- Linear relaxations should work for this system, since the system itself is approximately linear.
- A problem with formulation (1) is that there are large sets of minimizers.

Structural Analysis

A Plausible Formulation

- In our structural analysis models, the elements of A and b are actually nonlinear functions of uncertain parameters.
- Instead of making the elements of A and b variables in the optimization problem, we can make the original parameters (heights, elasticities, etc.) into variables.
- This may reduce some of the superabundance of global optimizers.
- This is preliminary; we will overcome some small coding hurdles and try this.