Formulation for Reliable Analysis of Structural Frames

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Abstract

Structural engineers use design codes formulated to consider uncertainty for both reinforced concrete and structural steel design. For a simple one-bay structural steel frame, we survey typical uncertainties and compute an interval solution for displacements and forces. The naive solutions have large over-estimations, so we explore the Mullen-Muhanna element-by-element strategy, scaling, and constraint propagation to achieve tight enclosures of the true ranges for displacements and forces in a fraction of the CPU time typically used for simulations. That we compute tight enclosures, even for large parameter uncertainties used in practice, suggests the promise of interval methods for much larger structures.
Uncertain Parameters

Typical parameter values (note wide intervals):

\[ E_b = E_c = 29,000,000 \pm 3,480,000 \text{ lbs/in}^2 \ (\pm 12\%) \quad \text{(Young modulus)} \]

\[ I_b = 510 \pm 51 \text{ in}^4; \quad I_c = 272 \pm 27.2 \text{ in}^4 \ (\pm 10\%) \quad \text{(Second moment)} \]

\[ A_b = 10.3 \pm 10.3 \text{ in}^2; \quad A_c = 14.4 \pm 1.44 \text{ in}^2 \ (\pm 10\%) \quad \text{(Area)} \]

\[ H = 5,305.5 \pm 2,203.5 \text{ lbs} \ (\pm 41.6\%) \quad \text{(External force)} \]

\[ \alpha = 277,461,000 \pm 126,504,000 \text{ lb-in/rad} \ (\pm 45.6\%) \quad \text{(Joint stiffness)} \]

\[ L_c = 144 \text{ in}; \quad L_b = 2L_c \quad \text{(Length)} \]
Attributes (in local (\(\hat{\cdot}\)) or global coordinates)*:

- Displacements: \(d_{N\hat{x}}\), \(d_{N\hat{y}}\), \(d_{F\hat{x}}\), \(d_{F\hat{y}}\) (local) or \(d_{Nx}\), \(d_{Ny}\), \(d_{Fx}\), \(d_{Fy}\) (global)
- Rotations: \(r_{N\hat{z}}\), \(d_{F\hat{z}}\) (local) or \(r_{Nz}\), \(d_{Fz}\) (global)
- Forces: \(q_{N\hat{x}}\), \(q_{N\hat{y}}\), \(q_{F\hat{x}}\), \(q_{F\hat{y}}\) (local) or \(q_{Nx}\), \(q_{Ny}\), \(q_{Fx}\), \(q_{Fy}\) (global)
- Moments: \(m_{N\hat{z}}\), \(m_{F\hat{z}}\) (local) or \(m_{Nz}\), \(m_{Fz}\) (global)

*Hibbeler, *Structural Analysis*, 2002
Component: Member

Properties: Frame-member stiffness equation:

\[
\begin{bmatrix}
\frac{AE}{L} & 0 & 0 & -\frac{AE}{L} & 0 & 0 \\
0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\
0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\
-\frac{AE}{L} & 0 & 0 & \frac{AE}{L} & 0 & 0 \\
0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\
0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L}
\end{bmatrix}
\begin{bmatrix}
\delta_{N\hat{x}} \\
\delta_{N\hat{y}} \\
r_{N\hat{z}} \\
\delta_{F\hat{x}} \\
\delta_{F\hat{y}} \\
r_{F\hat{z}}
\end{bmatrix}
= \begin{bmatrix}
q_{N\hat{x}} \\
q_{N\hat{y}} \\
m_{N\hat{z}} \\
q_{F\hat{x}} \\
q_{F\hat{y}} \\
m_{F\hat{z}}
\end{bmatrix}
\]

Local coordinates transformed to global coordinates
Component: End

“End” is an end of a Member or a Joint
Ends define the topology of the structure

Attributes (in global coordinates):

- Displacements: $d_x$, $d_y$; Rotations: $r_z$
- Forces: $q_x$, $q_y$, $q_{Ex}$, $q_{Ey}$; Moments: $m_z$, $m_{Ez}$,

Properties:

- Can be incident with two or more Members and Joints
- Displacements $d_x$, $d_y$, and $r_z$ are equal for all Members and Joints incident on an End
- Forces $q_{Ex} + q_x$, $q_{Ey} + q_y$, and moments $m_{Ez} + m_z$ each sum to zero for all Members and Joints incident on an End
Component: Joint

Attributes (in local or global coordinates):

- Displacements: \( d_{N\hat{x}}, d_{N\hat{y}}, d_{F\hat{x}}, d_{F\hat{y}} \); Rotations: \( r_{N\hat{z}}, d_{F\hat{z}} \)
- Forces: \( q_{N\hat{x}}, q_{N\hat{y}}, q_{F\hat{x}}, q_{F\hat{y}} \); Moments: \( m_{N\hat{z}}, m_{F\hat{z}} \)

Properties:

- Length = 0
- Joins one Member to another
- Global displacements \( d_x \) are equal for incident End and Member
- Global displacements \( d_y \) are equal for incident End and Member
- Global forces \( q_x \) are equal for incident End and Member
- Global forces \( q_y \) are equal for incident End and Member
- Local rotations and moments satisfy

\[
\begin{bmatrix}
\alpha & -\alpha \\
-\alpha & \alpha \\
\end{bmatrix}
\begin{bmatrix}
r_{N\hat{z}} \\
r_{F\hat{z}} \\
\end{bmatrix}
= 
\begin{bmatrix}
m_{N\hat{z}} \\
m_{F\hat{z}} \\
\end{bmatrix}
\]
Approximate Solution

Solve using mid-point values of parameters:

<table>
<thead>
<tr>
<th>Connection</th>
<th>Displacement $d_x$</th>
<th>Displacement $d_y$</th>
<th>Rotation $r_z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Connection 2</td>
<td>0.15356843</td>
<td>0.00033236</td>
<td>-0.00096285</td>
</tr>
<tr>
<td>Connection 3</td>
<td>0.15102784</td>
<td>-0.00033236</td>
<td>-0.00094313</td>
</tr>
<tr>
<td>Connection 5</td>
<td></td>
<td></td>
<td>-0.00045995</td>
</tr>
<tr>
<td>Connection 6</td>
<td></td>
<td></td>
<td>-0.00044556</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Connection</th>
<th>Force $q_x$</th>
<th>Force $q_y$</th>
<th>Moment $m_z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Connection 1</td>
<td>-2670.516</td>
<td>-963.856</td>
<td>245019.992</td>
</tr>
<tr>
<td>Connection 4</td>
<td>-2634.984</td>
<td>963.856</td>
<td>241381.602</td>
</tr>
</tbody>
</table>

$8 \times 8$ system, condition number $\text{cond}(K) = 4.7e+04$
Naive Interval Solution

With uncertainties 1% of given:

<table>
<thead>
<tr>
<th>Disp.</th>
<th>Float</th>
<th>Interval True range</th>
<th>Midpoint ± Radius Rel. overest.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_{2x}$</td>
<td>0.153568</td>
<td>[0.09375783, 0.21337873]</td>
<td>0.153568 ± 0.05981 76.34%</td>
</tr>
<tr>
<td>Tight:</td>
<td>[0.15237484, 0.15476814]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$d_{2y}e+3$</td>
<td>0.332364</td>
<td>[0.19060424, 0.47412283]</td>
<td>0.3323635 ± 0.1418 83.52%</td>
</tr>
<tr>
<td>Tight:</td>
<td>[0.32940418, 0.33533906]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$r_{2z}e+3$</td>
<td>-0.962852</td>
<td>[-1.3531968, -0.57250484]</td>
<td>-0.9628508 ± 0.3903 79.42%</td>
</tr>
<tr>
<td>Tight:</td>
<td>[-0.97085151, -0.95490139]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$r_{5z}e+3$</td>
<td>-0.459955</td>
<td>[-0.6557609, -0.26414725]</td>
<td>-0.4599541 ± 0.1958 83.47%</td>
</tr>
<tr>
<td>Tight:</td>
<td>[-0.4638112, -0.45611532]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$r_{6z}e+3$</td>
<td>-0.445563</td>
<td>[-0.64100045, -0.2501251]</td>
<td>-0.4455628 ± 0.1954 86.05%</td>
</tr>
<tr>
<td>Tight:</td>
<td>[-0.44930811, -0.4418354]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$d_{3x}$</td>
<td>0.151028</td>
<td>[0.091230936, 0.21082444]</td>
<td>0.1510277 ± 0.0598 77.62%</td>
</tr>
<tr>
<td>Tight:</td>
<td>[0.14985048, 0.15221127]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$d_{3y}e+3$</td>
<td>-0.332364</td>
<td>[-0.47412283, -0.19060424]</td>
<td>-0.3323635 ± 0.1418 83.52%</td>
</tr>
<tr>
<td>Tight:</td>
<td>[-0.33533906, -0.32940418]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$r_{3z}e+3$</td>
<td>-0.943133</td>
<td>[-1.3330326, -0.55323186]</td>
<td>-0.9431322 ± 0.3899 81.02%</td>
</tr>
<tr>
<td>Tight:</td>
<td>[-0.95100335, -0.93531196]</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Interval solutions are hopelessly pessimistic
At 4% of given uncertainties, stiffness matrix includes singular matrix
Element-By-Element

Mullen & Muhanna (1999):
Introduce extra variables and add extra equations to the system to reduce the interval dependencies. Each Member & Joint is separate

Member $M_1$ global stiffness:

$$
\begin{bmatrix}
\frac{12EJ_c}{L_c^3} & 0 & \frac{-6EJ_c}{L_c^2} & \frac{-12EJ_c}{L_c^3} & 0 & \frac{-6EJ_c}{L_c^2} \\
0 & \frac{AE_c}{L_c} & 0 & 0 & \frac{-AE_c}{L_c} & 0 \\
\frac{-6EJ_c}{L_c^2} & 0 & \frac{4EJ_c}{L_c^2} & \frac{6EJ_c}{L_c^2} & 0 & \frac{2EJ_c}{L_c^2} \\
\frac{-12EJ_c}{L_c^3} & 0 & \frac{6EJ_c}{L_c^2} & \frac{12EJ_c}{L_c^3} & 0 & \frac{6EJ_c}{L_c^2} \\
0 & \frac{-AE_c}{L_c} & 0 & 0 & \frac{AE_c}{L_c} & 0 \\
\frac{-6EJ_c}{L_c^2} & 0 & \frac{2EJ_c}{L_c} & \frac{6EJ_c}{L_c^2} & 0 & \frac{4EJ_c}{L_c^2}
\end{bmatrix}
\begin{bmatrix}
d1_x \\
d1_y \\
r1_z \\
d2_x \\
d2_y \\
r2_z
\end{bmatrix}
- 
\begin{bmatrix}
q1_x \\
q1_y \\
m1_z \\
q2_x \\
q2_y \\
m2_z
\end{bmatrix}
= 0
$$

60 x 60 system with Cond($K$) = 1.2e+17
Intervals essentially the same as naive interval solution
Subdistributivity

In interval arithmetic: \( a(b + c) \subseteq ab + ac \) (subdistributivity). E.g.,

\[
[-1, 2] \ast ([4, 5] + [-3, -2]) = [-3, 6] \subseteq [-1, 2] \ast [4, 5] + [-1, 2] \ast [-3, -2]
\]

To get tighter enclosures, factor (as Mullen & Muhanna)

Member \( M_1 \) global stiffness matrix:

\[
\begin{bmatrix}
\frac{12E_cI_c}{L_c^3} & 0 & -\frac{6E_cI_c}{L_c^2} & -\frac{12E_cI_c}{L_c^3} & 0 & -\frac{6E_cI_c}{L_c^2} \\
0 & \frac{A_cE_c}{L_c} & 0 & 0 & -\frac{A_cE_c}{L_c} & 0 \\
-\frac{6E_cI_c}{L_c^2} & 0 & \frac{4E_cI_c}{L_c} & \frac{6E_cI_c}{L_c^2} & 0 & \frac{2E_cI_c}{L_c} \\
-\frac{12E_cI_c}{L_c^3} & 0 & \frac{6E_cI_c}{L_c} & \frac{12E_cI_c}{L_c^3} & 0 & \frac{6E_cI_c}{L_c^2} \\
0 & -\frac{A_cE_c}{L_c} & 0 & 0 & \frac{A_cE_c}{L_c} & 0 \\
-\frac{6E_cI_c}{L_c^2} & 0 & \frac{2E_cI_c}{L_c} & \frac{6E_cI_c}{L_c^2} & 0 & \frac{4E_cI_c}{L_c}
\end{bmatrix}
\begin{bmatrix}
\begin{bmatrix}
d1_x \\
d1_y \\
r1_z \\
d2_x \\
d2_y \\
r2_z
\end{bmatrix} & -
\begin{bmatrix}
q1_x \\
q1_y \\
m1_z \\
q2_x \\
q2_y \\
m2_z
\end{bmatrix}
\end{bmatrix}
\]
Subdistributivity

Let

\[ d_{63} := \frac{A_c E_c}{L_c}(d_{1y} - d_{2y}) \]
\[ d_{64} := \frac{6E_c I_c}{L_c^2}(d_{1x} - d_{2x}) \]
\[ d_{65} := \frac{2E_c I_c}{L_c}(r_{1z} + r_{2z}) \]
\[ d_{66} := \frac{2E_c I_c}{L_c} r_{1z} \]

Leads to a considerably simpler system:

\[ \frac{2}{L_c} d_{64} - \frac{3}{L_c} d_{65} - q_{1x} = 0; \quad d_{63} - q_{1y} = 0 \]
\[ -d_{64} + d_{65} + d_{66} - m_{1z} = 0 \]
\[ q_{1x} + q_{2x} = 0; \quad q_{1y} + q_{2y} = 0 \]
\[ -d_{64} + 2d_{65} - d_{66} - m_{2z} = 0 \]
\[ d_{1y} - d_{2y} - \frac{L_c}{A_c E_c} d_{63} = 0; \quad d_{1x} - d_{2x} - \frac{L_c^2}{6 E_c I_c} d_{64} = 0 \]
\[ r_{1z} + r_{2z} - \frac{L_c}{2 E_c I_c} d_{65} = 0; \quad r_{1z} - \frac{L_c}{2 E_c I_c} d_{66} = 0 \]
Element-by-Element Interval Solution

<table>
<thead>
<tr>
<th>Disp.</th>
<th>Float</th>
<th>Interval True range</th>
<th>Midpoint ± Radius Rel. overest.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d2_x$</td>
<td>0.153568</td>
<td>[0.15206288, 0.15507492]</td>
<td>0.1535689 ± 0.001506</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Tight: [0.15237484, 0.15476814]</td>
<td>0.40%</td>
</tr>
<tr>
<td>$d2_y e+3$</td>
<td>0.332364</td>
<td>[0.32918317, 0.33554758]</td>
<td>0.3323654 ± 0.003182</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Tight: [0.32940418, 0.33533906]</td>
<td>0.13%</td>
</tr>
<tr>
<td>$r2_z e+3$</td>
<td>-0.962852</td>
<td>[-0.97485786, -0.95084958]</td>
<td>-0.9628537 ± 0.012</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Tight: [-0.97085151, -0.95490139]</td>
<td>0.84%</td>
</tr>
<tr>
<td>$r5_z e+3$</td>
<td>-0.459955</td>
<td>[-0.46757208, -0.45234116]</td>
<td>-0.4599566 ± 0.007615</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Tight: [-0.4638112, -0.45611532]</td>
<td>1.63%</td>
</tr>
</tbody>
</table>

$74 \times 74$ system, $\text{Cond}(K) = 1.2E+17$

Intervals are MUCH tighter
Scaling

Reduce Cond($K$) = 1.2e+17 by scaling?

Replace force $H$ by its midpoint $\tilde{H}$; multiply solution by $[H]/\tilde{H}$
Replace each variable force ($q_{1x}$, ...) by force/\tilde{H}
Eliminate known “variables”

Interval solution using scaled element-by-element approach with 1% given uncertainties

<table>
<thead>
<tr>
<th>Disp.</th>
<th>Float</th>
<th>[Interval True range]</th>
<th>Midpoint ± Radius Rel. overest.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_{2x}$</td>
<td>0.153568</td>
<td>[0.15294597, 0.15419182]</td>
<td>0.1535689 ± 0.0006229</td>
</tr>
<tr>
<td></td>
<td>Tight:</td>
<td>[0.1531698, 0.15396904]</td>
<td>0.29%</td>
</tr>
<tr>
<td>$d_{2y}$ $e+3$</td>
<td>0.332364</td>
<td>[0.33111682, 0.33361393]</td>
<td>0.3323654 ± 0.001249</td>
</tr>
<tr>
<td></td>
<td>Tight:</td>
<td>[0.3311227, 0.33360764]</td>
<td>0.004%</td>
</tr>
<tr>
<td>$r_{2z}$ $e+3$</td>
<td>-0.962852</td>
<td>[-0.96945816, -0.95624927]</td>
<td>-0.9628537 ± 0.006604</td>
</tr>
<tr>
<td></td>
<td>Tight:</td>
<td>[-0.96583881, -0.95988319]</td>
<td>0.75%</td>
</tr>
<tr>
<td>$r_{5z}$ $e+3$</td>
<td>-0.459955</td>
<td>[-0.46515166, -0.45476159]</td>
<td>-0.4599566 ± 0.005195</td>
</tr>
<tr>
<td></td>
<td>Tight:</td>
<td>[-0.46141645, -0.45849491]</td>
<td>1.62%</td>
</tr>
</tbody>
</table>
# Scaling

Interval solution using scaled element-by-element approach with GIVEN uncertainties

<table>
<thead>
<tr>
<th>Disp.</th>
<th>Float</th>
<th>Interval True range</th>
<th>Midpoint ± Radius Rel. overest.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_{2x}$</td>
<td>0.153568</td>
<td>[ 0.022924888, 0.29366922 ]</td>
<td>0.1582971 ± 0.1354 119.8%</td>
</tr>
<tr>
<td></td>
<td>Tight:</td>
<td>[ 0.12130751, 0.20804041 ]</td>
<td></td>
</tr>
<tr>
<td>$d_{2y}e+3$</td>
<td>0.332364</td>
<td>[ 0.11407836, 0.57891094 ]</td>
<td>0.3464947 ± 0.2324 62.51%</td>
</tr>
<tr>
<td></td>
<td>Tight:</td>
<td>[ 0.21526742, 0.47234932 ]</td>
<td></td>
</tr>
<tr>
<td>$r_{2z}e+3$</td>
<td>-0.962852</td>
<td>[ -2.5286276, 0.55857565 ]</td>
<td>-0.985026 ± 1.544 250.3%</td>
</tr>
<tr>
<td></td>
<td>Tight:</td>
<td>[ -1.4124783, -0.73502904 ]</td>
<td></td>
</tr>
<tr>
<td>$r_{5z}e+3$</td>
<td>-0.459955</td>
<td>[ -1.7359689, 0.77691797 ]</td>
<td>-0.4795255 ± 1.256 479.8%</td>
</tr>
<tr>
<td></td>
<td>Tight:</td>
<td>[ -0.6216869, -0.3157181 ]</td>
<td></td>
</tr>
</tbody>
</table>

$58 \times 58$ system, Cond($K$) = 2.8e+08
Successful solution with 1.7 times given uncertainties

+ Interval arithmetic can handle realistic uncertainties
- Enormous overestimations
Constraint Propagation

Find roots in \([-4, 4]\) of \(f(x) = x^2 + x - 5 = 0\)

Solve for \(x = 5 - x^2\)
Substitute \(x = [-4, 4]\): \(x = 5 - [-4, 4]^2 = [-11, 5]\)
Not especially helpful

Solve for \(x = \pm \sqrt{5 - x}\)
Substitute \(x = [-4, 4]\): \(x = \pm \sqrt{5 - [-4, 4]} = [-3, -1] \cup [1, 3]\)
Iterate \(x = \sqrt{5 - x}\) from \(x = [1, 3]\):

\[x = [1.41421356237309, 2.0000000000000000]\]
\[1.73205080756887, 1.89361728911280]\]
\[1.76249332222485, 1.80774699347866]\]
\[1.78668771936265, 1.79930727719730]\]

converging to the root \(x^* = 1.79128784747792\).
Constraint Propagation

From logic programming, see Van Hentenryck et al., *Numerica: A Modeling Language for Global Optimization*, MIT, 1997

Constraint propagation:
- Discard candidate solutions infeasible with respect to already known information
- Gauss-Seidel, except solve each equation for each variable
- Especially attractive for sparse systems, such as ours
Constraint Propagation

Start with [0.6, 1.4] times the approximate solution using midpoints
Use no interval system solver
Five iterations of constraint propagation

<table>
<thead>
<tr>
<th>Disp.</th>
<th>Float</th>
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</tr>
</thead>
<tbody>
<tr>
<td>$d2_x$</td>
<td>0.153568</td>
<td>[0.076784216, 0.23035265]</td>
<td>0.1535684 ± 0.07678</td>
</tr>
<tr>
<td></td>
<td>Tight:</td>
<td>[0.12130751, 0.20804041]</td>
<td>43.52%</td>
</tr>
<tr>
<td>$d2_y e+3$</td>
<td>0.332364</td>
<td>[0.166182, 0.49854599]</td>
<td>0.332364 ± 0.1662</td>
</tr>
<tr>
<td></td>
<td>Tight:</td>
<td>[0.21526742, 0.47234932]</td>
<td>22.65%</td>
</tr>
<tr>
<td>$r2_z e+3$</td>
<td>-0.962852</td>
<td>[-1.4442773, -0.48142577]</td>
<td>-0.9628515 ± 0.4814</td>
</tr>
<tr>
<td></td>
<td>Tight:</td>
<td>[-1.4124783, -0.73502904]</td>
<td>29.64%</td>
</tr>
<tr>
<td>$r5_z e+3$</td>
<td>-0.459955</td>
<td>[-0.68993206, -0.22997735]</td>
<td>-0.4599547 ± 0.23</td>
</tr>
<tr>
<td></td>
<td>Tight:</td>
<td>[-0.6216869, -0.3157181]</td>
<td>33.48%</td>
</tr>
</tbody>
</table>

We achieve relative over-estimations that are comparable to the relative uncertainties in the parameters, in spite of a condition numb of 2.8e+08
Conclusions: To Structural Engineers

- Reliable interval techniques are feasible
- One set of interval computations can guarantee to enclose the displacements, rotations, forces, and moments that could be observed from any combination of values of cross-sectional properties, loading, material properties, and connections, even for quite large uncertainties.
- Alternative to Monte Carlo simulations

Next: Extend to non-linear behaviors and to larger structures
Conclusions: To Interval Researchers

- Variety of techniques to achieve relatively tight enclosures of the solution to a realistic (although small) problem, even in the face of parameter uncertainties over 40%.
- Element-by-element adds equations specifying that two variables are the same, rather than simplifying by identifying them with the same variable.
- Used symbolic rearrangement, scaling of the equations, and constraint propagation.

Next: larger systems, sparsity-preserving preconditioning, more effective and efficient constraint propagation, and branch-and-bound-like strategies for subdividing the ranges of wide interval-valued parameters.