Recent Advances in the Validated Integration of ODEs

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Integration of the Volterra eq. COSY-VI and AWA
Definitions - Taylor Models and Operations

We begin with a review of the definitions of the basic operations.

Definition (Taylor Model) Let \( f : D \subset \mathbb{R}^v \rightarrow \mathbb{R} \) be a function that is \((n+1)\) times continuously partially differentiable on an open set containing the domain \( v\)-dimensional domain \( D \). Let \( x_0 \) be a point in \( D \) and \( P \) the \( n\)-th order Taylor polynomial of \( f \) around \( x_0 \). Let \( I \) be an interval such that
\[
f(x) \in P(x - x_0) + I \text{ for all } x \in D.
\]
Then we call the pair \((P, I)\) an \( n\)-th order Taylor model of \( f \) around \( x_0 \) on \( D \).

Definition (Addition and Multiplication) Let \( T_{1,2} = (P_{1,2}, I_{1,2}) \) be \( n\)-th order Taylor models around \( x_0 \) over the domain \( D \). We define
\[
T_1 + T_2 = (P_1 + P_2, I_1 + I_2)
\]
\[
T_1 \cdot T_2 = (P_{1,2}, I_{1,2})
\]
where \( P_{1,2} \) is the part of the polynomial \( P_1 \cdot P_2 \) up to order \( n \) and
\[
I_{1,2} = B(P_e) + B(P_1) \cdot I_2 + B(P_2) \cdot I_1 + I_1 \cdot I_2
\]
where \( P_e \) is the part of the polynomial \( P_1 \cdot P_2 \) of orders \((n + 1)\) to \( 2n \), and \( B(P) \) denotes a bound of \( P \) on the domain \( D \). We demand that \( B(P) \) is at least as sharp as direct interval evaluation of \( P(x - x_0) \) on \( D \).
The Operator $\partial^{-1}$ on Taylor Models

Let $(P_n, I_n)$ be an $n$-th order Taylor model of $f$. From this we can obtain a Taylor model for the indefinite integral $\partial^{-1}_i f = \int f \, dx'_i$ with respect to variable $x_i$.

Taylor polynomial part: $\int_0^{x_i} P_{n-1} \, dx'_i$,

Remainder Bound: $(B(P_n - P_{n-1}) + I_n) \cdot B(x_i)$, where $B(P)$ is a polynomial bound.

So define the operator $\partial^{-1}_i$ on space of Taylor models as

$$\partial^{-1}_i(P_n, I_n) = \left( \int_0^{x_i} P_{n-1} \, dx'_i, \ (B(P_n - P_{n-1}) + I_n) \cdot B(x_i) \right)$$
**TM Scaling Theorem**

**Theorem (Scaling Theorem)** Let \( f, g \in C^{n+1}(D) \) and \((P_{f,h}, I_{f,h})\) and \((P_{g,h}, I_{g,h})\) be \( n \)-th order Taylor models for \( f \) and \( g \) around \( x_h \) on \( x_h + [-h,h]^v \subset D \). Let the remainder bounds \( I_{f,h} \) and \( I_{g,h} \) satisfy \( I_{f,h} = O(h^{n+1}) \) and \( I_{g,h} = O(h^{n+1}) \). Then the Taylor models \((P_{f+g}, I_{f+g,h})\) and \((P_{f\cdot g}, I_{f\cdot g,h})\) for the sum and products of \( f \) and \( g \) obtained via addition and multiplication of Taylor models satisfy

\[
I_{f+g,h} = O(h^{n+1}), \quad \text{and} \quad I_{f\cdot g,h} = O(h^{n+1}).
\]

Furthermore, let \( s \) be any of the intrinsic functions defined above, then the Taylor model \((P_{s(f)}, I_{s(f),h})\) for \( s(f) \) obtained by the above definition satisfies

\[
I_{s(f),h} = O(h^{n+1}).
\]

We say the Taylor model arithmetic has the \((n+1)\)-st order scaling property.

**Proof.** The proof for the binary operations follows directly from the definition of the remainder bounds for the binaries. Similarly, the proof for the intrinsics follows because all intrinsics are composed of binary operations as well as an additional interval, the width of which scales at least with the \((n+1)\)-st power of a bound \( B \) of a function that scales at least linearly with \( h \).
Important TM Algorithms

- **Range Bounding** (Evaluate $f$ as TM, bound polynomial, add remainder bound)
- **Quadrature** (Evaluate $f$ as TM, integrate polynomial and remainder bound)
- **Implicit Equations** (Obtain TMs for implicit solutions of TM equations)
- **Superconvergent** Interval Newton Method (Application of Implicit Equations)
- **ODEs** (Obtain TMs describing dependence of final coordinates on initial coordinates)
- **Implicit ODEs and DAEs**
- **Complex Arithmetic** (Describe complex ranges as two-dimensional TMs)
Implementation of TM Arithmetic

Validated Implementation of TM Arithmetic exists. The following points are important:

• Strict requirements for **underlying FP arithmetic**
• Taylor models require cutoff threshold (**garbage collection**)
• Coefficients remain FP, not intervals
• Package quite **extensively tested** by Corliss et al.

For practical considerations, the following is important:

• Need **sparsity** support
• Need efficient coefficient **addressing** scheme
• About 50,000 lines of code
• **Language Independent** Platform, coexistence in F77, C, F90, C++
Multiplication - Weighting

Sometimes important: Carry different variables $x_i$ to different orders $w_i$. Can be achieved by simply "seeding" original variables as

$$P(x) = (x_1^{w_1}, x_2^{w_2}, \ldots, x_v^{w_v}).$$

Then in all subsequent operations, only multiples of $w_i$ appear as powers of $x_i$. Optimal reduction of speed by sparsity.

Use weighted coding:

$$n_1(x_1^{i_1} \cdots x_v^{i_v}) = \frac{i_1}{w_1} + \frac{i_2}{w_2} \left( \left[ \frac{n}{w_1} \right] + 1 \right) + \frac{i_3}{w_3} \left( \left[ \frac{n}{w_1} \right] + 1 \right) \left( \left[ \frac{n}{w_2} \right] + 1 \right)$$

$$+ \cdots + \frac{i_v}{w_v} \left( \prod_{k=1}^{v-2} \left( \left[ \frac{n}{w_k} \right] + 1 \right) \right)$$

$$n_2(x_1^{i_1} \cdots x_v^{i_v}) = \frac{i_v}{w_{v+1}} + \cdots + \frac{i_v}{w_v} \left( \prod_{k=v+1}^{v-1} \left( \left[ \frac{n}{w_k} \right] + 1 \right) \right).$$

"[ ]": Gauss bracket. So, exponents are divided by their weighting factor, and resulting quotients are "digits" in a "variable-base" representation.
**Table 1.** Number of floating point numbers necessary to store all appearing partial derivatives in COSY to order $n_i$ in initial conditions, and in the first order code AWA

<table>
<thead>
<tr>
<th>Order $n$</th>
<th>Variables $v$</th>
<th>Weighting $w$</th>
<th>Order $n_i$</th>
<th>Cosy Coefs</th>
<th>AWA Coefs</th>
</tr>
</thead>
<tbody>
<tr>
<td>17</td>
<td>3</td>
<td>9</td>
<td>1</td>
<td>41</td>
<td>144</td>
</tr>
<tr>
<td>17</td>
<td>5</td>
<td>9</td>
<td>1</td>
<td>57</td>
<td>216</td>
</tr>
<tr>
<td>17</td>
<td>10</td>
<td>9</td>
<td>1</td>
<td>97</td>
<td>396</td>
</tr>
<tr>
<td>17</td>
<td>20</td>
<td>9</td>
<td>1</td>
<td>177</td>
<td>756</td>
</tr>
<tr>
<td>17</td>
<td>3</td>
<td>5</td>
<td>3</td>
<td>135</td>
<td>144</td>
</tr>
<tr>
<td>17</td>
<td>5</td>
<td>5</td>
<td>3</td>
<td>308</td>
<td>216</td>
</tr>
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<td>5</td>
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<td>10</td>
<td>3</td>
<td>5</td>
<td>7098</td>
<td>352</td>
</tr>
</tbody>
</table>

all interval endpoints of the $n_i = 1$ representation used in AWA is also given. The first four rows show the situation for the case most similar to the performance of the $n_i = 1$ case of AWA; the smaller number of COSY coefficients is due to the fact that on the one hand, instead of interval coefficients only real numbers are stored, and on the other hand that the expansion order in time for the dependence on initial conditions is reduced. The other rows show the situation for other choices of weights, which of course is more expensive; yet in the COSY scheme third order $n_i$ at least for low dimensions can still be achieved with a similar number of coefficients of AWA.

In the following section, we will study in detail the two fundamental questions of validated integration, the accurate representation of flows of ODEs, and methods to prevent growth of the remainder bound, and illustrate the behavior with a large number of examples.

### 2. Faithful Representation of Flows by Taylor Models

As discussed in the previous section, the successful use of validated methods requires on the one hand the accurate representation of the solution sets over short time scales, and on the other hand the ability to suppress the long-term build up of errors. In this section we study the behavior of the Taylor model method with respect to the first question, which is directly connected to and characteristic of the mathematical behavior of the ODE being studied. We observe that for linear systems, this first source of errors is particularly easy to control, since the flows of linear ODEs are merely linear transformations of the initial coordinates. However, as simple as the matter is for linear ODEs, as complicated it is for nonlinear ODEs. In this case, except for special cases there is no simple representation of the dependency of final conditions on initial conditions. This is the prime reason why nonlinear ODEs represent the real challenge in the validated integration of differential equations, and results obtained for the purely linear case are often not characteristic for the behavior in nonlinear cases.
Taylor Models for the Flow

Goal: Determine a Taylor model, consisting of a Taylor Polynomial and an interval bound for the remainder, for the flow of the differential equation

$$\frac{d}{dt}\vec{r}(t) = \vec{F}(\vec{r}(t), t)$$

where $\vec{F}$ is sufficiently differentiable. The Remainder Bound should be fully rigorous for all initial conditions $\vec{r}_0$ and times $t$ that satisfy

$$\vec{r}_0 \in [\vec{r}_{01}, \vec{r}_{02}] = \vec{B}$$

$$t \in [t_0, t_1].$$

In particular, $\vec{r}_0$ itself may be a Taylor model, as long as its range is known to lie in $\vec{B}$. 
The Volterra Equation

Describe dynamics of two conflicting populations

\[
\frac{dx_1}{dt} = 2x_1(1 - x_2), \quad \frac{dx_2}{dt} = -x_2(1 - x_1)
\]

Interested in initial condition

\[x_{01} \in 1 + [-0.05, 0.05], \quad x_{02} \in 3 + [-0.05, 0.05] \quad \text{at } t = 0.\]

Satisfies constraint condition

\[C(x_1, x_2) = x_1 x_2^2 e^{-x_1 - 2x_2} = \text{Constant}\]
Integration of the Volterra eq. COSY-VI and AWA
Solution Enclosure Box Width

Time

COSY-VI

AWA
Volterra 18th, IC: P1, Result: \( P_n + \{ B(P_{n+1} \text{ to } P_{18}) + IR \} \), same \( P \)
Volterra 18th, IC: P1, Result: $P_n + \{B(P_{n+1} \text{ to } P_{18}) + IR\}$, same $T$

- $2P$
- $P$
- $\frac{1}{2}P$
- $\frac{1}{4}P$
- $\frac{1}{8}P$
Shrink-Wrapping I

A method to remove the remainder bound of a Taylor model by increasing the polynomial part.

After the $k$th step of the integration, the region occupied by the final variables is given by

$$A \equiv \tilde{I}_0 + \bigcup_{\tilde{x}_0 \in \tilde{B}} \mathcal{M}_0(\tilde{x}_0),$$

where $\tilde{x}_0$ are the initial variables, $\tilde{B}$ is the original box of initial conditions, $\mathcal{M}_0$ is the polynomial part of the Taylor model, and $\tilde{I}_0$ is the remainder bound interval. $\mathcal{M}_0$ is scaled such that the original box $\tilde{B}$ is unity, i.e. $\tilde{B} = [-1, 1]^v$. $\tilde{I}_0$ accounts for the local approximation error of the expansion in time carried out in the $k$th step as well as floating point errors and potentially other accumulated errors from previous steps; it is usually very small. Try to “absorb” the small remainder interval into a set very similar to the first part via

$$A \subset A^* \equiv \tilde{I}_0^* + \bigcup_{\tilde{x}_0 \in \tilde{B}} \mathcal{M}_0^*(\tilde{x}_0),$$

where $\mathcal{M}_0^*$ is a slightly modified polynomial, and $\tilde{I}_0^*$ is significantly reduced.
**Shrink Wrapping**

**Theorem (Shrink Wrapping)** Let $\mathcal{M} = \mathcal{I} + \mathcal{S}(\vec{x})$, where $\mathcal{I}$ is the identity. Let $\mathcal{I} = d \cdot [-1, 1]^v$, and

$$A \equiv \mathcal{I} + \bigcup_{\vec{x} \in \bar{B}} \mathcal{M}(\vec{x})$$

be the set sum of the interval $\mathcal{I} = [-d, d]^v$ and the range of $\mathcal{M}$ over the original domain box $\bar{B}$. So $A$ is the range enclosure of the flow of the ODE over the interval $\bar{B}$ provided by the Taylor model. Let $q$ be the shrink wrap factor of $\mathcal{M}$; then we have

$$A \subset \bigcup_{\vec{x} \in \bar{B}} (q\mathcal{M})(\vec{x})$$

and hence multiplying $\mathcal{M}$ with the number $q$ allows to set the remainder bound to zero.
Shrink Wrapping

We define $q$, the so-called shrink wrap factor, as

$$q = 1 + d \cdot \frac{1}{(1 - (v - 1)t) \cdot (1 - s)}.$$

The bounds $s$ and $t$ for the polynomials $S_i$ and $\partial S_i / \partial x_j$ can be computed by interval evaluation. The factor $q$ will prove to be a factor by which the Taylor polynomial $I + S$ has to be multiplied in order to absorb the remainder bound interval.

**Remark (Typical values for $q$)** To put the various numbers in perspective, in the case of the verified integration of the Asteroid 1997 XF11, we typically have $d = 10^{-7}$, $s = 10^{-4}$, $t = 10^{-4}$, and thus $q \approx 1 + 10^{-7}$. It is interesting to note that the values for $s$ and $t$ are determined by the non-linearity in the problem at hand, while in the absence of “noise” terms in the ODEs described by intervals, the value of $d$ is determined mostly by the accuracy of the arithmetic. Rough estimates of the expected performance in quadruple precision arithmetic indicate that with an accompanying decrease in step size, if desired $d$ can be decreased below $10^{-12}$, resulting in $q \approx 1 + 10^{-12}$. 
Consider very simple two-state dynamical system:

\[
x_{n+1} = x_n \cdot \sqrt{1 + x_n^2 + y_n^2} \quad \text{and} \quad y_{n+1} = y_n \cdot \sqrt{1 + x_n^2 + y_n^2}
\]

\[
x_{n+2} = x_{n+1} \cdot \sqrt{\frac{2}{1 + \sqrt{1 + 4(x_{n+1}^2 + y_{n+1}^2)}}}
\quad \text{and} \quad
\]

\[
y_{n+2} = y_{n+1} \cdot \sqrt{\frac{2}{1 + \sqrt{1 + 4(x_{n+1}^2 + y_{n+1}^2)}}}.
\]

Simple arithmetic shows that, also here we have \((x_{n+2}, y_{n+2}) = (x_n, y_n)\).
Stretch by $\sqrt{1+x^2+y^2}$ and unstretch back, $DX=0.05$, $(0,0)$, noSW
Stretch by $\sqrt{1+x^2+y^2}$ and unstretch back, $DX=0.05$, (0,0), SW

- 1st order TM
- 5th order TM
- 10th order TM
- 20th order TM
Shrink Wrapping II

Let us consider the practical limitations of the method; apparently the measures of the nonlinearities $s$ and $t$ must not become too large.

**Remark (Limitations of shrink wrapping)** Apparently the shrink wrap method discussed above has the following limitations.

**Remark 1** 1. The measures of nonlinearities $s$ and $t$ must not become too large.

2. The application of the inverse of the linear part should not lead to large increases in the size of remainder bounds.

Apparently the first requirement limits the domain size that can be covered by the Taylor model, and it will thus happen only in extreme cases. Furthermore, in practice the case of $s$ and $t$ becoming large is connected to also having accumulated a large remainder bound, since the remainder bounds are calculated from the bounds of the various orders of $s$. In the light of this, not much additional harm is done by removing the offending $s$ into the remainder bound and create a linearized Taylor model.

**Definition (Blunting of an Ill-Conditioned Matrix)**

Let $\hat{A}$ be a regular matrix that is potentially ill-conditioned and $\vec{q} = (q_1, ..., q_n)$ a vector with $q_i > 0$. Arrange the column vectors $\vec{a}_i$ of $\hat{A}$ by size.
Let $\vec{e}_i$ be the familiar orthonormal vectors obtained through the Gram-Schmidt procedure, i.e.

$$
\vec{e}_i = \frac{\vec{a}_i - \sum_{k=1}^{i-1} \vec{e}_k \cdot \vec{a}_i \cdot \vec{e}_k}{\left| \vec{a}_i - \sum_{k=1}^{i-1} \vec{e}_k \cdot \vec{a}_i \cdot \vec{e}_k \right|}.
$$

We form vectors $\vec{b}_i$ via

$$
\vec{b}_i = \vec{a}_i + q_i \vec{e}_i
$$

and assemble them columnwise into the matrix $\hat{B}$. We call $\hat{B}$ the $\vec{q}$-blunted matrix belonging to $\hat{A}$.
Intuitively, the effect of blunting is that each vector $\vec{b}_i$ is being "pulled away" from the space spanned by the previous vectors $\vec{b}_1, ..., \vec{b}_{i-1}$, and more strongly so if $q_i$ becomes bigger and bigger. In fact, we have the following result:
Preconditioning the Flow

Idea: write the Taylor model of the solution as a composition of two Taylor models \((P_l + I_l)\) and \((P_r + I_r)\), and then choose \(P_l + I_l\) in such a way that in each step, the operations appearing on \(I_r\) are minimized. At the same time, \(I_l\) will be chosen as small as possible. Can be viewed as a coordinate transformation.

In the factorization, we impose that \((P_r + I_r)\) is normalized such that each of its components has a range in \([-1, 1]\), and even near the boundaries.

**Definition (Preconditioning the Flow)** Let \((P + I)\) be a Taylor model. We say that \((P_l + I_l), (P_r + I_r)\) is a factorization of \((P + I)\) if \(B(P_r + I_r) \in [-1, 1]\) and

\[
(P + I) \in (P_l \uplus I_l) \circ (P_r \uplus I_r) \text{ for all } x \in D
\]

where \(D\) is the domain of the Taylor model \((P_r + I_r)\).

The composition of the Taylor models is here to be understood as insertion of the Taylor model \(P_r + I_r\) into the polynomial \(P_l\) via Taylor model addition and multiplication and subsequent addition of the remainder bound \(I_l\). For the study of the solutions of ODEs, the following result is important.
Preconditioning the Flow II

**Proposition** Let \((P_{l,n} + I_{l,n}) \circ (P_{r,n} + I_{r,n})\) be a factored Taylor model that encloses the flow of the ODE at time \(t_n\). Let \((P_{l,n+1}^*, I_{l,n+1}^*)\) be the result of integrating \((P_{l,n} \pm I_{l,n})\) from \(t_n\) to \(t_{n+1}\). Then solution is in
\[
(P_{l,n+1}^*, I_{l,n+1}^*) \circ (P_{r,n} + I_{r,n})
\]

**Definition (Curvilinear Preconditioning)** Let \(x^{(m)} = f(x, x', ..., x^{(m-1)}, t)\) be an \(m\)-th order ODE in \(n\) variables. Let \(x_r(t)\) be a solution of the ODE and \(x'_r(t), ..., x_r^{(k)}(t)\) its first \(k\) time derivatives. Let \(\vec{e}_1, ..., \vec{e}_l\) be the \(l\) unit vectors not in the span of \(x'_r(t), ..., x_r^{(k)}(t)\), sorted by distance from the span. Then we call the Gram-Schmidt orthonormalization of the set \((x'_r(t), ..., x_r^{(k)}(t), \vec{e}_1, ..., \vec{e}_l)\) the curvilinear basis of depth \(k\).

Curvilinear coordinates have long history. Study of solar system, Beam Physics, ...

**Example (Curvilinear Solar System and Particle Accelerators)** In this case, \(n = 3\), and one usually chooses \(k = 2\). The first basis vector points in the direction of motion of the reference orbit. The second vector is perpendicular to it and points approximately to the sun or the center of the accelerator. The third vector is chosen perpendicular to the plane of the previous two.
Volterra - QR based preconditioning
Volterra - Curvilinear preconditioning
needle IC(1.5,-1) - QR based preconditioning
needle IC(1.5,-1) - Curvilinear preconditioning
Random Matrices - Discrete

Select 1000 twodimensional random matrices with coefficients in $[-1, 1]$. Sort according to eigenvalues into seven sub-cases.

Perform iteration in the following ways:

- Naive Interval
- Naive Taylormodel
- Parallelepiped-preconditioned Taylormodel
- QR-preconditioned Taylormodel
- Blunted preconditioned TM, various blunting factors
- Set of four floating point corner points for volume estimation

Perform the following tasks:

- Iterations through matrix
- Sets of iterations through matrix and its inverse
80 Real EVs (5 ≤ ratio < 10) Random Matrices

- Log_10(Mean)
- Step Number

- VE
- IN
- TMN
- TMP
- TMQ
- TMB
Random Matrices - Discrete

Select 1000 twodimensional random matrices with coefficients in $[-1, 1]$. Sort according to eigenvalues into seven sub-cases.

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- Naive Interval
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Perform the following tasks:

- Iterations through matrix
- Sets of iterations through matrix and its inverse
325 Conjugate EVs Random Matrices

log_{10}(Mean) vs Step Number
325 Conjugate EVs Random Matrices

log_{10}(Mean)

Step Number (showing every 20th step)
80 Real EVs (5 <= ratio < 10) Random Matrices

Step Number (showing every 20th step)
The Henon Map

Henon Map: frequently used elementary example that exhibits many of the well-known effects of nonlinear dynamics, including chaos, periodic fixed points, islands and symplectic motion. The dynamics is two-dimensional, and given by

\[ x_{n+1} = 1 - \alpha x_n^2 + y_n \]
\[ y_{n+1} = \beta x_n. \]

It can easily be seen that the motion is area preserving for \(|\beta| = 1\). We consider

\[ \alpha = 2.4 \text{ and } \beta = -1, \]

and concentrate on initial boxes of the form \((x_0, y_0) \in (0.4, -0.4) + [-d, d]^2\).
Henon system, $x_n = 1-2.4x^2+y$, $y_n = -x$, the positions at each step
Henon system, \( x_n = 1-2.4x^2+y \), \( y_n = -x \), corner points (+-0.01) the first 5 steps
Henon system, $x_n = 1 - 2.4x^2 + y$, $y_n = -x$, corner points (+-0.01) the first 120 steps
Henon system, \( x_n = 1 - 2.4 x^2 + y \), \( y_n = -x \), NO=1, SW
Henon system, $x_n = 1-2.4x^2+y$, $y_n = -x$, NO=1, SW
Henon system, $x_n = 1-2.4x^2+y$, $y_n = -x$, NO=20, SW
Henon system, $x_n = 1-2.4x^2+y$, $y_n = -x$, NO=20, SW
--- TM 5th

$m = 6$

$m = 7$

$m = 9$

$m = 15$

center at $(0.2123944, -0.4880256)$

TM center at $(0.4143362, -0.3895275)$

--- TM 5th

--- TM 1st

center at $(0.4143362, -0.3895275)$
Henon system, $x_n = 1 - 2.4 x^2 + y$, $y_n = -x$, $DX = 1 \times 10^{-14}$, $(0.4, -0.4)$, SW

Limit of zonotope method
A Muon Cooling Ring

Example from Beam Physics: Simple model of muon cooling ring, using curvilinear preconditioning.

Simultaneous damping via matter, and azimuthal accelerations. Equations of motion:

\[
\begin{align*}
\dot{x} &= p_x \\
\dot{y} &= p_y \\
\dot{p}_x &= p_y - \frac{\alpha}{\sqrt{p_x^2 + p_y^2}} \cdot p_x + \frac{\alpha}{\sqrt{x^2 + y^2}} \cdot y \\
\dot{p}_y &= -p_x - \frac{\alpha}{\sqrt{p_x^2 + p_y^2}} \cdot p_y - \frac{\alpha}{\sqrt{x^2 + y^2}} \cdot x
\end{align*}
\]

Has invariant solution

\[
(x, y, p_x, p_y)_I(t) = (\cos t, -\sin t, -\sin t, -\cos t),
\]

ODE asymptotically approach circular motion of the form

\[
(x, y, p_x, p_y)_a(t) = (\cos (t - \phi), -\sin (t - \phi), -\sin (t - \phi), -\cos (t - \phi)),
\]

where \(\phi\) is a characteristic angle for each particle.
MUON COOLING IN THE RFOFO RING*

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Dated: April 29, 2003

Abstract

Practical muon cooling rings could lead to better performance or lower cost designs of neutrino factories or muon colliders. The performance of the ring described here compares favorably with the linear cooling channel used in the second U.S. Neutrino Factory Study[1]. The 6D phase space density of an idealized ring is increased by a factor of 238, compared with the linear channel’s factor of only 15. The simulations make use of fully realistic magnetic fields, and include absorber and rf cavity windows, and empty lattice cells for injection/extraction.

INTRODUCTION

In the present U.S. Neutrino Factory design [2] the muon beam 6D phase space density must be reduced in order to be able to accelerate it and inject into the storage ring pointing to a long distance neutrino detector. Ionization cooling is currently the only feasible option for cooling the beam within the muon lifetime ($\tau_0 = 2.19 \mu s$). If muons alternately pass through a material absorber, and are then reaccelerated, and if there is sufficient focusing at the absorber, then the muon’s transverse phase space is reduced, i.e. the muons are cooled in the transverse dimension. A consequence of the transverse cooling is an increase of the longitudinal phase space caused by energy straggling in the material. The consequent momentum spread can be reduced if dispersion is introduced and a wedge absorber placed such that high momentum particles pass through more material than low momentum particles. However, in this process the beam width is increased. The process is thus primarily an exchange of emittance between the longitudinal and transverse dimensions, but when combined with transverse cooling in the material, all three dimensions can be cooled.

Recently there has been considerable progress in the design of cooling rings for neutrino factories and muon colliders[3]. The ring described here is based on an earlier design that used a simplified model of the magnetic field[4]. The ring consists of twelve 2.75 m long cells, each of which provide transverse cooling and emittance exchange. The focusing comes from a so-called RFOFO lattice of alternating direction solenoids, giving large angular and momentum acceptances. The axial magnetic field changes direction in the center of the cell. Ionization cooling is provided in 6D by sending the muon beam through wedge-shaped absorbers containing liquid H$_2$. The energy lost in the absorbers is replaced using 201.25 MHz rf cavities in each lattice cell.

Figure 1 shows the layout of the cooling ring drawn to scale. The RFOFO lattice was chosen because, apart from questions of injection/extraction, all cells are strictly identical, and the presence of an integer betatron resonance within the momentum acceptance is eliminated. The ring design parameters are given in Tb. 1.

Table 1: RFOFO Basic Ring Parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Circumference (m)</td>
<td>33</td>
</tr>
<tr>
<td>Cells</td>
<td>12</td>
</tr>
<tr>
<td>Max $B_z$ (T)</td>
<td>2.77</td>
</tr>
<tr>
<td>Coil Tilts (deg.)</td>
<td>3.0</td>
</tr>
<tr>
<td>Ave Momentum (MeV/c)</td>
<td>220</td>
</tr>
<tr>
<td>Min Trans. Beta (cm)</td>
<td>38</td>
</tr>
<tr>
<td>Max. Dispersion (cm)</td>
<td>8</td>
</tr>
<tr>
<td>Momentum Compaction</td>
<td>0.0037</td>
</tr>
<tr>
<td>Wedge Absorber Material</td>
<td>H$_2$</td>
</tr>
<tr>
<td>Wedge Thickness on axis (cm)</td>
<td>25.4</td>
</tr>
<tr>
<td>Wedge Angle (°)</td>
<td>90</td>
</tr>
<tr>
<td>Wedge Vertex position (cm)</td>
<td>12.7</td>
</tr>
<tr>
<td>Wedge Azimuthal angle (°)</td>
<td>230</td>
</tr>
<tr>
<td>Frequency (MHz)</td>
<td>201.25</td>
</tr>
<tr>
<td>Gradient (MV/m)</td>
<td>12</td>
</tr>
<tr>
<td>Phase (°) from 0-crossing</td>
<td>25</td>
</tr>
</tbody>
</table>

* MUC-NOTE-COOL-THEORY-273
Figure 2 shows a detailed view of three cells of the lattice, in plan (a) and side (b) views. The solenoids are not evenly spaced; those on either side of the absorbers are closer in order to increase the focusing at the absorber. The longitudinal field on-axis has an approximately sinusoidal dependence on position, as shown in Fig. 3. The beam axis is displaced laterally with respect to the coil centers (as shown in Fig. 2a) to minimize horizontal fields that cause vertical beam deviations.

Figure 4a) shows the beta function along a cell; the minimum value at the center of the absorber and at the central momentum is about 38 cm. Figure 4b) plots the beta function as a function of the muon beam momentum showing that the lattice transmits particles in the momentum band 160 - 260 MeV/c.

The bending field for the ring, and the required dispersion, are provided by alternately tilting the solenoids by $3.0^\circ$ (as shown in Fig. 2b).

It is found that the acceptance is reduced as the bending field is increased. We thus use a wedge with maximum possible angle (giving zero thickness on one side), and the lowest bending field consistent with adequate emittance exchange. The dispersion at the absorber is approximately 8 cm in a direction $26^\circ$ from the $y$ axis. The dispersion at the center of the rf cavity region has the opposite sign, and is mostly in the $y$ direction (see Fig. 5).

The rf cavities have a frequency of 201.25 MHz and an accelerating gradient of 12 MV/m.

**MODELING THE RING**

The RFOFO ring was modeled using the ICOOL code[5]. The magnetic field from the tipped solenoids was calculated in an independent code that found the result:
mucool, DX=0.01, preconditioned TM 12th, noSW
A Muon Cooling Ring - Results

1. Need to treat a large box of \([-10^{-2}, 10^{-2}]^4\]

2. Because of damping action towards the invariant limit cycle, the linear part of the motion is more and more ill-conditioned.

COSY easily integrates 10 cycles for \(d = 10^{-2}\) with curvilinear preconditioning and QR preconditioning. AWA (method 4) behaves as follows:

<table>
<thead>
<tr>
<th>(d)</th>
<th>Cycles</th>
</tr>
</thead>
<tbody>
<tr>
<td>(10^{-2})</td>
<td>0.22</td>
</tr>
<tr>
<td>(10^{-3})</td>
<td>1.25</td>
</tr>
<tr>
<td>(10^{-4})</td>
<td>9.5</td>
</tr>
</tbody>
</table>

Thus, trying to simulate the system with AWA requires \(> (10^2)^4 = 10^8\) subdivisions of the box that COSY can transport in one piece.
mucool ODE, (1,0,0,-1), Pre-conditioned TM 12th, noSW

Determinant L vs Time

- DX=1e-6 (red solid line)
- DX=1e-4 (green dashed line)
- DX=1e-2 (blue dotted line)
mucool ODE, (1,0,0,-1), Pre-conditioned TM 12th, noSW

Remainder Error Size of x

Time

DX=1e-6
DX=1e-4
DX=1e-2
mucool ODE, (1,0,0,-1), Pre-conditioned TM 12th, noSW

Condition Number L vs. Time

- Red line: \( DX=1e-6 \)
- Green dashed line: \( DX=1e-4 \)
- Blue dotted line: \( DX=1e-2 \)
Third International Workshop on Taylor Methods
Miami Beach, Florida
December 16-20, 2004

Topics:
High-Order Methods
Automatic Differentiation
Validated Methods
Taylor Models
ODE and PDE Solvers
Global Optimization
Constraint Satisfaction
Beam Physics
Optics

Website: http://bt.pa.msu.edu/TM/Miami2004/
Companion Workshop: Muon Collider Simulation 2004
Support: Department of Energy, Michigan State University