Recent Advances in the Validated Integration of ODEs

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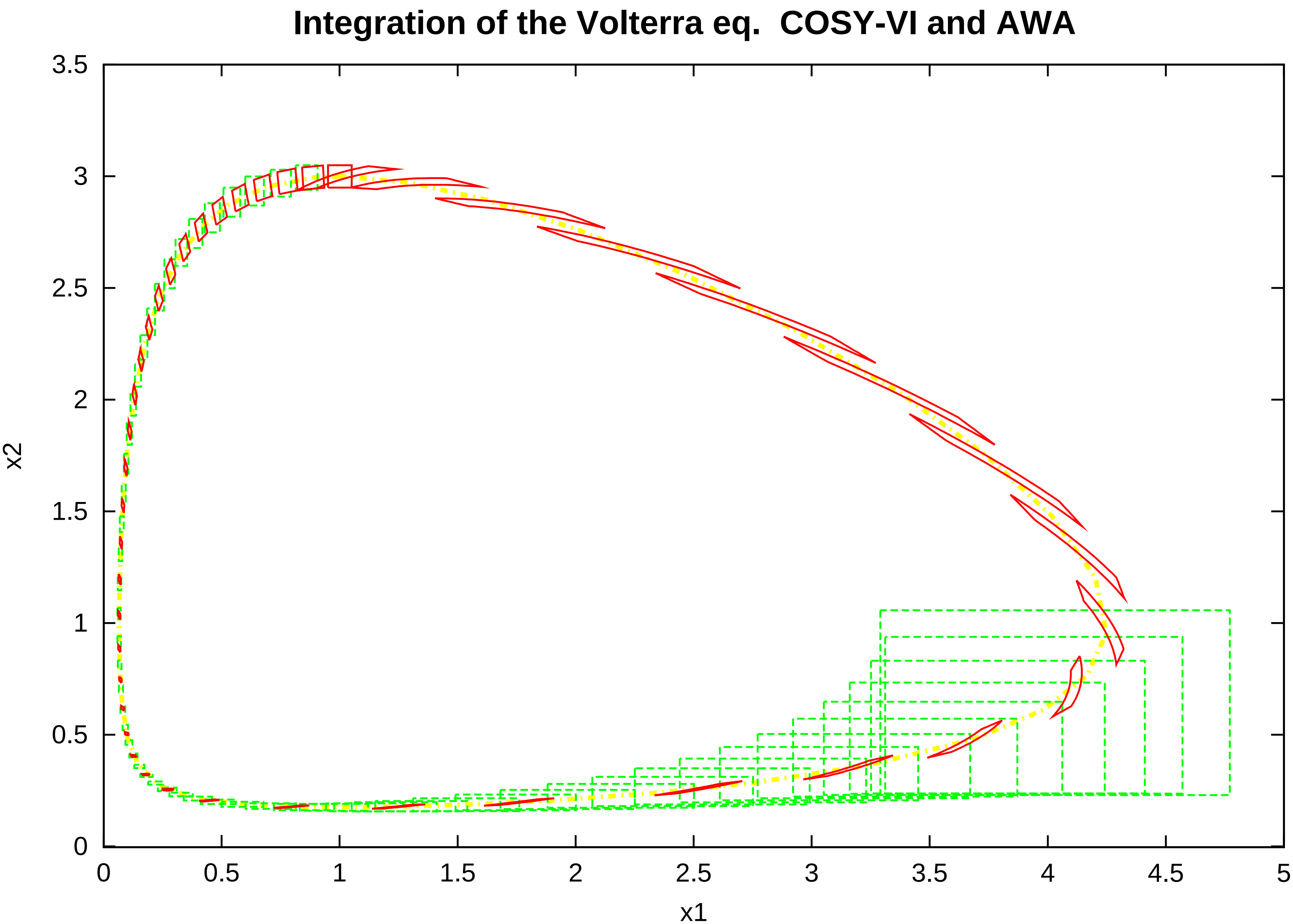

Integration of the Volterra eq. COSY-VI and AWA
x1
x2
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
0
0.5
1
1.5
2
2.5
3
3.5

Definitions - Taylor Models and Operations

We begin with a review of the definitions of the basic operations.

Definition (Taylor Model) Let $f : D \subset R^v \to R$ be a function that is $(n+1)$ times continuously partially differentiable on an open set containing the domain v-dimensional domain D. Let x_0 be a point in D and P the n-th order Taylor polynomial of f around x_0. Let I be an interval such that

$$f(x) \in P(x - x_0) + I \text{ for all } x \in D.$$

Then we call the pair (P, I) an n-th order Taylor model of f around x_0 on D.

Definition (Addition and Multiplication) Let $T_{1,2} = (P_{1,2}, I_{1,2})$ be n-th order Taylor models around x_0 over the domain D. We define

$$T_1 + T_2 = (P_1 + P_2, I_1 + I_2)$$
$$T_1 \cdot T_2 = (P_{1\cdot 2}, I_{1\cdot 2})$$

where $P_{1\cdot 2}$ is the part of the polynomial $P_1 \cdot P_2$ up to order n and

$$I_{1\cdot 2} = B(P_e) + B(P_1) \cdot I_2 + B(P_2) \cdot I_1 + I_1 \cdot I_2$$

where P_e is the part of the polynomial $P_1 \cdot P_2$ of orders $(n+1)$ to $2n$, and $B(P)$ denotes a bound of P on the domain D. We demand that $B(P)$ is at least as sharp as direct interval evaluation of $P(x - x_0)$ on D.

The Operator ∂^{-1}on Taylor Models

Let (P_n, I_n) be an n-th order Taylor model of f. From this we can obtain a Taylor model for the indefinite integral $\partial_i^{-1} f = \int f \; dx_i'$ with respect to variable x_i.

Taylor polynomial part: $\int_0^{x_i} P_{n-1} dx_i'$,

Remainder Bound: $(B(P_n - P_{n-1}) + I_n) \cdot B(x_i)$, where $B(P)$ is a polynomial bound.

So define the operator ∂_i^{-1}on space of Taylor models as

$$\partial_i^{-1}(P_n, I_n)$$
$$= \left(\int_0^{x_i} P_{n-1} dx_i' \;\; , \quad (B(P_n - P_{n-1}) + I_n) \cdot B(x_i) \right)$$

TM Scaling Theorem

Theorem (Scaling Theorem) Let $f,\ g \in C^{n+1}(D)$ and $(P_{f,h}, I_{f,h})$ and $(P_{g,h}, I_{g,h})$ be n-th order Taylor models for f and g around x_h on $x_h + [-h, h]^v \subset D$. Let the remainder bounds $I_{f,h}$ and $I_{g,h}$ satisfy $I_{f,h} = O(h^{n+1})$ and $I_{g,h} = O(h^{n+1})$. Then the Taylor models $(P_{f+g}, I_{f+g,h})$ and $(P_{f\cdot g}, I_{f\cdot g,h})$ for the sum and products of f and g obtained via addition and multiplication of Taylor models satisfy

$$I_{f+g,h} = O(h^{n+1}),\ \text{and}\ I_{f\cdot g,h} = O(h^{n+1}).$$

Furthermore, let s be any of the intrinsic functions defined above, then the Taylor model $(P_{s(f)}, I_{s(f),h})$ for $s(f)$ obtained by the above definition satisfies

$$I_{s(f),h} = O(h^{n+1}).$$

We say the Taylor model arithmetic has the $(n+1)$-st order scaling property.

Proof. The proof for the binary operations follows directly from the definition of the remainder bounds for the binaries. Similarly, the proof for the intrinsics follows because all intrinsics are composed of binary operations as well as an additional interval, the width of which scales at least with the $(n+1)$-st power of a bound B of a function that scales at least linearly with h.

Important TM Algorithms

- **Range Bounding** (Evaluate f as TM, bound polynomial, add remainder bound)
- **Quadrature** (Evaluate f as TM, integrate polynomial and remainder bound)
- **Implicit Equations** (Obtain TMs for implicit solutions of TM equations)
- **Superconvergent** Interval Newton Method (Application of Implicit Equations)
- **ODEs** (Obtain TMs describing dependence of final coordinates on initial coordinates)
- **Implicit ODEs and DAEs**
- **Complex Arithmetic** (Describe complex ranges as two-dimensional TMs)

Implementation of TM Arithmetic

Validated Implementation of TM Arithmetic exists. The following points are important

- Strict requirements for **underlying FP arithmetic**
- Taylor models require cutoff threshold (**garbage collection**)
- Coefficients remain FP, not intervals
- Package quite **extensively tested** by Corliss et al.

For practical considerations, the following is important:

- Need **sparsity** support
- Need efficient coefficient **addressing** scheme
- About $50,000$ lines of code
- **Language Independent** Platform, coexistence in F77, C, F90, C++

Multiplication - Weighting

Sometimes important: Carry different variables x_i to different orders w_i.

Can be achieved by simply "seeding" original variables as

$$P(x) = (x_1^{w_1}, x_2^{w_2}, ..., x_v^{w_v}).$$

Then in all subsequent operations, only multiples of w_i appear as powers of x_i. Optimal reduction of speed by sparsity.

Order n	Variables v	Weighting w	Order n_i	Cosy Coefs	AWA Coefs
17	3	9	1	41	144
17	5	9	1	57	216
17	10	9	1	97	396
17	20	9	1	177	756
17	3	5	3	135	144
17	5	5	3	308	216
17	10	5	3	1248	396
17	20	5	3	6578	756
13	5	3	4	504	168
13	10	3	4	3094	308
15	5	3	5	882	192
15	10	3	5	7098	352

TABLE 1. Number of floating point numbers necessary to store all appearing partial derivaties in COSY to order n_i in initial conditions, and in the first order code AWA

Taylor Models for the Flow

Goal: Determine a Taylor model, consisting of a Taylor Polynomial and an interval bound for the remainder, for the flow of the differential equation

$$\frac{d}{dt}\vec{r}(t) = \vec{F}(\vec{r}(t), t)$$

where $\vec{F}$ is sufficiently differentiable. The Remainder Bound should be fully rigorous for all initial conditions $\vec{r}_0$ and times t that satisfy

$$\begin{aligned} \vec{r}_0 &\in [\vec{r}_{01}, \vec{r}_{02}] = \vec{B} \\ t &\in [t_0, t_1]. \end{aligned}$$

In particular, $\vec{r}_0$ itself may be a Taylor model, as long as its range is known to lie in $\vec{B}$.

The Volterra Equation

Describe dynamics of two conflicting populations

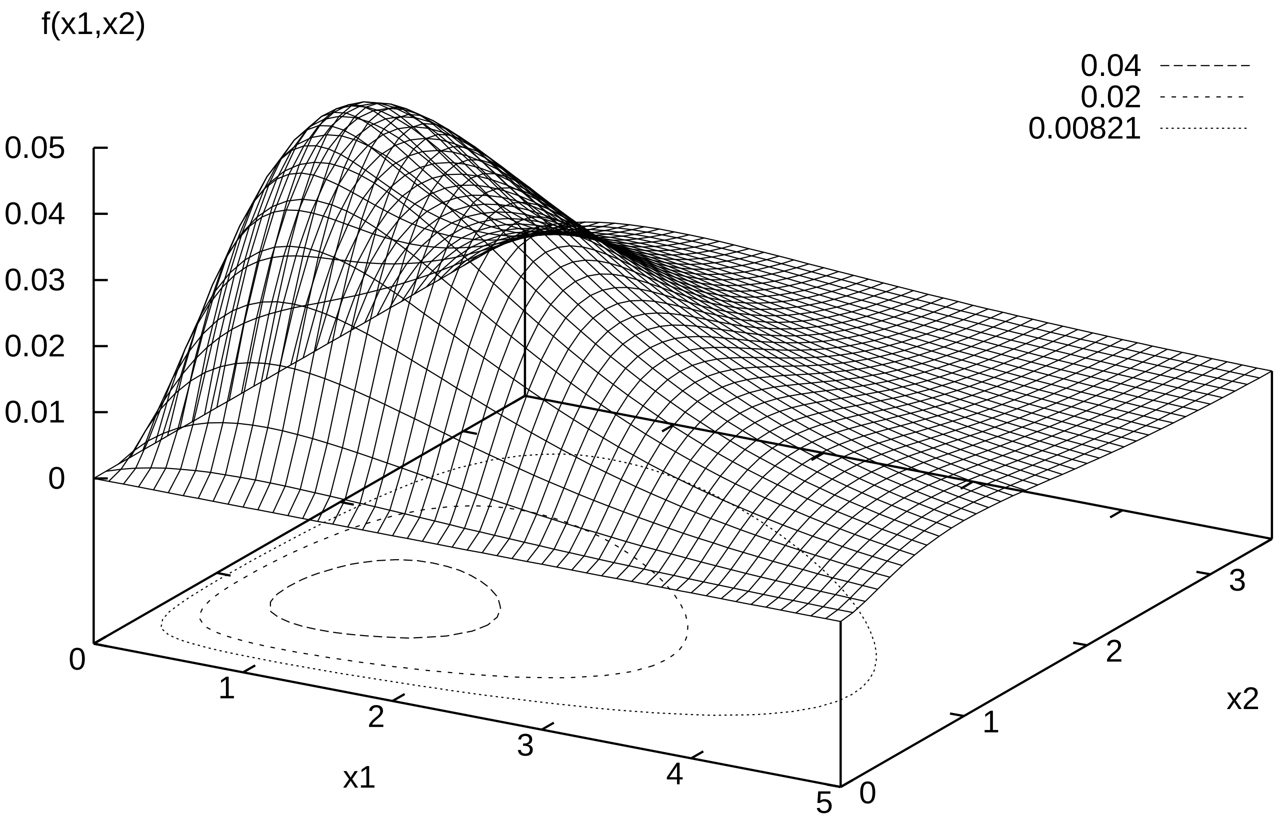
$$\frac{dx_1}{dt} = 2x_1(1 - x_2), \quad \frac{dx_2}{dt} = -x_2(1 - x_1)$$

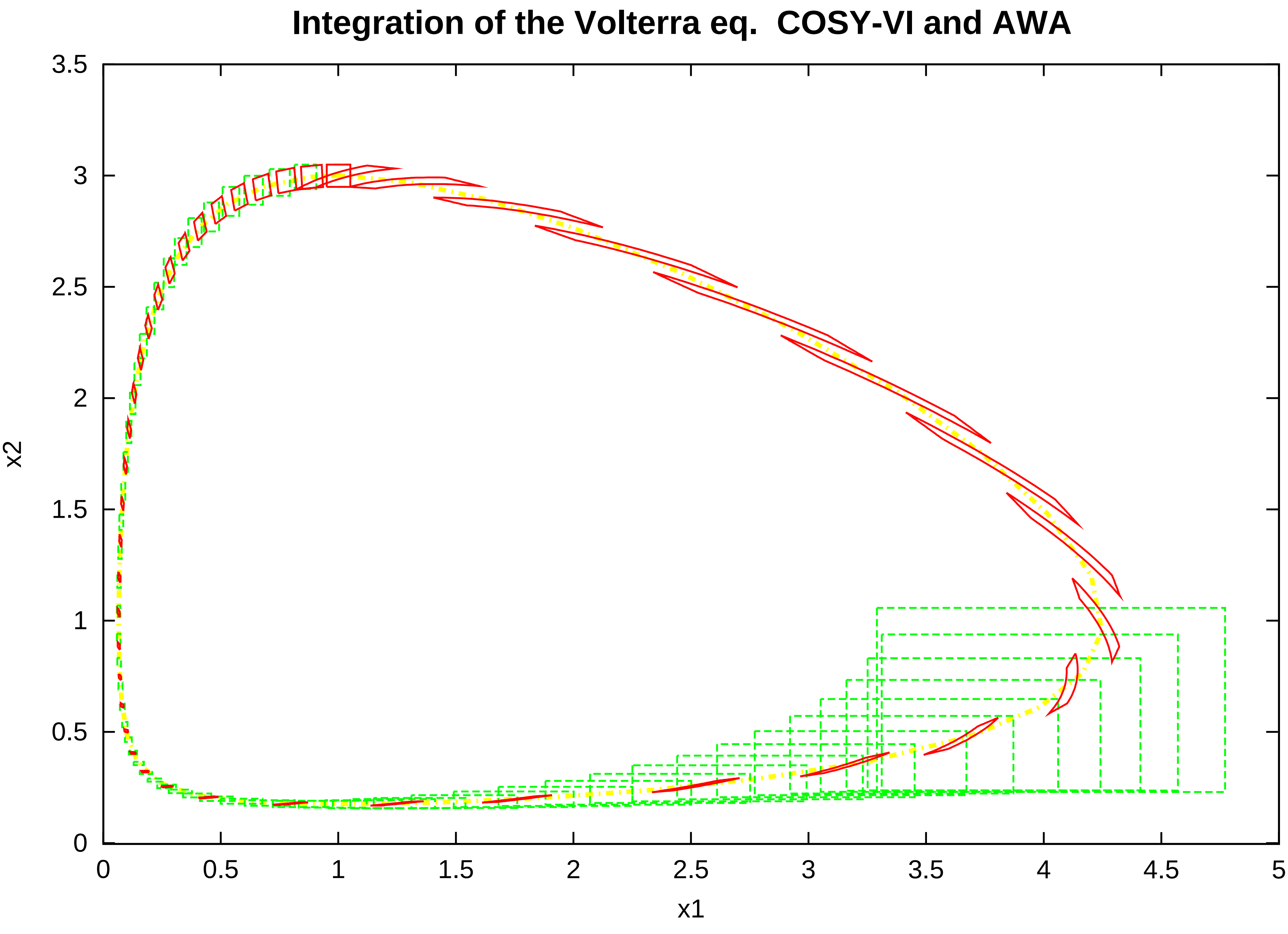
Interested in initial condition

$$x_{01} \in 1 + [-0.05, 0.05], \quad x_{02} \in 3 + [-0.05, 0.05] \quad \text{at } t = 0.$$

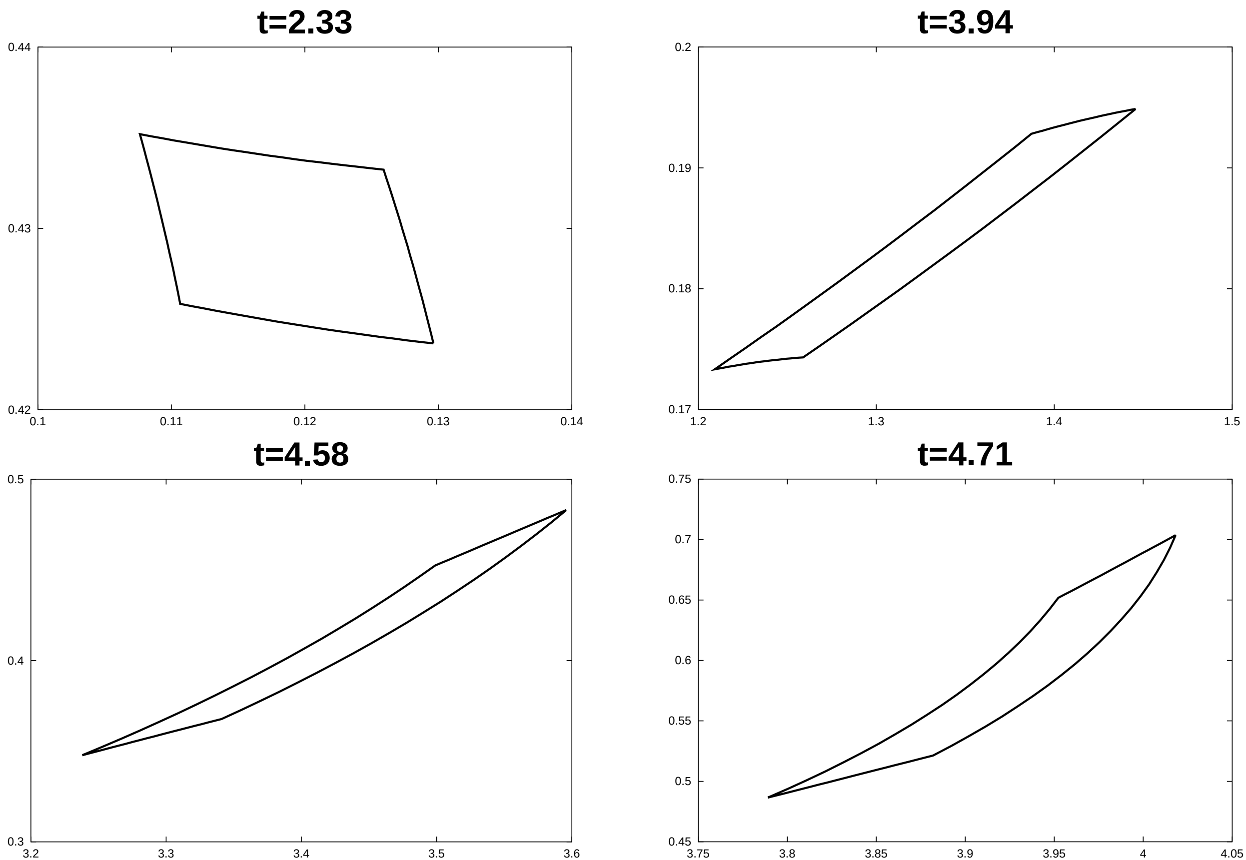
Satisfies constraint condition

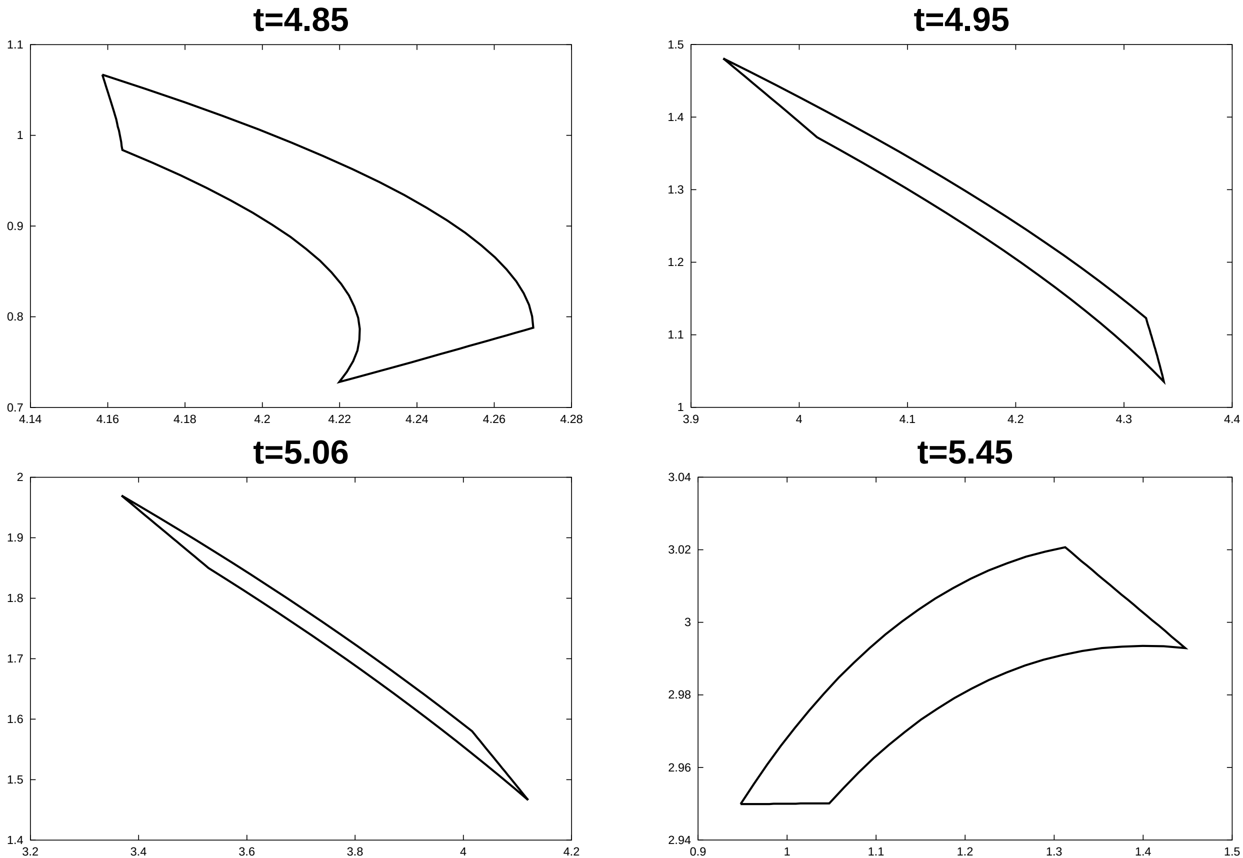
$$C(x_1, x_2) = x_1 x_2^2 e^{-x_1 - 2x_2} = \text{Constant}$$

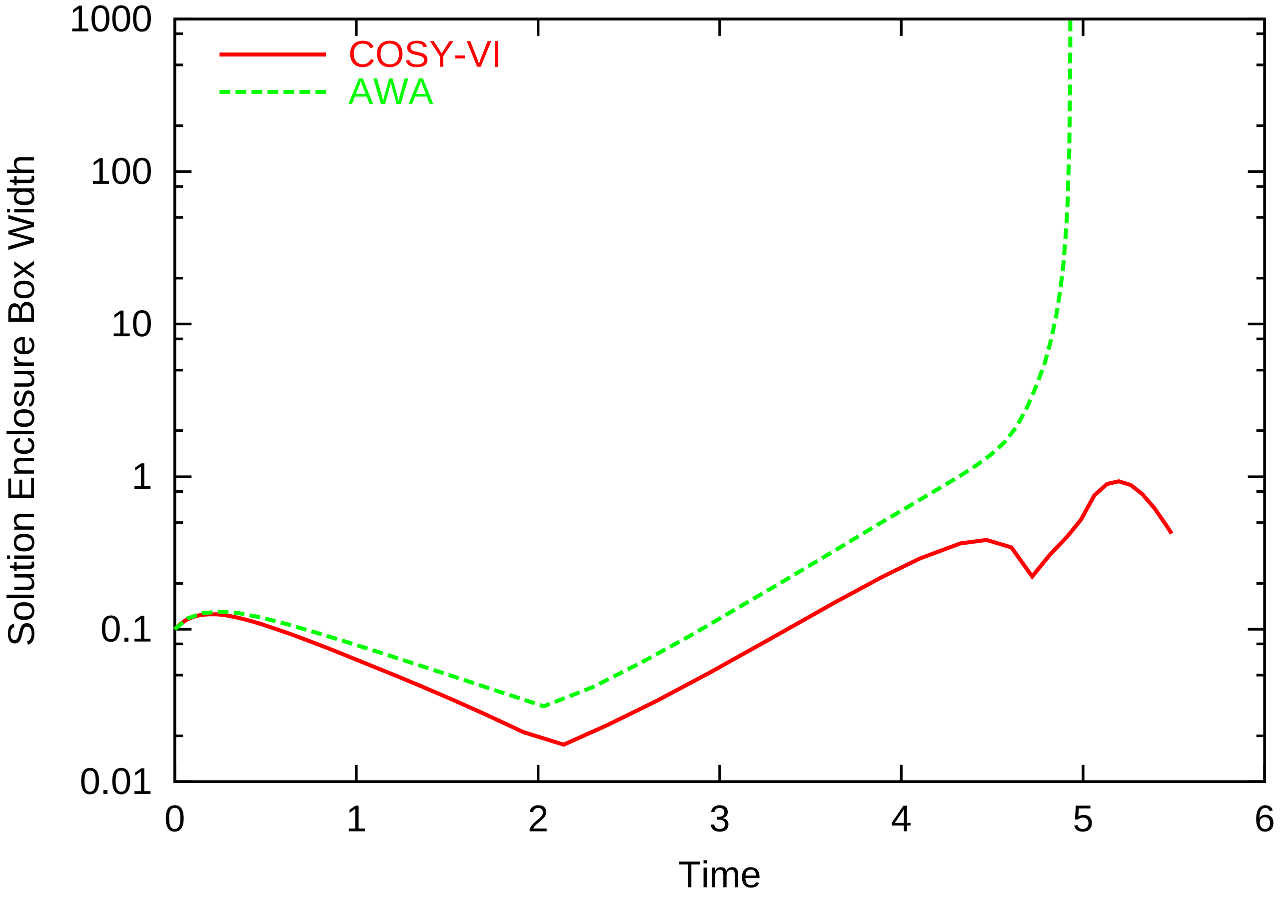

f(x1,x2)
0.04
0.02
0.00821
0.05
0.04
0.03
0.02
0.01
0
0
1
2
3
4
5
x1
0
1
2
3
x2



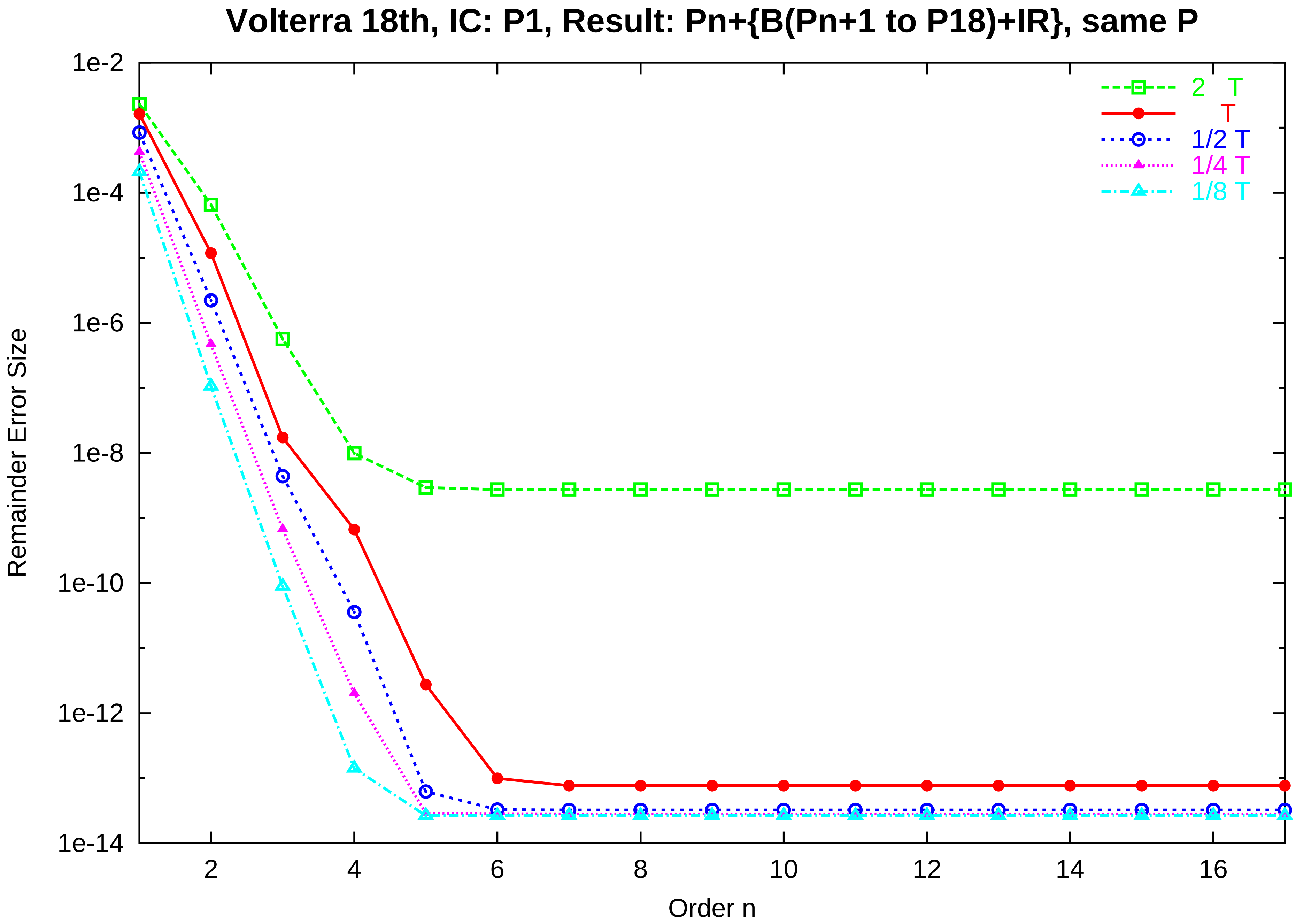
Integration of the Volterra eq. COSY-VI and AWA
x1
x2
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
0
0.5
1
1.5
2
2.5
3
3.5


t=2.33
t=3.94
t=4.58
t=4.71
0.44
0.43
0.42
0.1
0.11
0.12
0.13
0.14
0.2
0.19
0.18
0.17
1.2
1.3
1.4
1.5
0.5
0.4
0.3
3.2
3.3
3.4
3.5
3.6
0.75
0.7
0.65
0.6
0.55
0.5
0.45
3.75
3.8
3.85
3.9
3.95
4
4.05

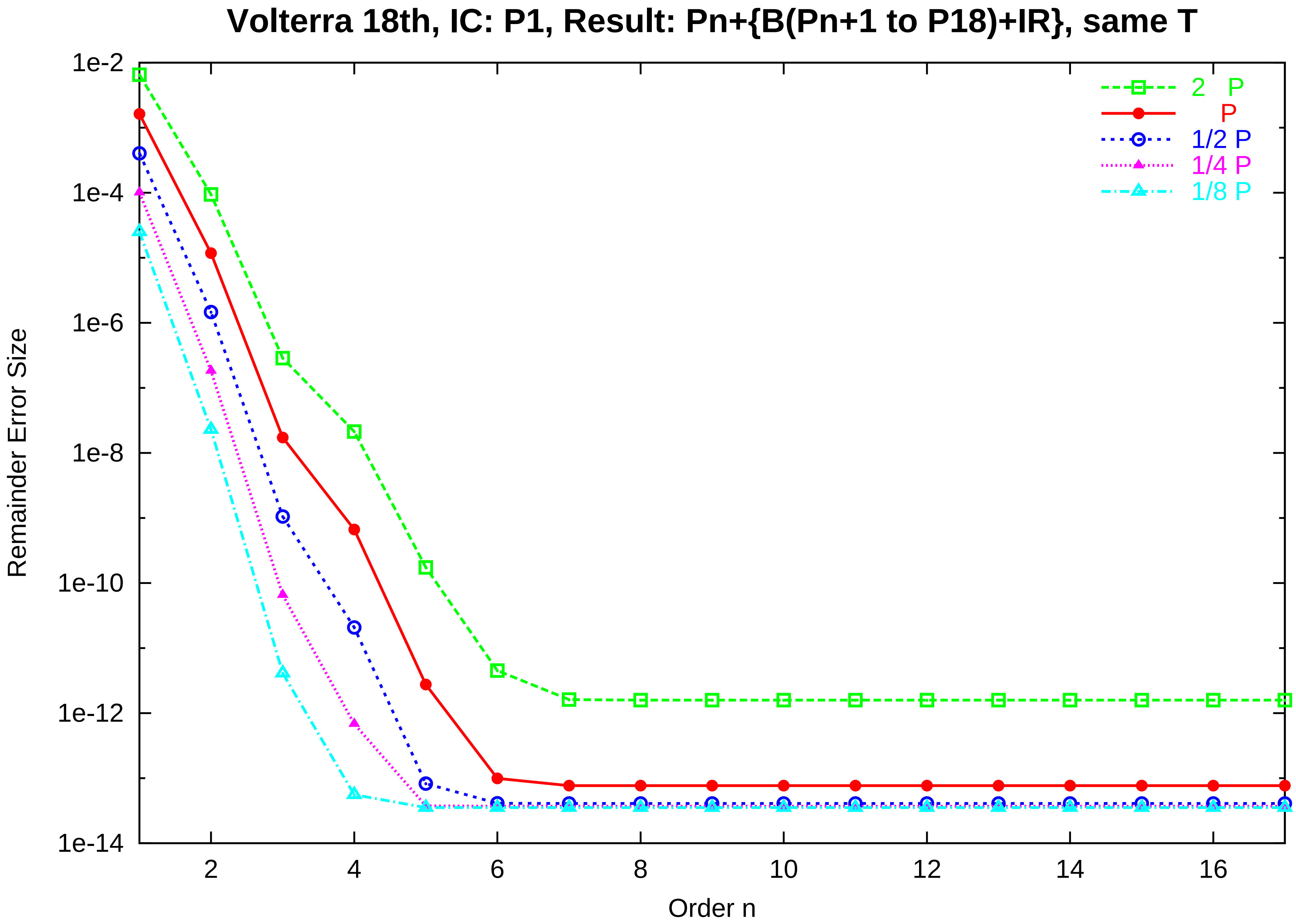

t=4.85
1.1
1
0.9
0.8
0.7
4.14
4.16
4.18
4.2
4.22
4.24
4.26
4.28
t=4.95
1.5
1.4
1.3
1.2
1.1
1
3.9
4
4.1
4.2
4.3
4.4
t=5.06
2
1.9
1.8
1.7
1.6
1.5
1.4
3.2
3.4
3.6
3.8
4
4.2
t=5.45
3.04
3.02
3
2.98
2.96
2.94
0.9
1
1.1
1.2
1.3
1.4
1.5



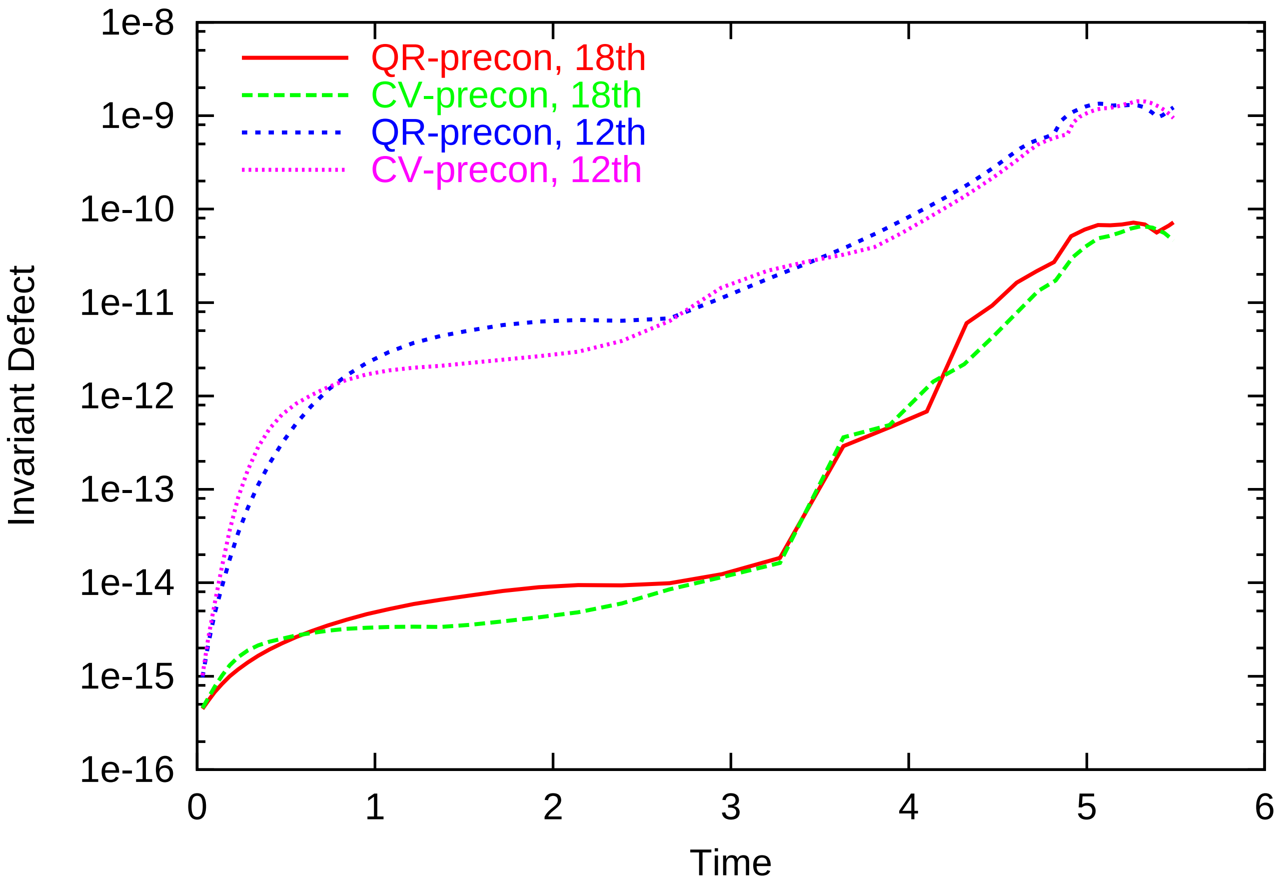
COSY-VI
AWA
Solution Enclosure Box Width
1000
100
10
1
0.1
0.01
0
1
2
3
4
5
6
Time



Volterra 18th, IC: P1, Result: Pn+{B(Pn+1 to P18)+IR}, same P
Remainder Error Size
Order n
1e-2
1e-4
1e-6
1e-8
1e-10
1e-12
1e-14
2
4
6
8
10
12
14
16
2 T
T
1/2 T
1/4 T
1/8 T



Volterra 18th, IC: P1, Result: Pn+{B(Pn+1 to P18)+IR}, same T
Remainder Error Size
Order n
1e-2
1e-4
1e-6
1e-8
1e-10
1e-12
1e-14
2
4
6
8
10
12
14
16
2 P
P
1/2 P
1/4 P
1/8 P



QR-precon, 18th
CV-precon, 18th
QR-precon, 12th
CV-precon, 12th
Invariant Defect
Time
1e-8
1e-9
1e-10
1e-11
1e-12
1e-13
1e-14
1e-15
1e-16
0
1
2
3
4
5
6

Shrink Wrapping I

A method to remove the remainder bound of a Taylor model by increasing the polynomial part.

After the kth step of the integration, the region occupied by the final variables is given by

$$A = \vec{I}_0 + \bigcup_{\vec{x}_0 \in \vec{B}} \mathcal{M}_0(\vec{x}_0),$$

where $\vec{x}_0$ are the initial variables, $\vec{B}$ is the original box of initial conditions, $\mathcal{M}_0$ is the polynomial part of the Taylor model, and $\vec{I}_0$ is the remainder bound interval. $\mathcal{M}_0$ is scaled such that the original box $\vec{B}$ is unity, i.e. $\vec{B} = [-1, 1]^v$. $\vec{I}_0$ accounts for the local approximation error of the expansion in time carried out in the kth step as well as floating point errors and potentially other accumulated errors from previous steps; it is usually very small. Try to "absorb" the small remainder interval into a set very similar to the first part via

$$A \subset A^* = \vec{I}_0^* + \bigcup_{\vec{x}_0 \in \vec{B}} \mathcal{M}_0^*(\vec{x}_0),$$

where $\mathcal{M}_0^*$ is a slightly modified polynomial, and $\vec{I}_0^*$ is significantly reduced

Shrink Wrapping

Theorem (Shrink Wrapping) Let $\mathcal{M} = \mathcal{I} + \mathcal{S}(\vec{x})$, where $\mathcal{I}$ is the identity. Let $\vec{I} = d \cdot [-1, 1]^v$, and

$$A = \vec{I} + \bigcup_{\vec{x} \in \vec{B}} \mathcal{M}(\vec{x})$$

be the set sum of the interval $\vec{I} = [-d, d]^v$ and the range of $\mathcal{M}$ over the original domain box $\vec{B}$. So A is the range enclosure of the flow of the ODE over the interval $\vec{B}$ provided by the Taylor model. Let q be the shrink wrap factor of $\mathcal{M}$; then we have

$$A \subset \bigcup_{\vec{x} \in \vec{B}} (q\mathcal{M})(\vec{x}),$$

and hence multiplying $\mathcal{M}$ with the number q allows to set the remainder bound to zero.

Shrink Wrapping

We define q, the so-called shrink wrap factor, as

$$q = 1 + d \cdot \frac{1}{(1-(v-1)t)\cdot(1-s)}.$$

The bounds s and t for the polynomials $\mathcal{S}_i$ and $\partial \mathcal{S}_i / \partial x_j$ can be computed by interval evaluation. The factor q will prove to be a factor by which the Taylor polynomial $\mathcal{I} + \mathcal{S}$ has to be multiplied in order to absorb the remainder bound interval.

Remark (Typical values for q) To put the various numbers in perspective, in the case of the verified integration of the Asteroid 1997 XF11, we typically have $d = 10^{-7}$, $s = 10^{-4}$, $t = 10^{-4}$, and thus $q \approx 1 + 10^{-7}$. It is interesting to note that the values for s and t are determined by the non-linearity in the problem at hand, while in the absence of "noise" terms in the ODEs described by intervals, the value of d is determined mostly by the accuracy of the arithmetic. Rough estimates of the expected performance in quadruple precision arithmetic indicate that with an accompanying decrease in step size, if desired d can be decreased below 10^{-12}, resulting in $q \approx 1 + 10^{-12}$.

Long-Term Behavior - Validated Case

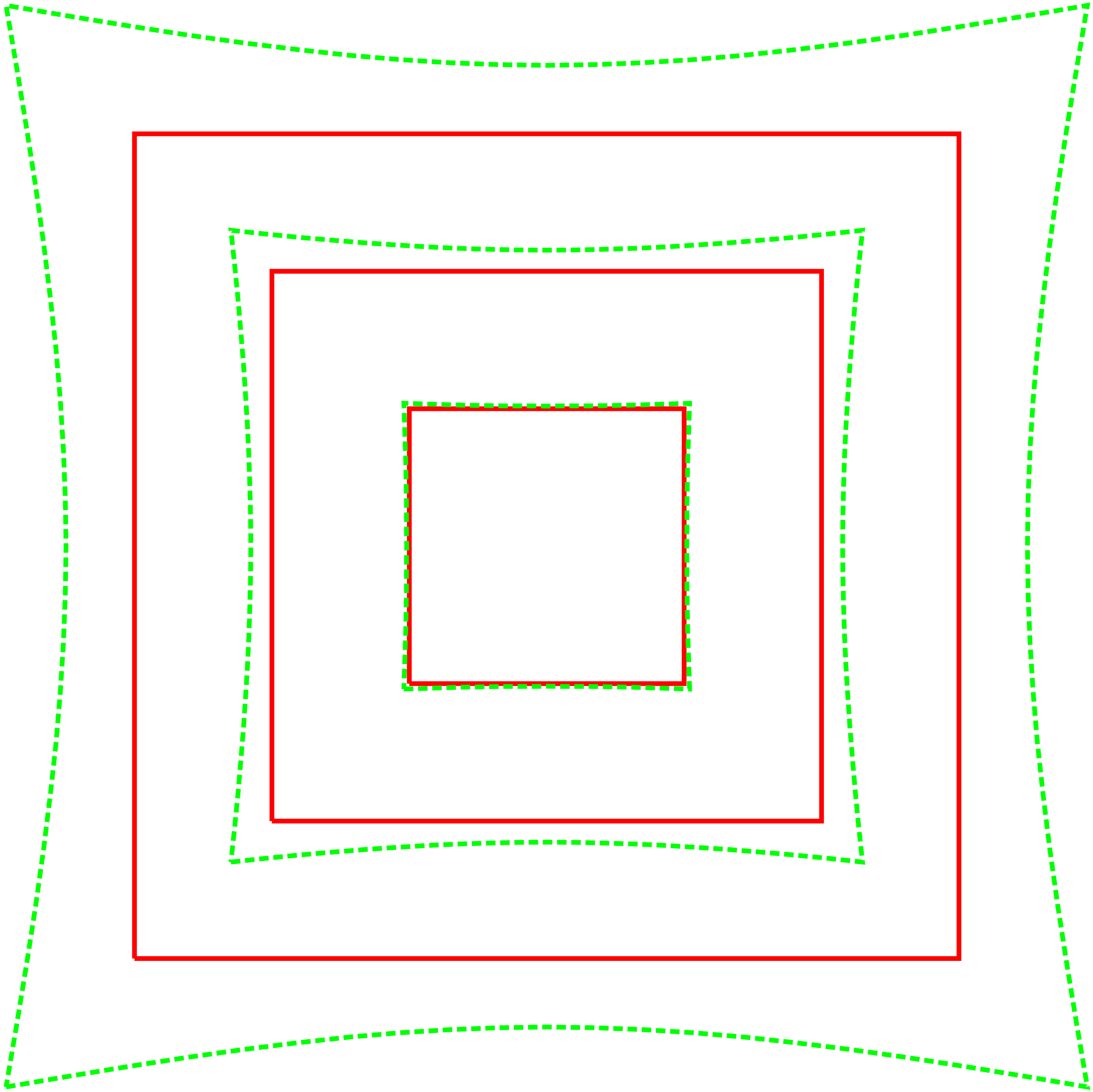
Consider very simple two-state dynamical system:

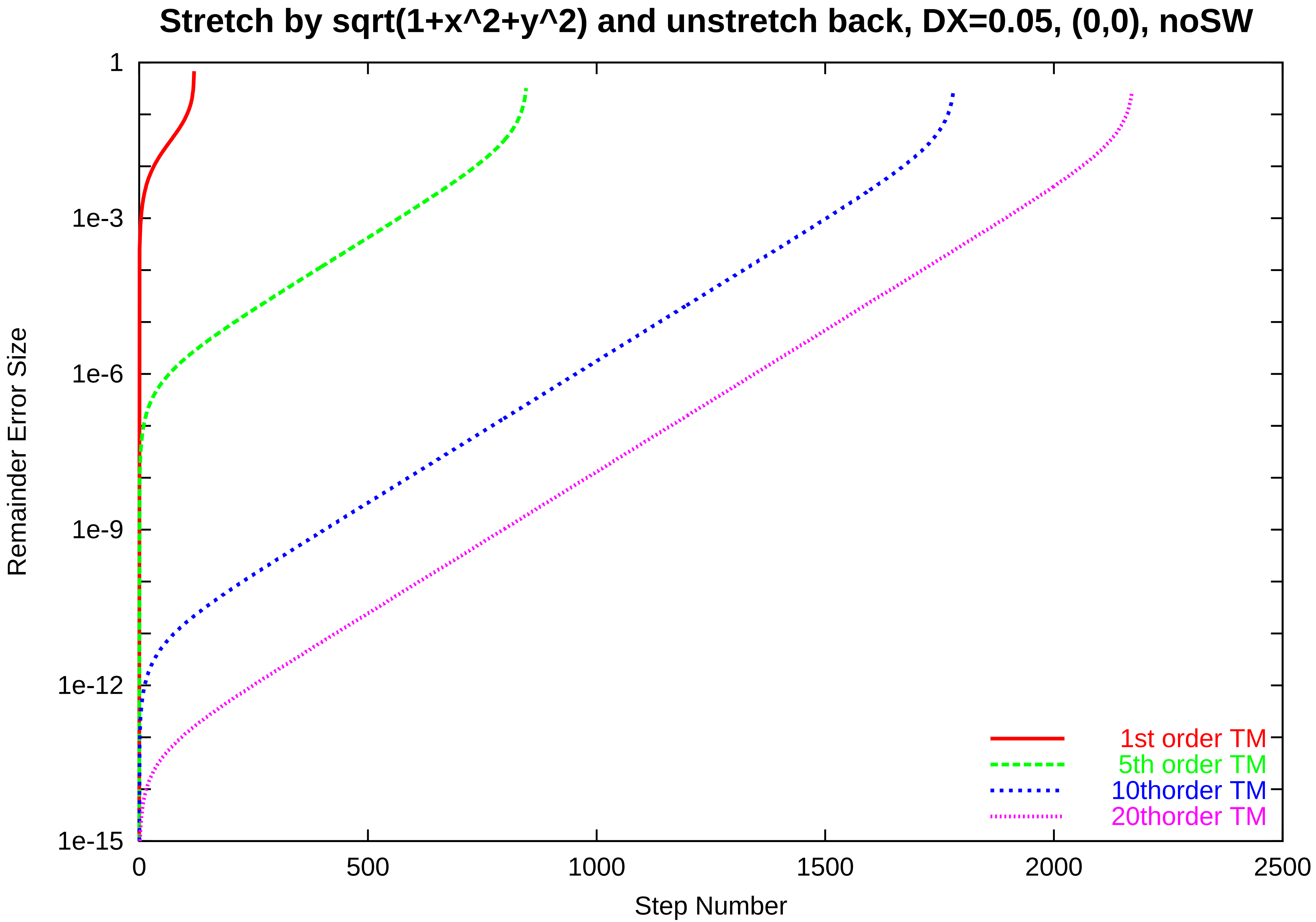
$$x_{n+1} = x_n \cdot \sqrt{1 + x_n^2 + y_n^2} \text{ and } y_{n+1} = y_n \cdot \sqrt{1 + x_n^2 + y_n^2}$$

$$x_{n+2} = x_{n+1} \cdot \sqrt{\frac{2}{1 + \sqrt{1 + 4(x_{n+1}^2 + y_{n+1}^2)}}} \text{ and}$$

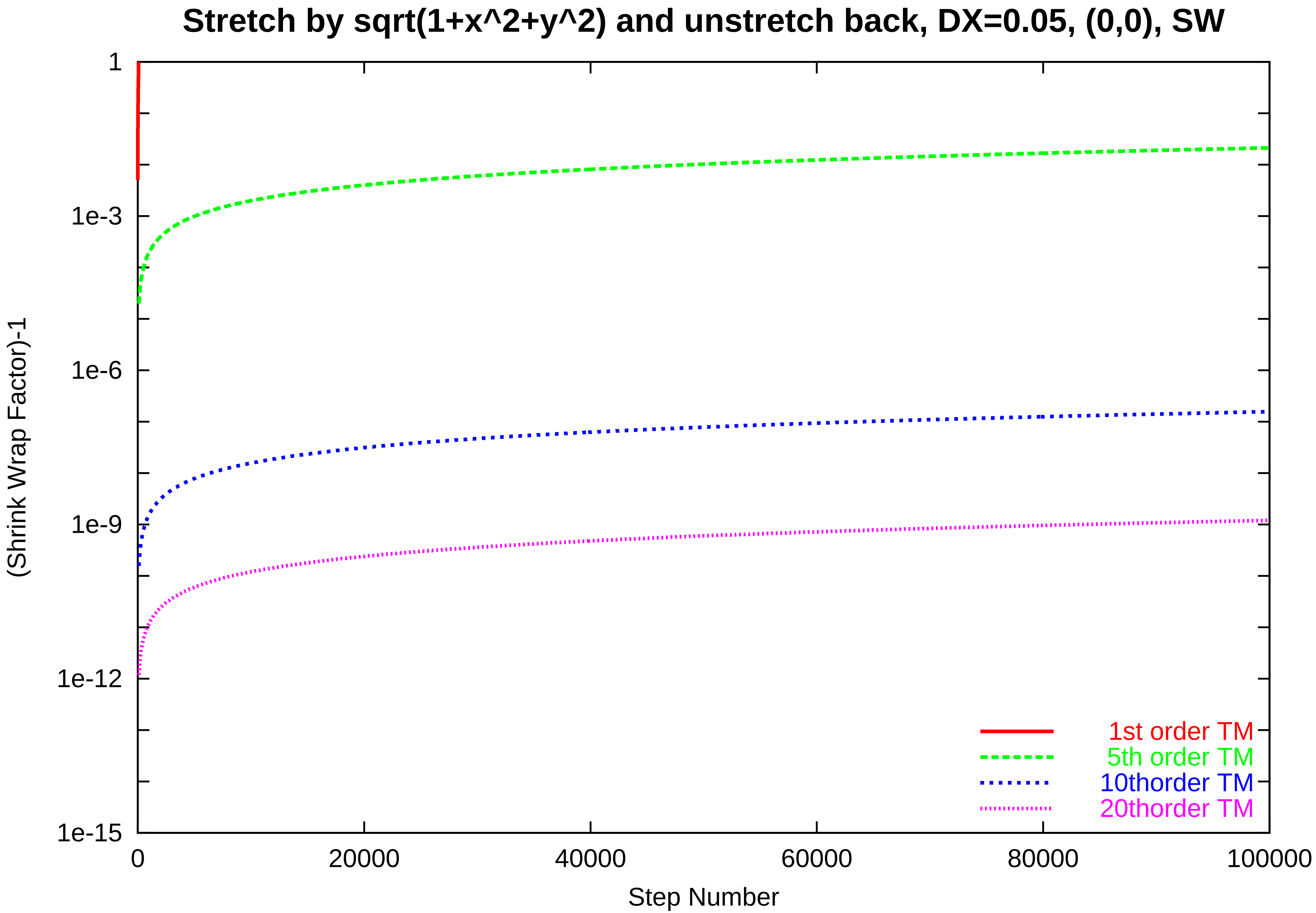
$$y_{n+2} = y_{n+1} \cdot \sqrt{\frac{2}{1 + \sqrt{1 + 4(x_{n+1}^2 + y_{n+1}^2)}}}.$$

Simple arithmetic shows that, also here we have $(x_{n+2}, y_{n+2}) = (x_n, y_n)$.





Stretch by sqrt(1+x^2+y^2) and unstretch back, DX=0.05, (0,0), noSW
Remainder Error Size
Step Number
1
1e-3
1e-6
1e-9
1e-12
1e-15
0
500
1000
1500
2000
2500
1st order TM
5th order TM
10thorder TM
20thorder TM



Stretch by sqrt(1+x^2+y^2) and unstretch back, DX=0.05, (0,0), SW
1
1e-3
1e-6
1e-9
1e-12
1e-15
(Shrink Wrap Factor)-1
0
20000
40000
60000
80000
100000
Step Number
1st order TM
5th order TM
10thorder TM
20thorder TM

Shrink Wrapping II

Let us consider the practical limiations of the method; apparently the measures of the nonlinearities s and t must not become too large

Remark (Limitations of shrink wrapping) Apparently the shrink wrap method discussed above has the following limitations

Remark 1 *1. The measures of nonlinearities s and t must not become too large*

2. The application of the inverse of the linear part should not lead to large increases in the size of remainder bounds.

Apparently the first requirement limits the domain size that can be covered by the Taylor model, and it will thus happen only in extreme cases. Furthermore, in practice the case of s and t becoming large is connected to also having accumulated a large remainder bound, since the remainder bounds are calculated from the bounds of the various orders of s. In the light of this, not much additional harm is done by removing the offending s into the remainder bound and create a linearized Taylor model.

Definition (Blunting of an Ill-Conditioned Matrix)

Let $\hat{A}$ be a regular matrix that is potentially ill-conditioned and $\vec{q} = (q_1, ...q_n)$ a vector with $q_i > 0$. Arrange the column vectors $\vec{a}_i$ of $\hat{A}$ by size.

Let $\vec{e}_i$ be the familiar orthonormal vectors obtained through the Gram-Schmidt procedure, i.e.

$$\vec{e}_i = \frac{\vec{a}_i - \sum_{k=1}^{i-1} \vec{e}_k \ (\vec{a}_i \cdot \vec{e}_k)}{\left| \vec{a}_i - \sum_{k=1}^{i-1} \vec{e}_k \ (\vec{a}_i \cdot \vec{e}_k) \right|}.$$

We form vectors $\vec{b}_i$ via

$$\vec{b}_i = \vec{a}_i + q_i \vec{e}_i$$

and assemble them columnwise into the matrix $\hat{B}$. We call $\hat{B}$ the $\vec{q}$-blunted matrix belonging to $\hat{A}$

Intuitively, the effect of blunting is that each vector $\vec{b}_i$ is being "pulled away" from the space spanned by the previous vectors $\vec{b}_1, ..., \vec{b}_{i-1}$, and more strongly so if q_i becomes bigger and bigger. In fact, we have the following result: .

Preconditioning the Flow

Idea: write the Taylor model of the solution as a composition of two Taylor models $(P_l + I_l)$ and $(P_r + I_r)$, and then choose $P_l + I_l$ in such a way that in each step, the operations appearing on I_r are minimized. At the same time, I_l will be chosen as small as possible. Can be viewed as a coordinate transformation.

In the factorization, we impose that $(P_r + I_r)$ is normalized such that each of its components has a range in $[-1, 1]$, and even near the boundaries.

Definition (Preconditioning the Flow) Let $(P + I)$ be a Taylor model. We say that $(P_l + I_l), (P_r + I_r)$ is a factorization of $(P + I)$ if $B(P_r + I_r) \in [-1, 1]$ and

$$(P + I) \in (P_l + I_l) \circ (P_r + I_r) \text{ for all } x \in \mathcal{D}$$

where D is the domain of the Taylor model $(P_r + I_r)$.

The composition of the Taylor models is here to be understood as insertion of the Taylor model $P_r + I_r$ into the polynomial P_l via Taylor model addition and multiplication and subsequent addition of the remainder bound I_l. For the study of the solutions of ODEs, the following result is important

Preconditioning the Flow II

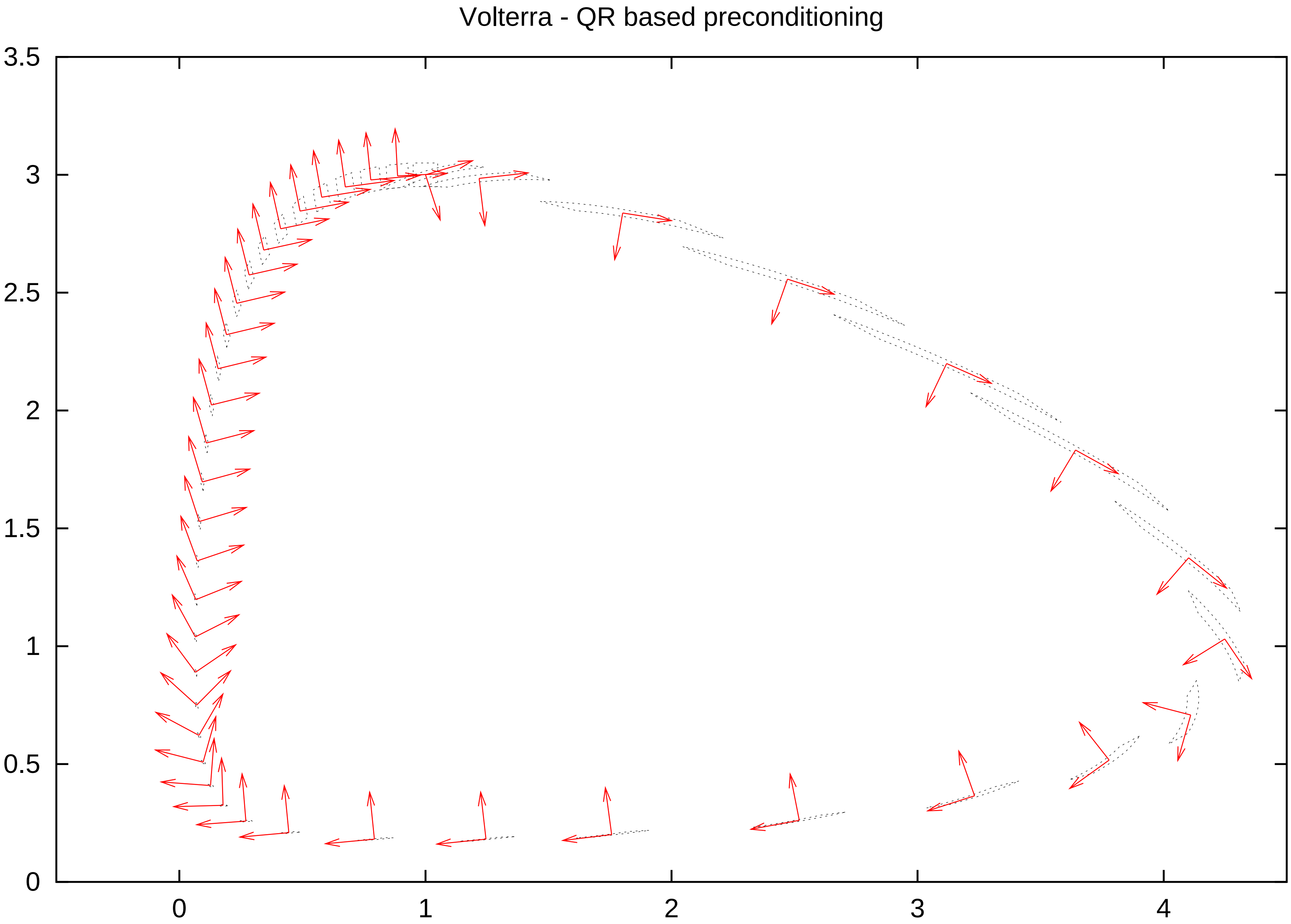
Proposition Let $(P_{l,n} + I_{l,n}) \circ (P_{r.n} + I_{r,n})$ be a factored Taylor model that encloses the flow of the ODE at time t_n. Let $(P^*_{l,n+1}, I^*_{l,n+1})$ be the result of integrating $(P_{l,n} + I_{l,n})$ from t_n to t_{n+1}. Then solution is in

$$(P^*_{l,n+1}, I^*_{l,n+1}) \circ (P_{r.n} + I_{r,n})$$

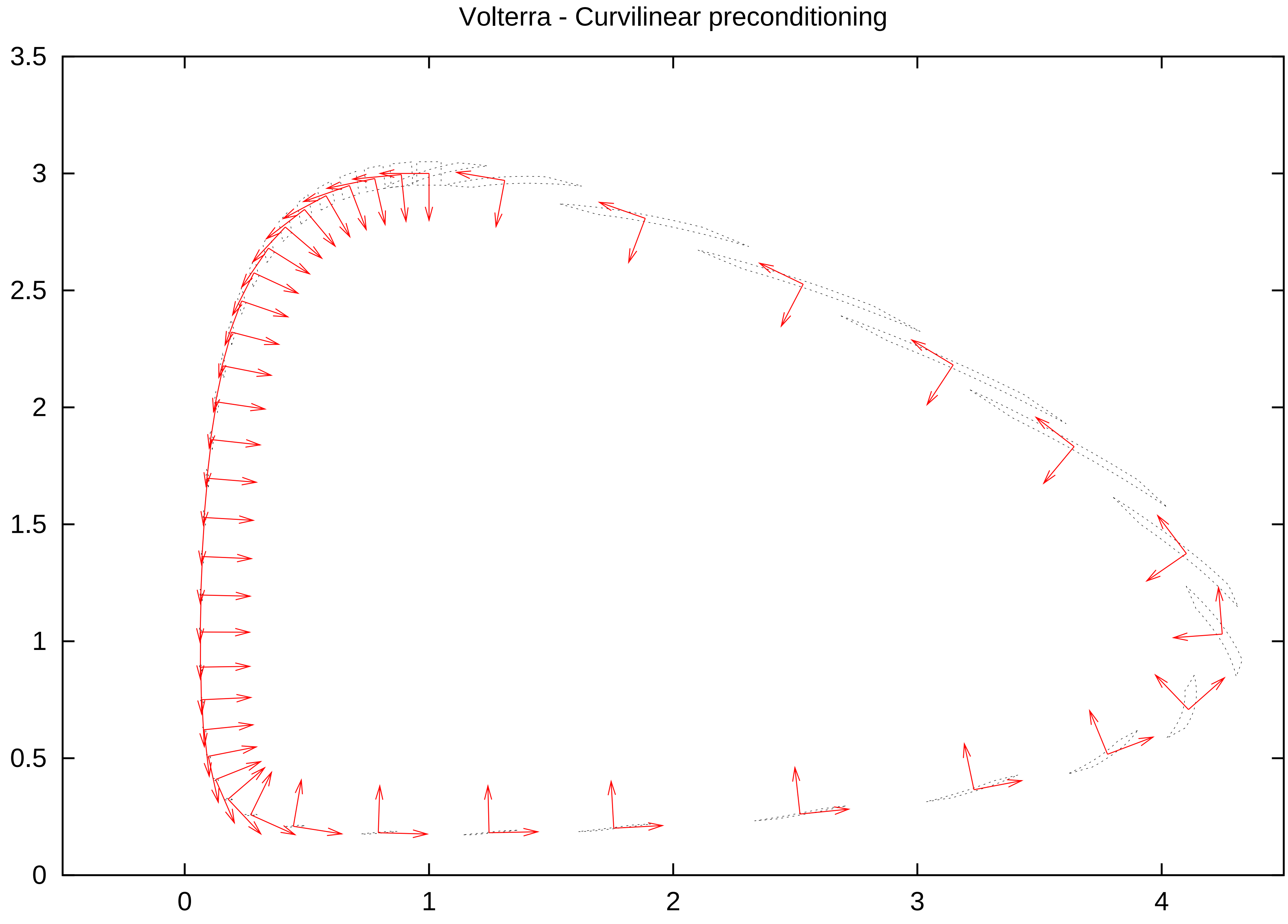
Definition (Curvilinear Preconditioning) Let $x^{(m)} = f(x, x', ...x^{(m-1)}, t)$ be an m-th order ODE in n variables. Let $x_r(t)$ be a solution of the ODE and $x'_r(t)$, ..., $x_r^{(k)}(t)$ its first k time derivatives. Let $\vec{e}_1, ..., \vec{e}_l$ be the l unit vectors not in the span of $x'_r(t)$, ..., $x_r^{(k)}(t)$, sorted by distance from the span. Then we call the Gram-Schmidt orthonormalization of the set $(x'_r(t)$, ..., $x_r^{(k)}(t), \vec{e}_1, ..., \vec{e}_l)$ the curvilinear basis of depth k.

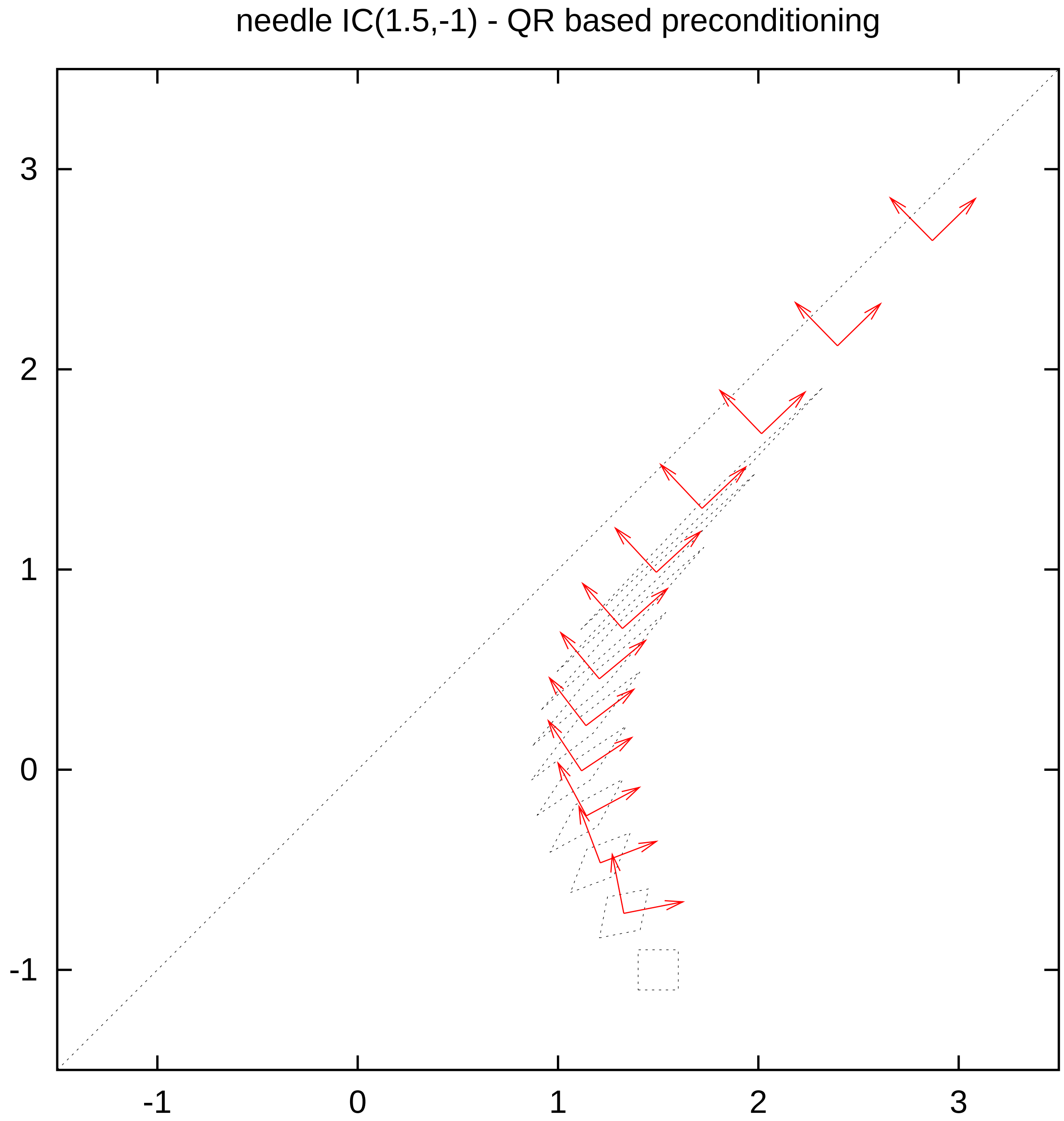
Curvilinear coordinates have long history. Study of solar system, Beam Physics, ...

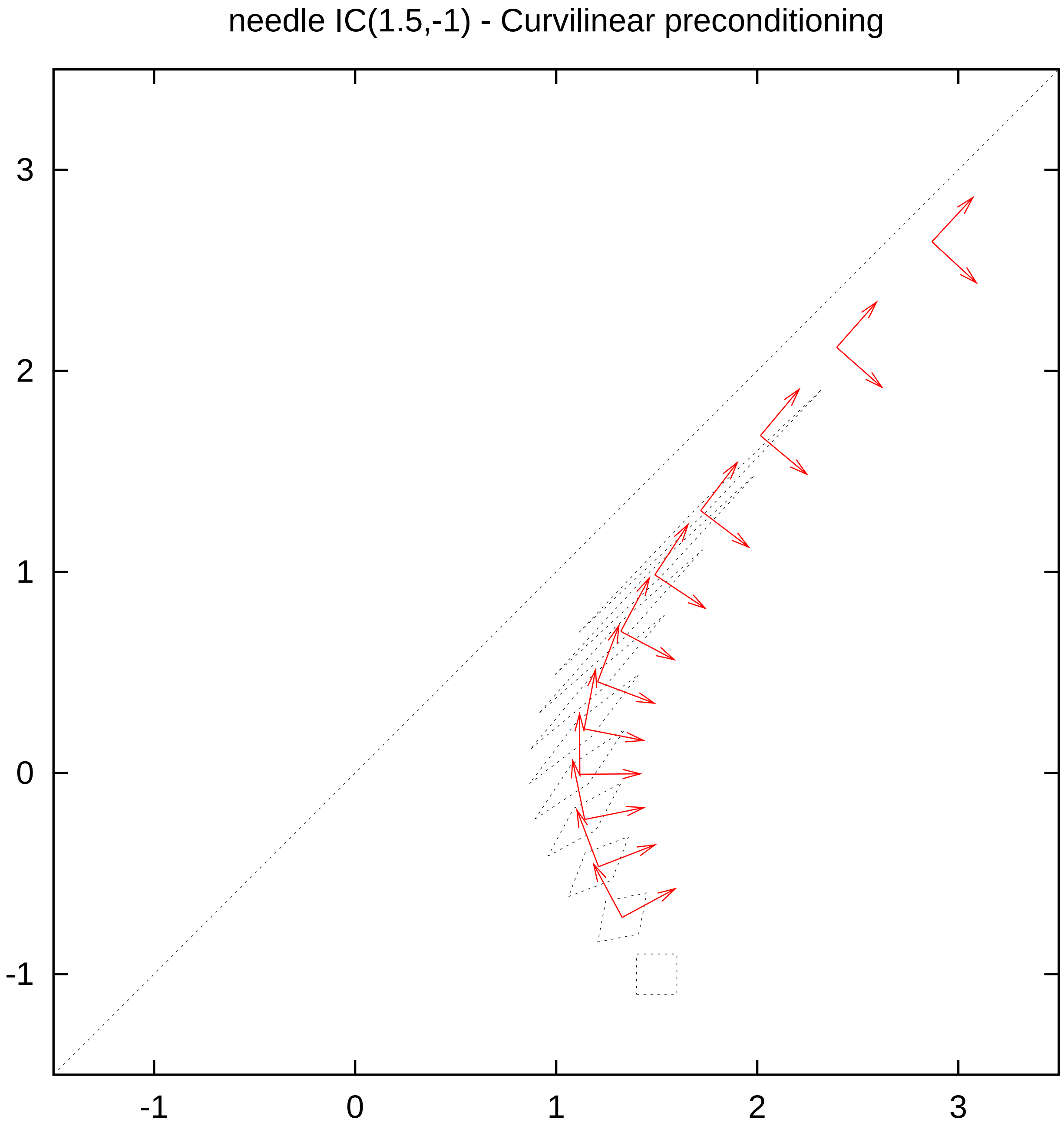
Example (Curvilinear Solar System and Particle Accelerators) In this case, $n = 3$, and one usually chooses $k = 2$. The first basis vector points in the direction of motion of the reference orbit. The second vector is perpendicular to it and points approximately to the sun or the center of the accelerator. The third vector is chosen perpendicular to the plane of the previous two.



Volterra - QR based preconditioning
3.5
3
2.5
2
1.5
1
0.5
0
0
1
2
3
4


Volterra - Curvilinear preconditioning
3.5
3
2.5
2
1.5
1
0.5
0
0
1
2
3
4


needle IC(1.5,-1) - QR based preconditioning
3
2
1
0
-1
-1
0
1
2
3



needle IC(1.5,-1) - Curvilinear preconditioning
3
2
1
0
-1
-1
0
1
2
3

Random Matrices - Discrete

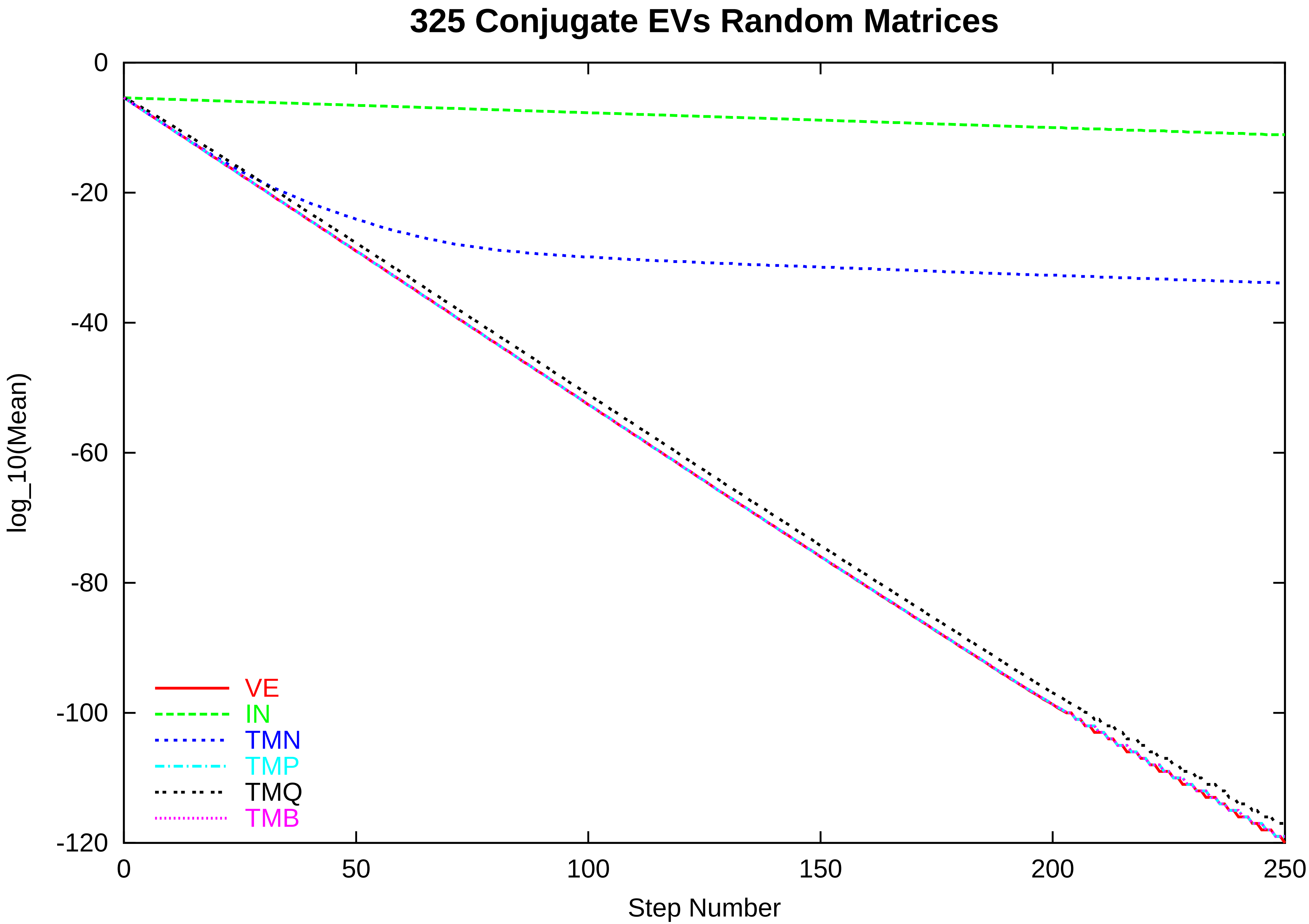
Select 1000 twodimensional random matrices with coefficients in $[-1, 1]$.Sort according to eigenvalues into seven sub-cases.

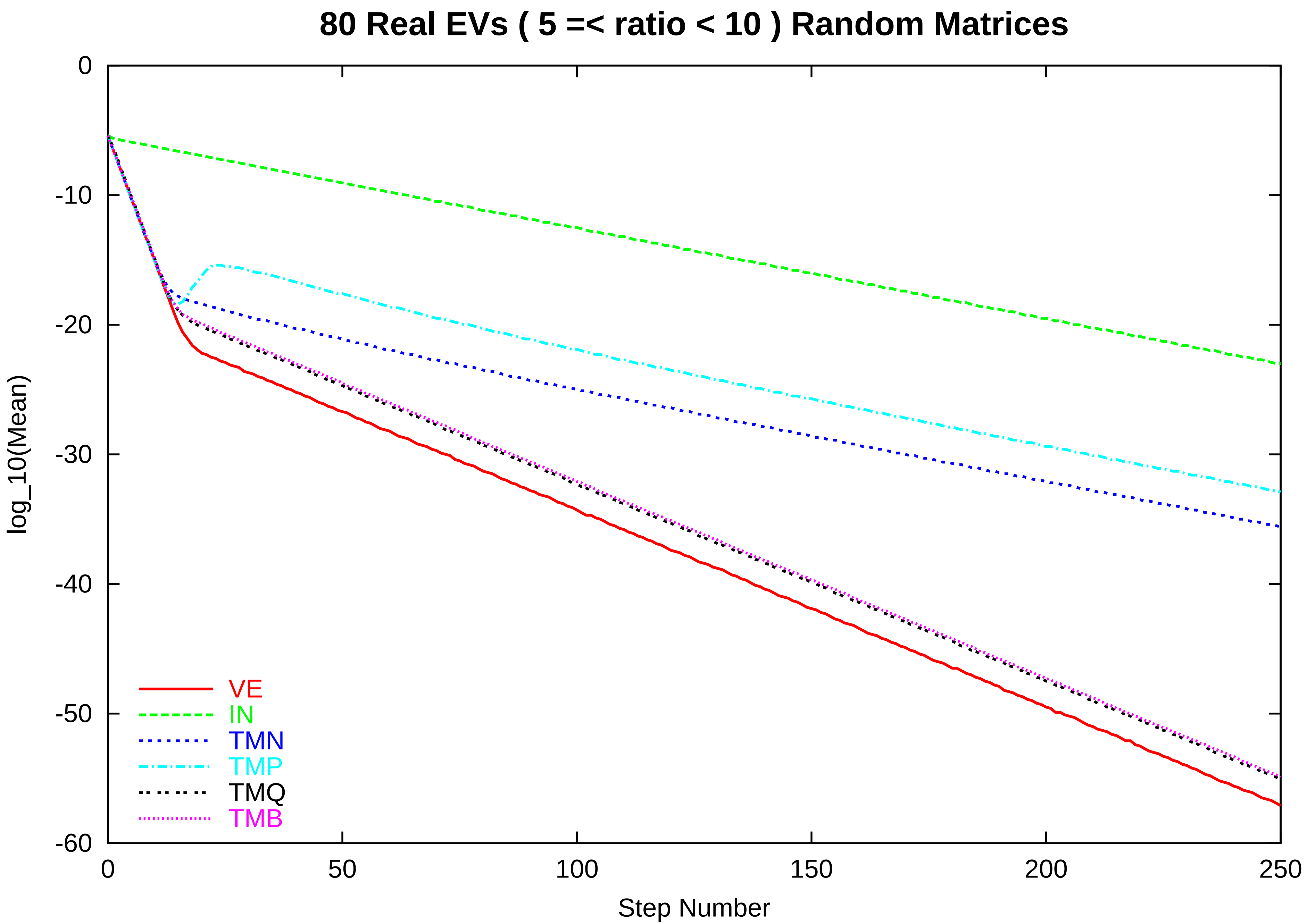
Perform iteration in the following ways:

- Naive Interval
- Naive Taylormodel
- Parallelepiped-preconditioned Taylormodel
- QR-preconditioned Taylormodel
- Blunted preconditioned TM, various blunting factors
- Set of four floating point corner points for volume estimation

Perform the following tasks:

- Iterations through matrix
- Sets of iterations through matrix and its inverse


325 Conjugate EVs Random Matrices
log_10(Mean)
Step Number
0
-20
-40
-60
-80
-100
-120
0
50
100
150
200
250
VE
IN
TMN
TMP
TMQ
TMB



80 Real EVs (5 =< ratio < 10) Random Matrices
log_10(Mean)
Step Number
0
-10
-20
-30
-40
-50
-60
0
50
100
150
200
250
VE
IN
TMN
TMP
TMQ
TMB

Random Matrices - Discrete

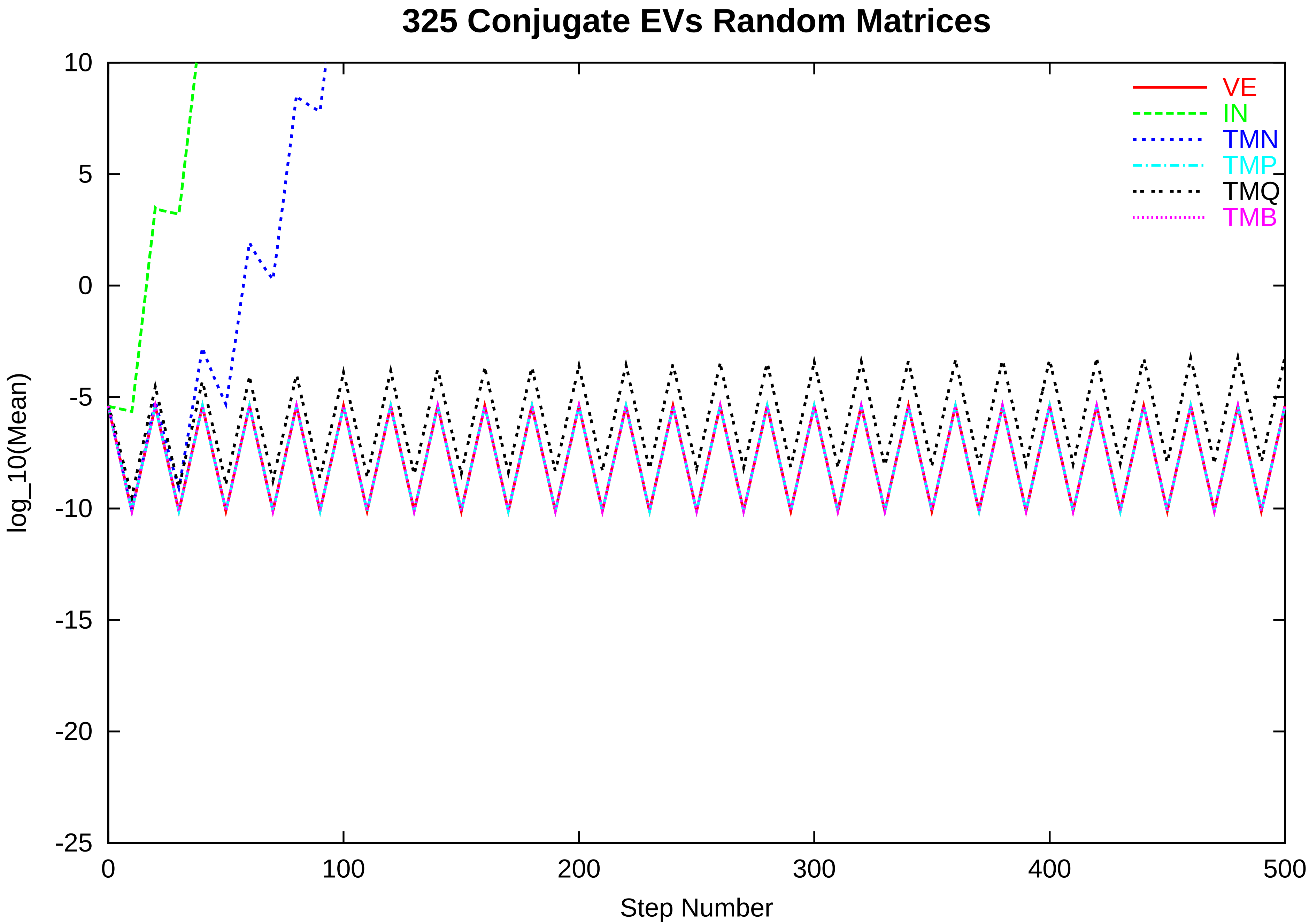
Select 1000 twodimensional random matrices with coefficients in $[-1, 1]$.Sort according to eigenvalues into seven sub-cases.

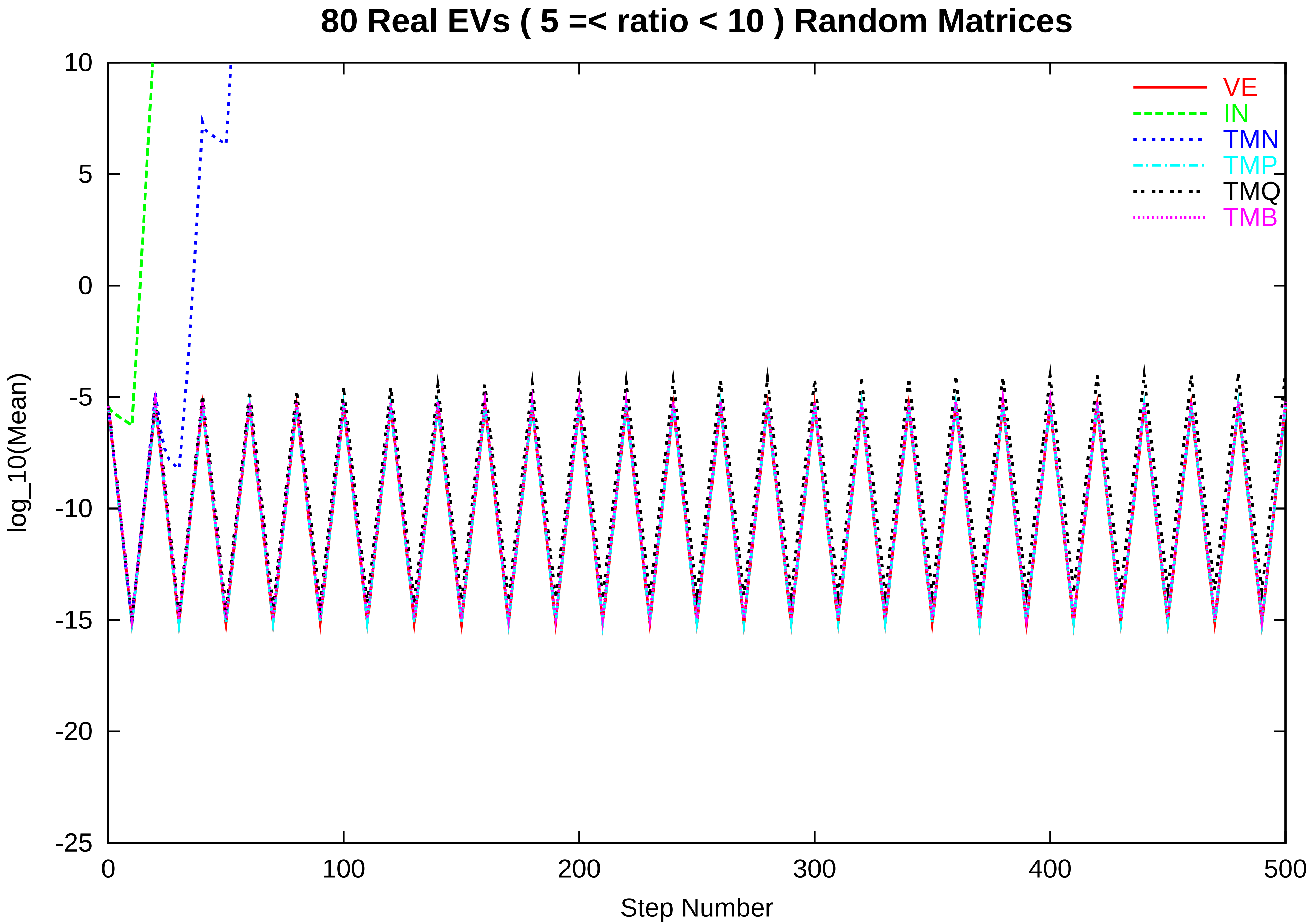
Perform iteration in the following ways:

- Naive Interval
- Naive Taylormodel
- Parallelepiped-preconditioned Taylormodel
- QR-preconditioned Taylormodel
- Blunted preconditioned TM, various blunting factors
- Set of four floating point corner points for volume estimation

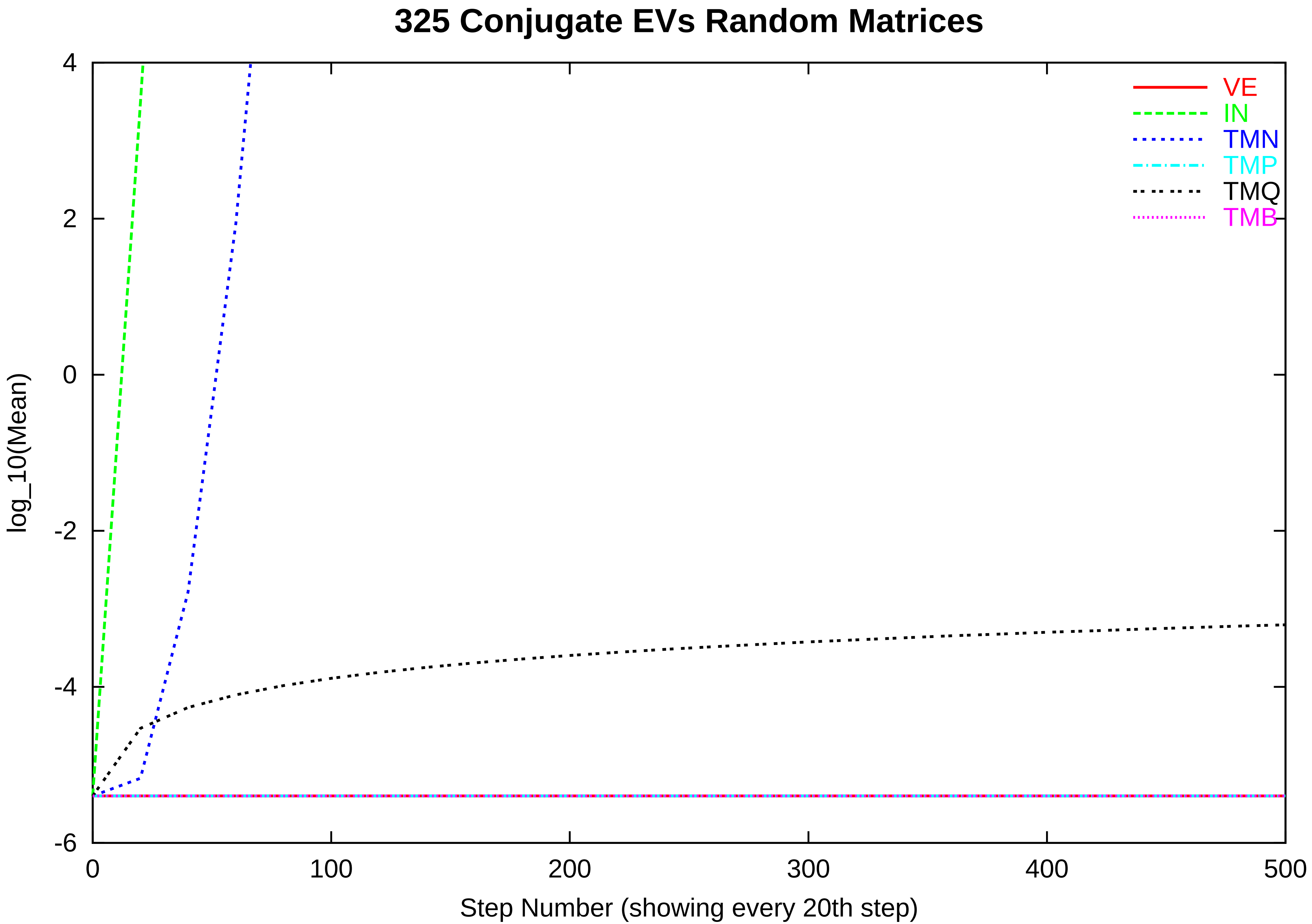
Perform the following tasks:

- Iterations through matrix
- Sets of iterations through matrix and its inverse

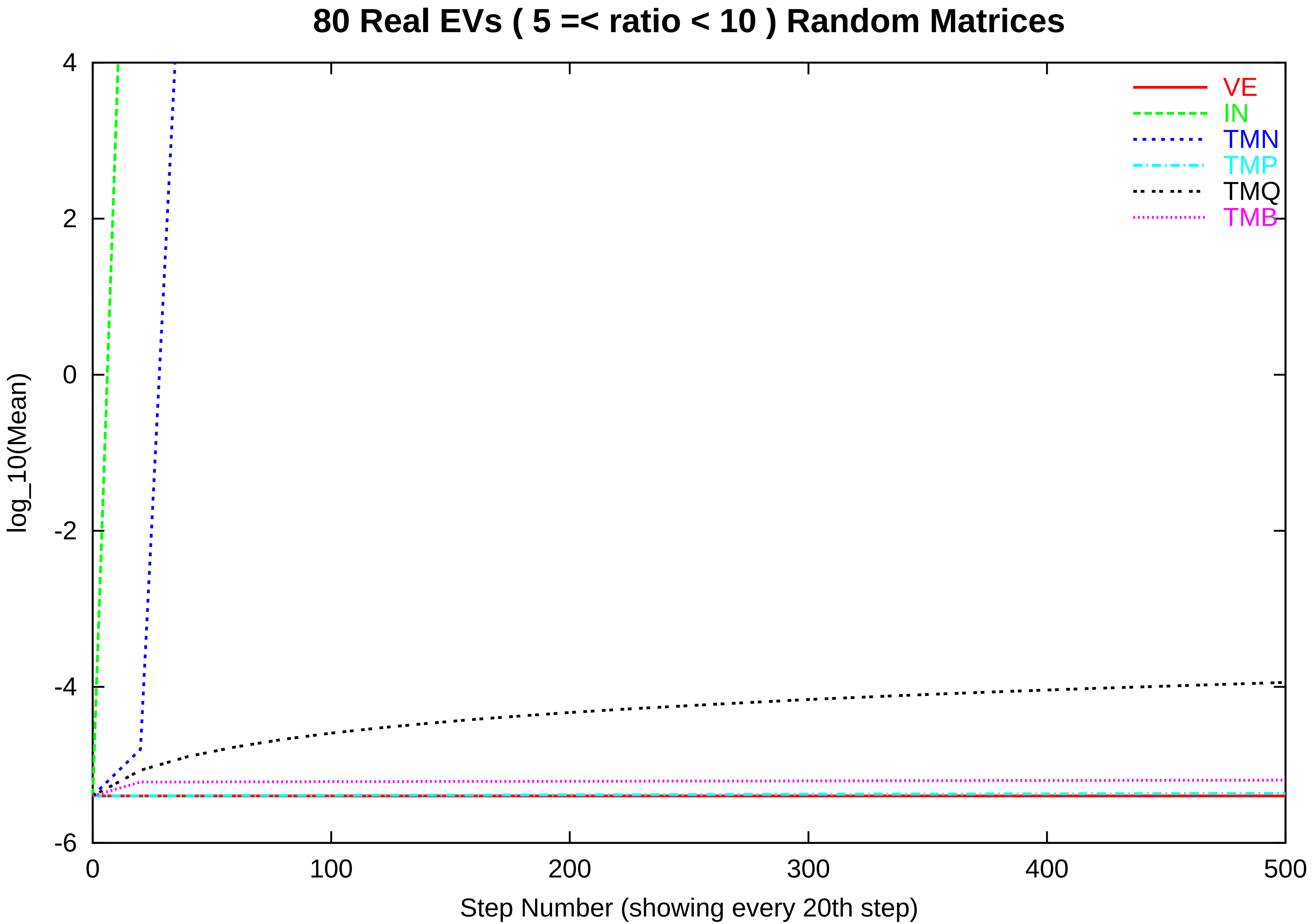

325 Conjugate EVs Random Matrices
log_10(Mean)
Step Number
VE
IN
TMN
TMP
TMQ
TMB
10
5
0
-5
-10
-15
-20
-25
0
100
200
300
400
500



80 Real EVs (5 =< ratio < 10) Random Matrices
log_10(Mean)
Step Number
10
5
0
-5
-10
-15
-20
-25
0
100
200
300
400
500
VE
IN
TMN
TMP
TMQ
TMB



325 Conjugate EVs Random Matrices
log_10(Mean)
Step Number (showing every 20th step)
4
2
0
-2
-4
-6
0
100
200
300
400
500
VE
IN
TMN
TMP
TMQ
TMB



80 Real EVs (5 =< ratio < 10) Random Matrices
VE
IN
TMN
TMP
TMQ
TMB
log_10(Mean)
Step Number (showing every 20th step)
4
2
0
-2
-4
-6
0
100
200
300
400
500

The Henon Map

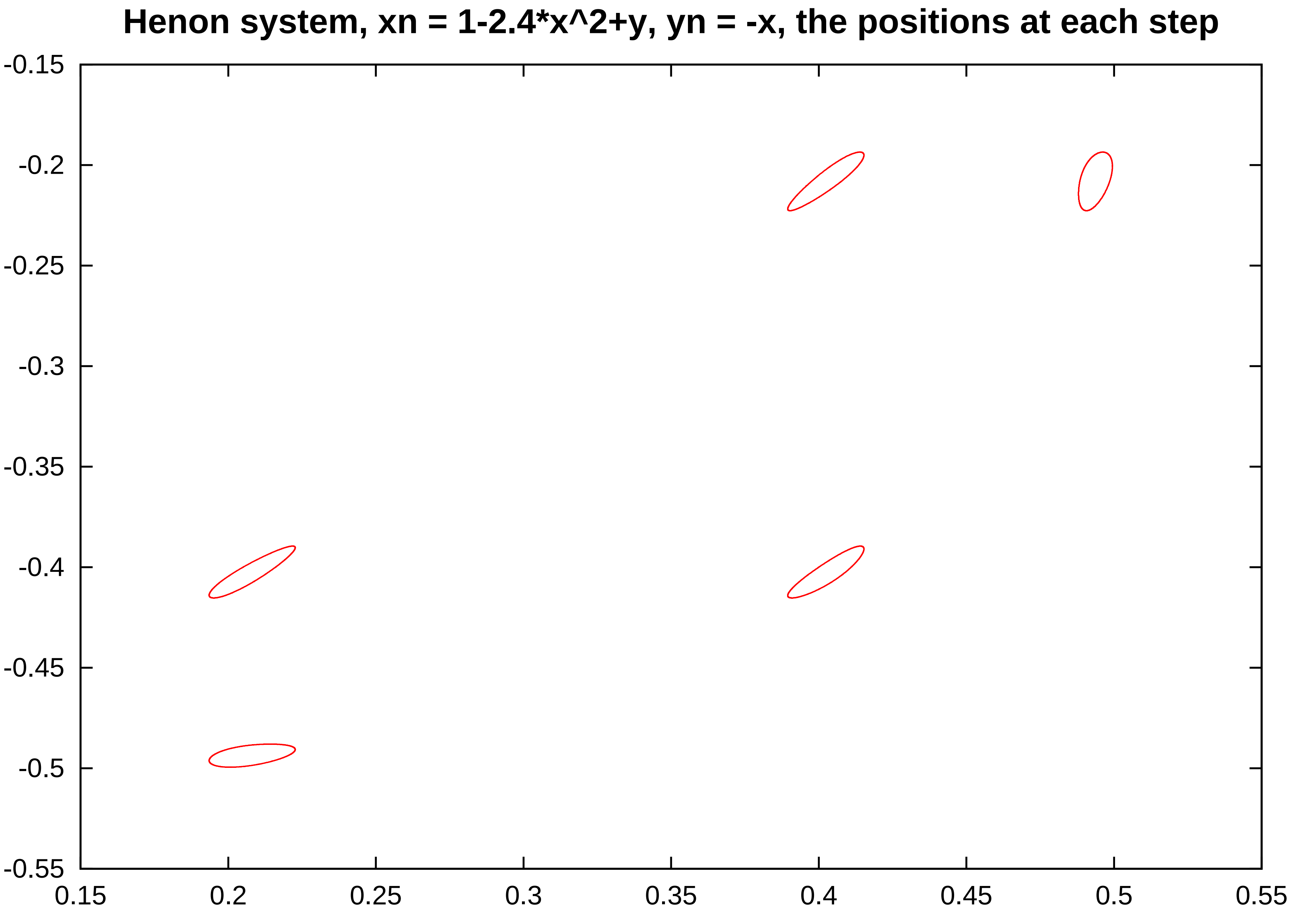
Henon Map: frequently used elementary example that exhibits many of the well-known effects of nonlinear dynamics, including chaos, periodic fixed points, islands and symplectic motion. The dynamics is two-dimensional, and given by

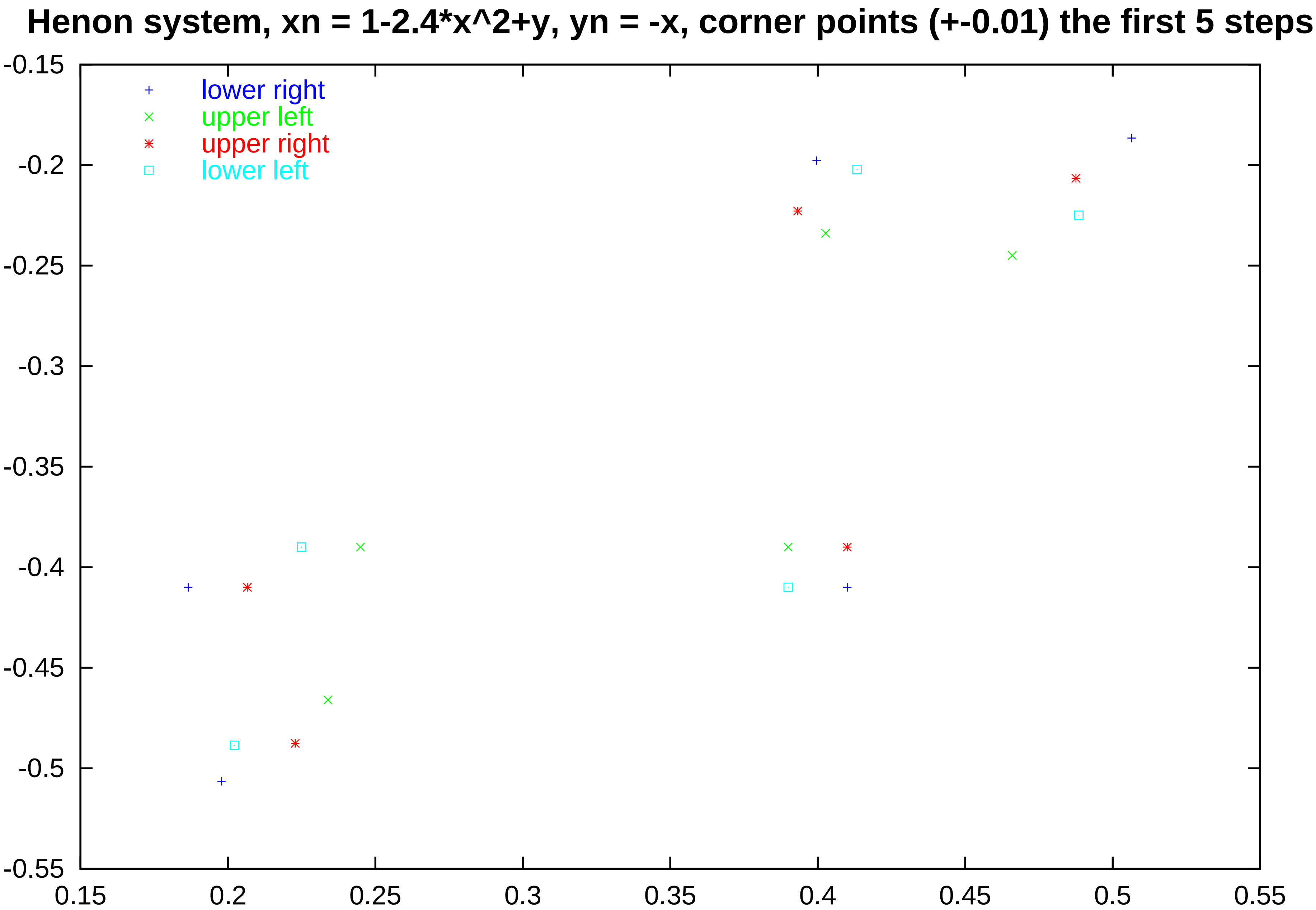
$$\begin{aligned} x_{n+1} &= 1 - \alpha x_n^2 + y_n \\ y_{n+1} &= \beta x_n. \end{aligned}$$

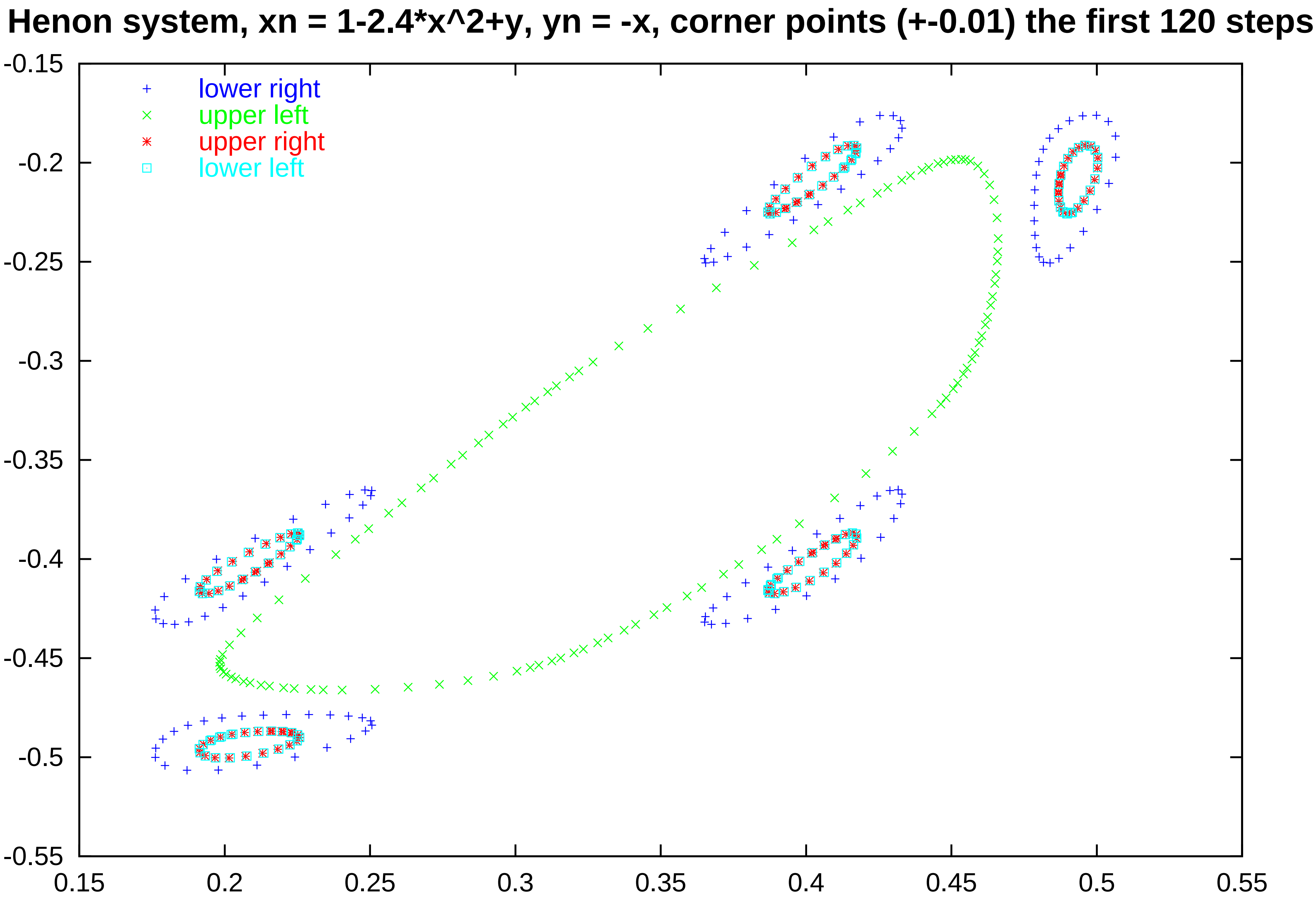
It can easily be seen that the motion is area preserving for $|\beta| = 1$.We consider

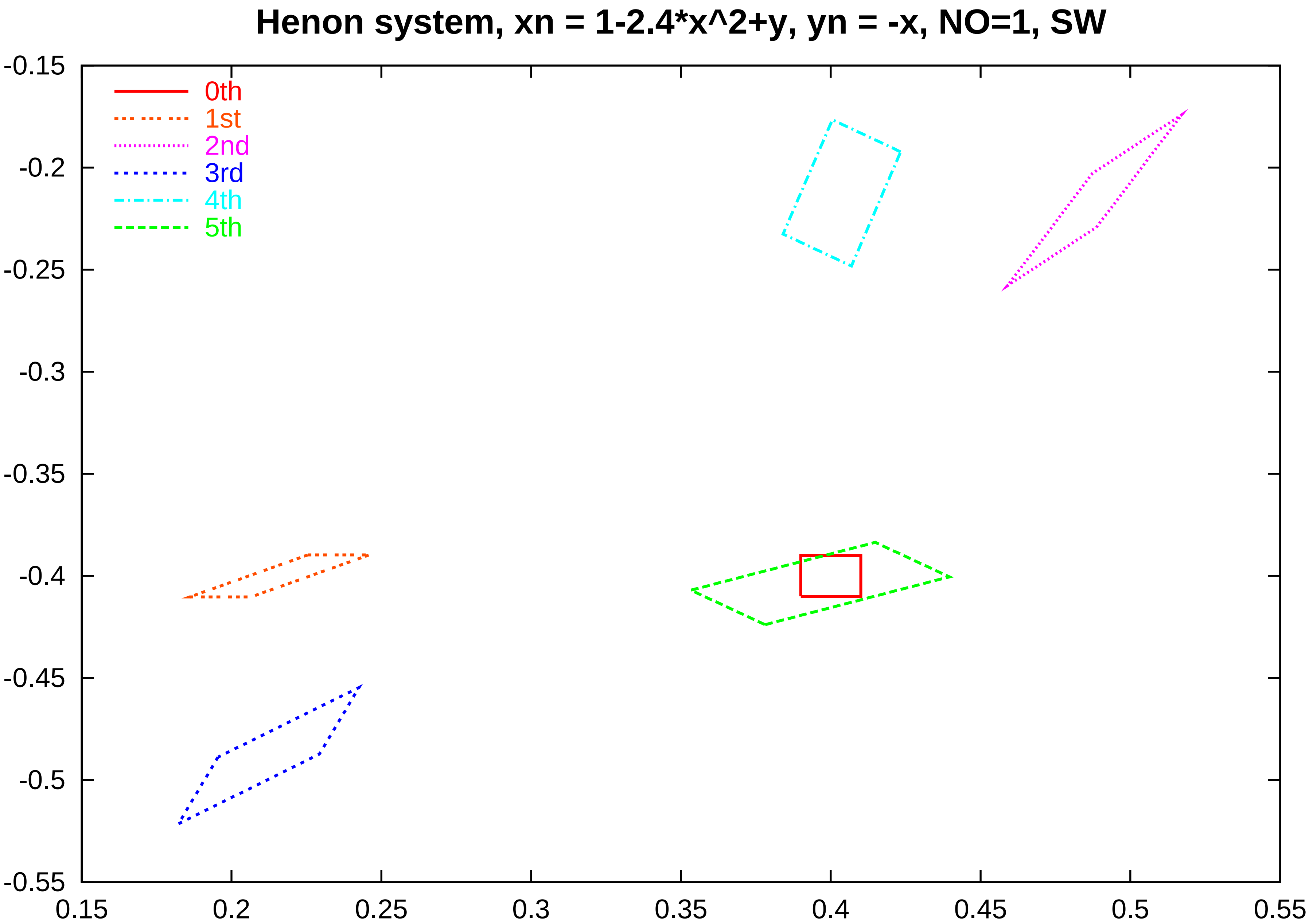
$$\alpha = 2.4 \text{ and } \beta = -1,$$

and concentrate on initial boxes of the from $(x_0, y_0) \in (0.4, \, -0.4) + [-d, d]^2$.

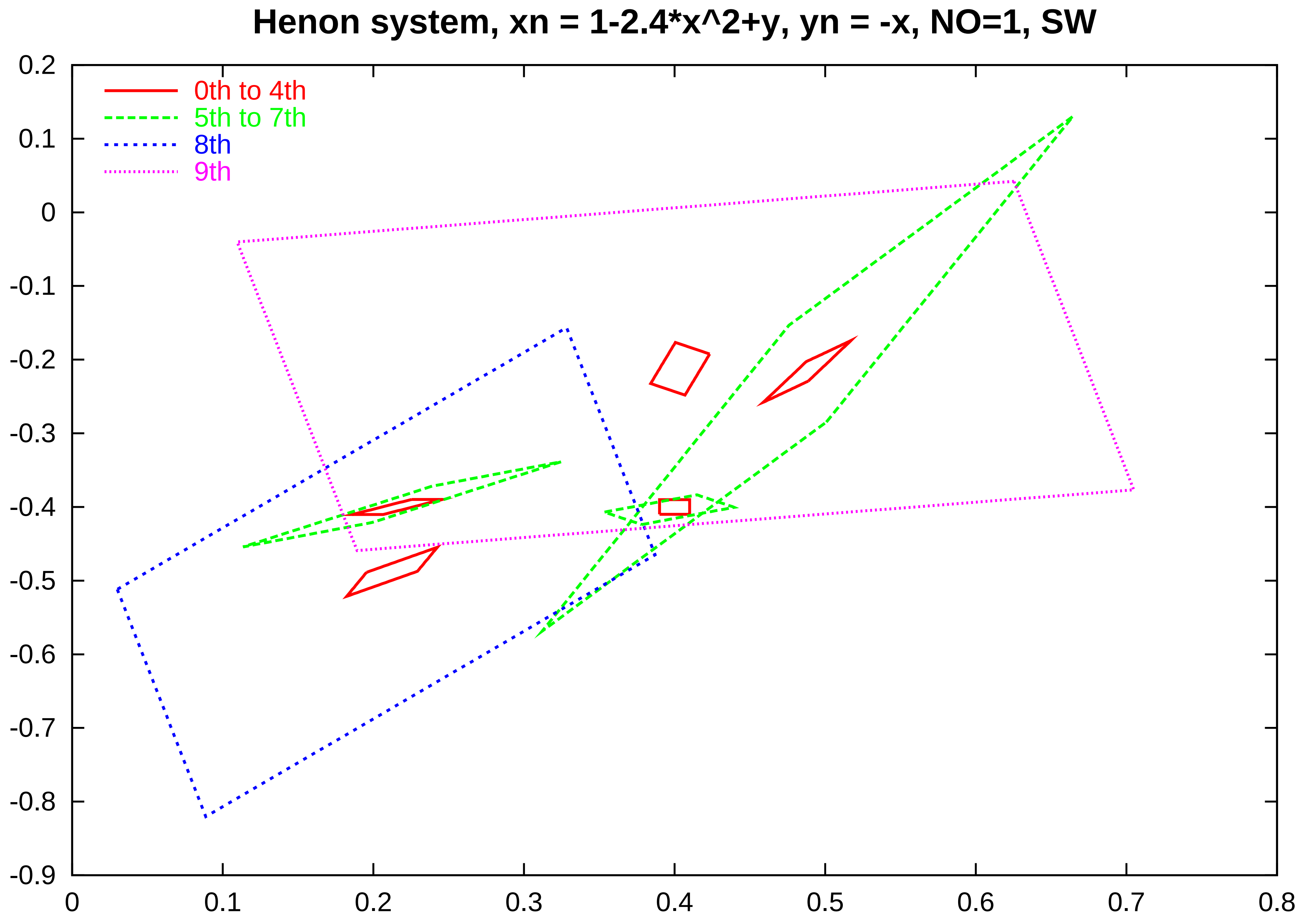

Henon system, xn = 1-2.4*x^2+y, yn = -x, the positions at each step
-0.15
-0.2
-0.25
-0.3
-0.35
-0.4
-0.45
-0.5
-0.55
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
0.55

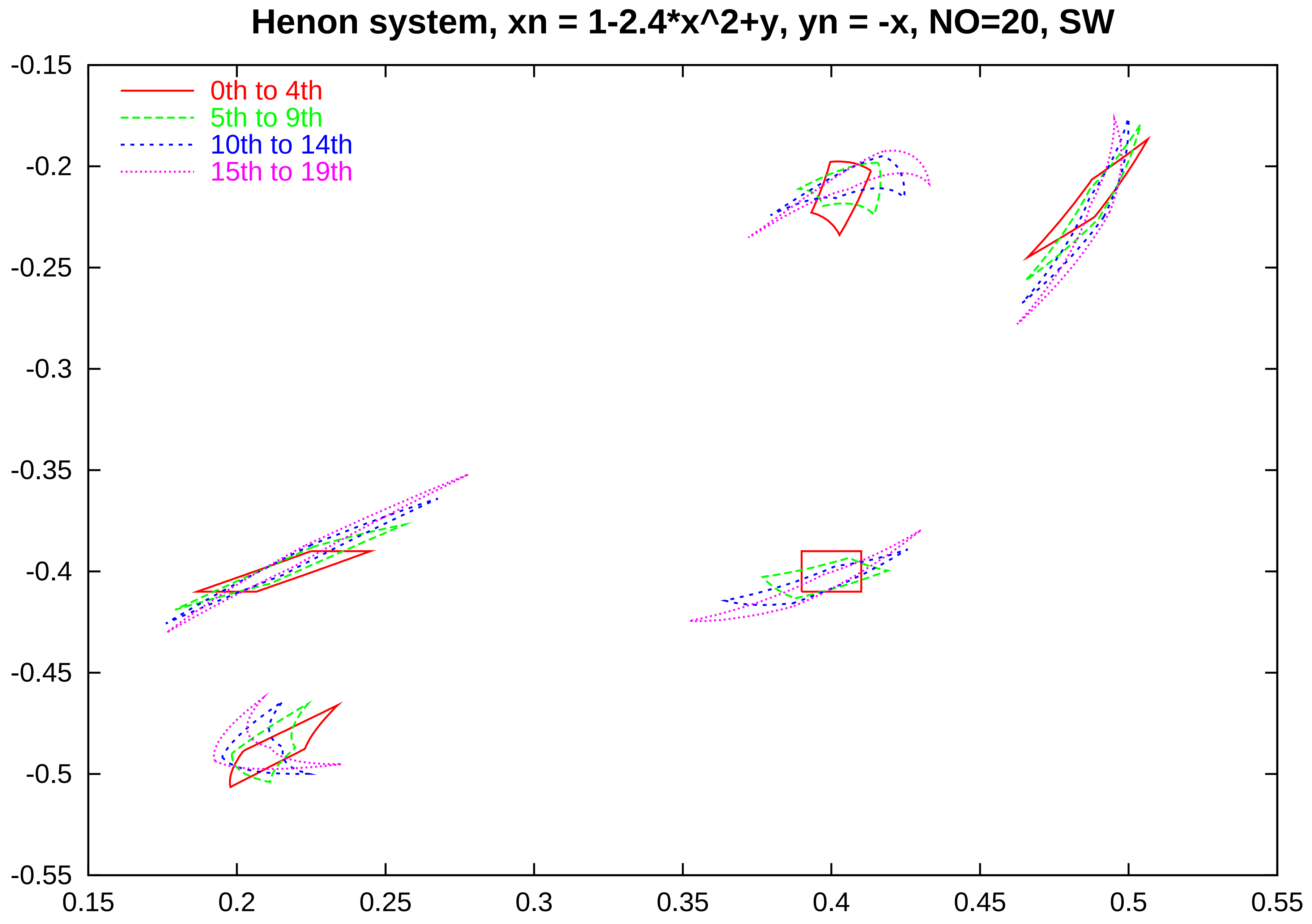

Henon system, xn = 1-2.4*x^2+y, yn = -x, corner points (+-0.01) the first 5 steps
lower right
upper left
upper right
lower left
-0.15
-0.2
-0.25
-0.3
-0.35
-0.4
-0.45
-0.5
-0.55
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
0.55


Henon system, xn = 1-2.4*x^2+y, yn = -x, corner points (+-0.01) the first 120 steps
lower right
upper left
upper right
lower left
-0.15
-0.2
-0.25
-0.3
-0.35
-0.4
-0.45
-0.5
-0.55
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
0.55

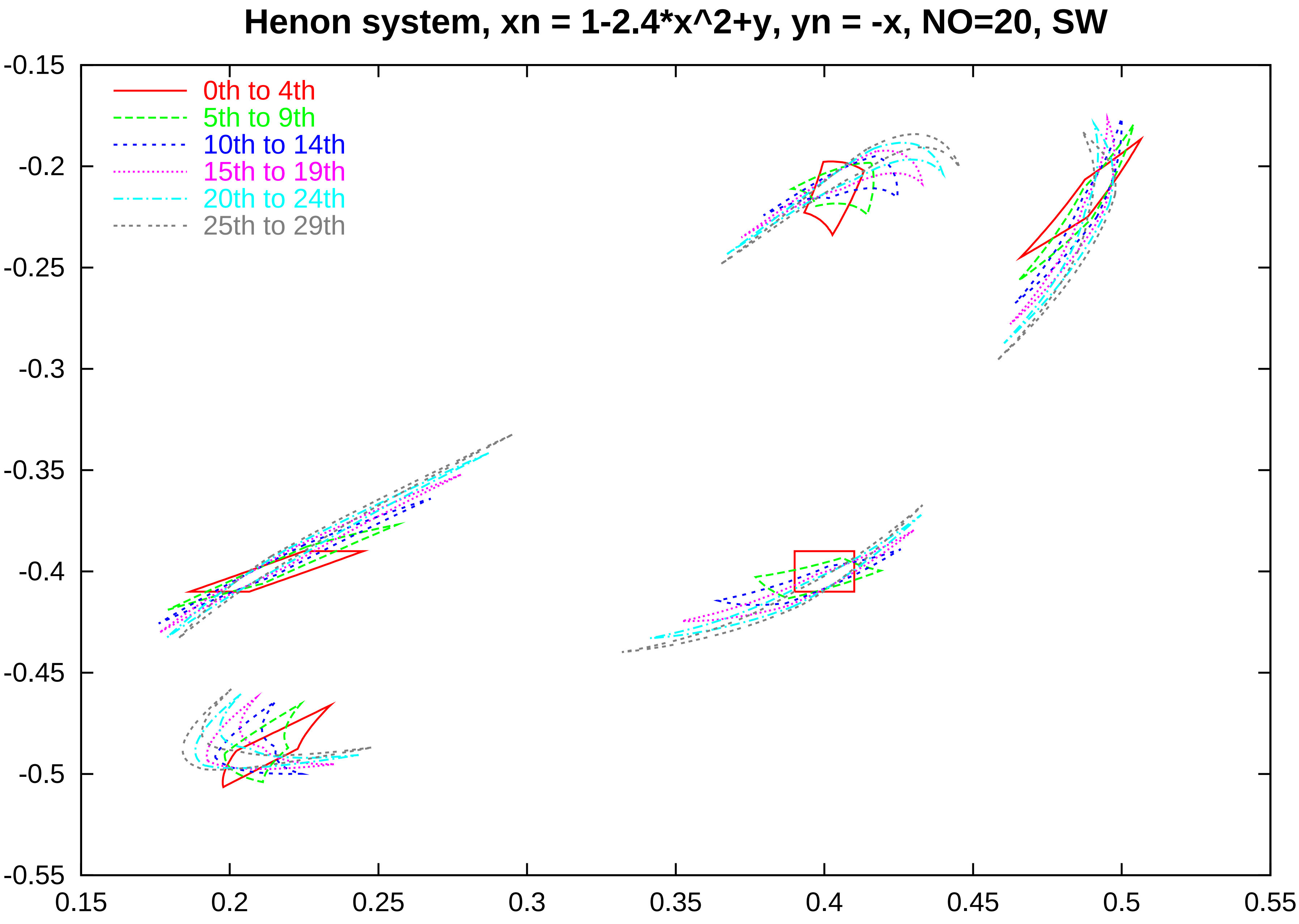


Henon system, xn = 1-2.4*x^2+y, yn = -x, NO=1, SW
0th
1st
2nd
3rd
4th
5th
-0.15
-0.2
-0.25
-0.3
-0.35
-0.4
-0.45
-0.5
-0.55
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
0.55

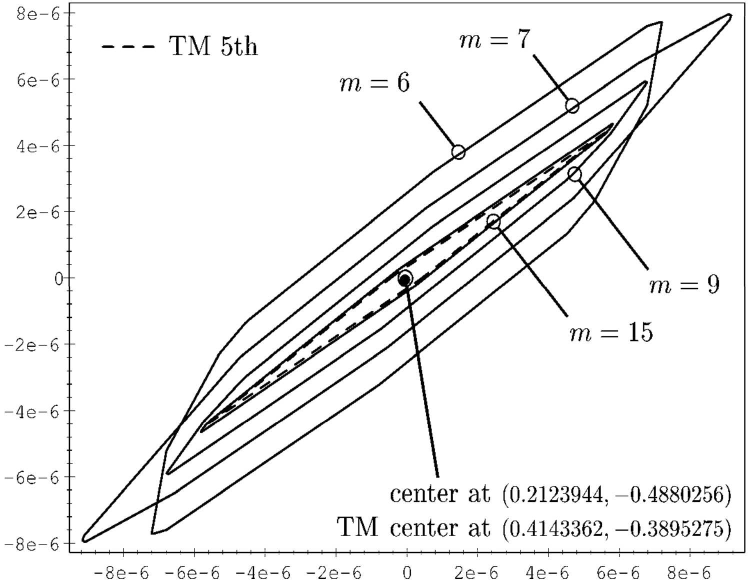

Henon system, xn = 1-2.4*x^2+y, yn = -x, NO=1, SW
0th to 4th
5th to 7th
8th
9th
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
-0.7
-0.8
-0.9
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8



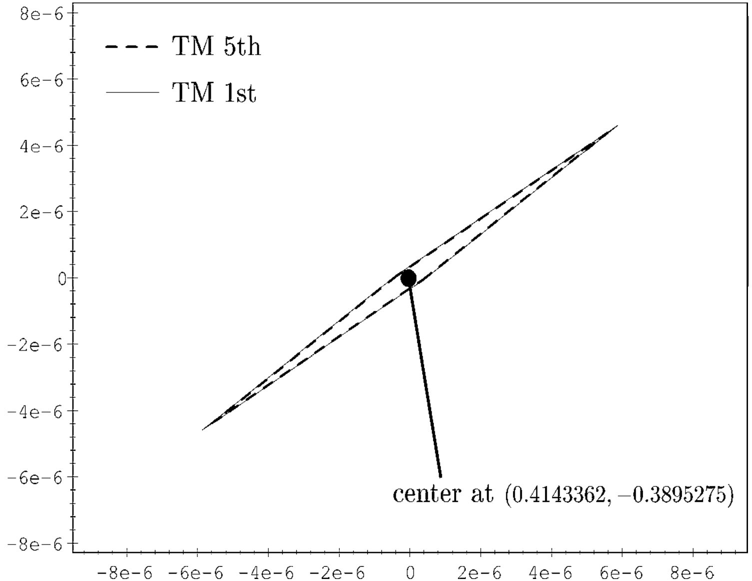
Henon system, xn = 1-2.4*x^2+y, yn = -x, NO=20, SW
0th to 4th
5th to 9th
10th to 14th
15th to 19th
-0.15
-0.2
-0.25
-0.3
-0.35
-0.4
-0.45
-0.5
-0.55
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
0.55



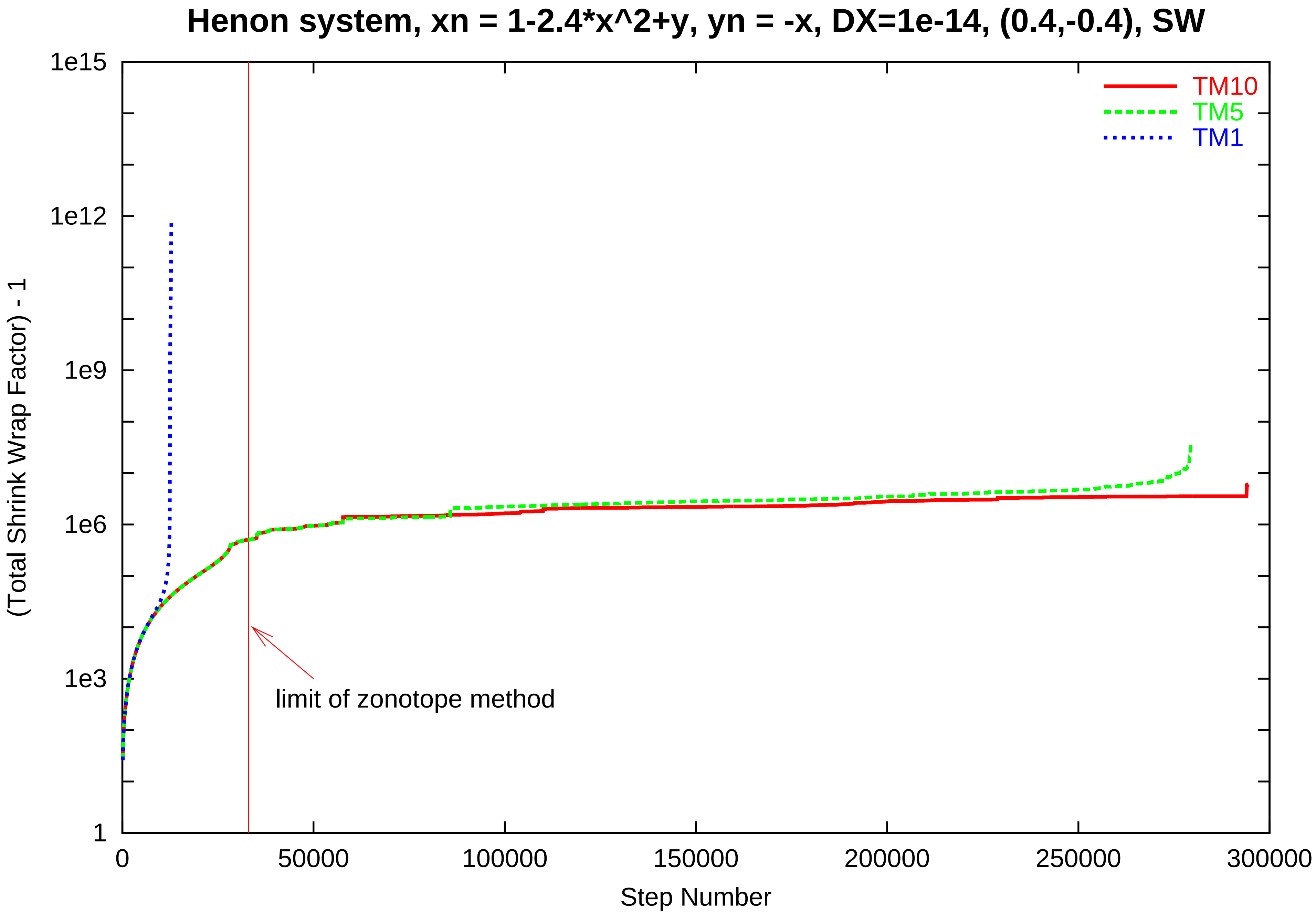
Henon system, xn = 1-2.4*x^2+y, yn = -x, NO=20, SW
0th to 4th
5th to 9th
10th to 14th
15th to 19th
20th to 24th
25th to 29th
-0.15
-0.2
-0.25
-0.3
-0.35
-0.4
-0.45
-0.5
-0.55
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
0.55



TM 5th
$m = 7$
$m = 6$
$m = 9$
$m = 15$
center at $(0.2123944, -0.4880256)$
TM center at $(0.4143362, -0.3895275)$



TM 5th
TM 1st
center at $(0.4143362, -0.3895275)$



Henon system, xn = 1-2.4*x^2+y, yn = -x, DX=1e-14, (0.4,-0.4), SW
1e15
1e12
1e9
1e6
1e3
1
0
50000
100000
150000
200000
250000
300000
Step Number
(Total Shrink Wrap Factor) - 1
TM10
TM5
TM1
limit of zonotope method

A Muon Cooling Ring

Example from Beam Physics: Simple model of muon cooling ring, using curvilinear preconditioning.

Simultaneous damping via matter, and azimuthal accelerations. Equations of motion:

$$\begin{aligned}
\dot{x} &= p_x \\
\dot{y} &= p_y \\
\dot{p}_x &= p_y - \frac{\alpha}{\sqrt{p_x^2 + p_y^2}} \cdot p_x + \frac{\alpha}{\sqrt{x^2 + y^2}} \cdot y \\
\dot{p}_y &= -p_x - \frac{\alpha}{\sqrt{p_x^2 + p_y^2}} \cdot p_y - \frac{\alpha}{\sqrt{x^2 + y^2}} \cdot x
\end{aligned}$$

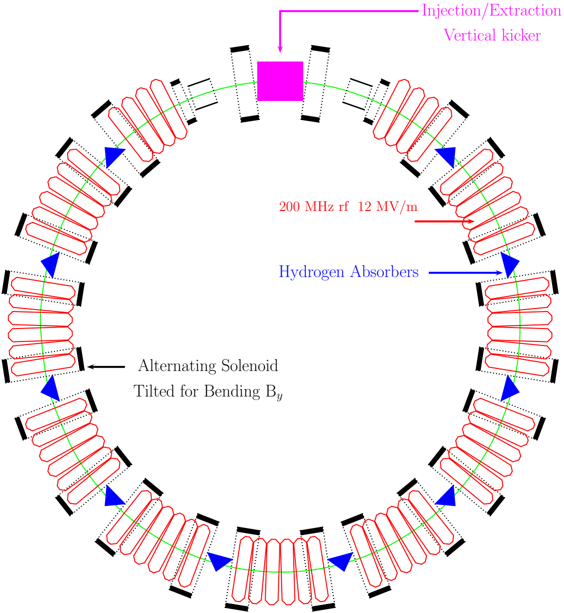
Has invariant solution

$$(x, y, p_x, p_y)_I(t) = (\cos t, -\sin t, -\sin t, -\cos t),$$

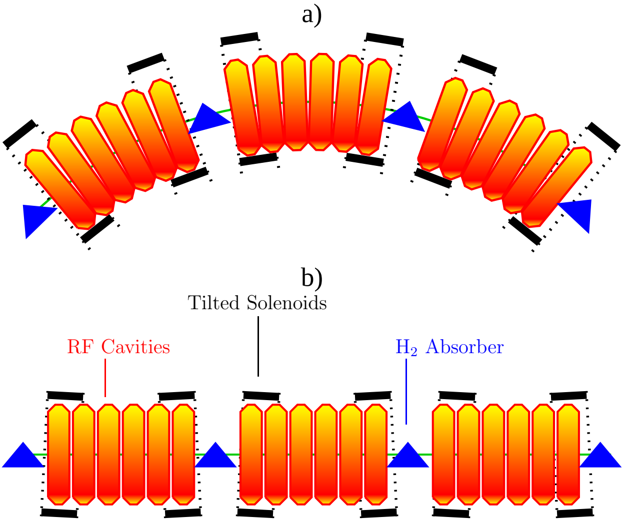
ODE asymptotically approach circular motion of the form

$$(x, y, p_x, p_y)_a(t) = (\cos(t-\phi), -\sin(t-\phi), -\sin(t-\phi), -\cos(t-\phi)),$$

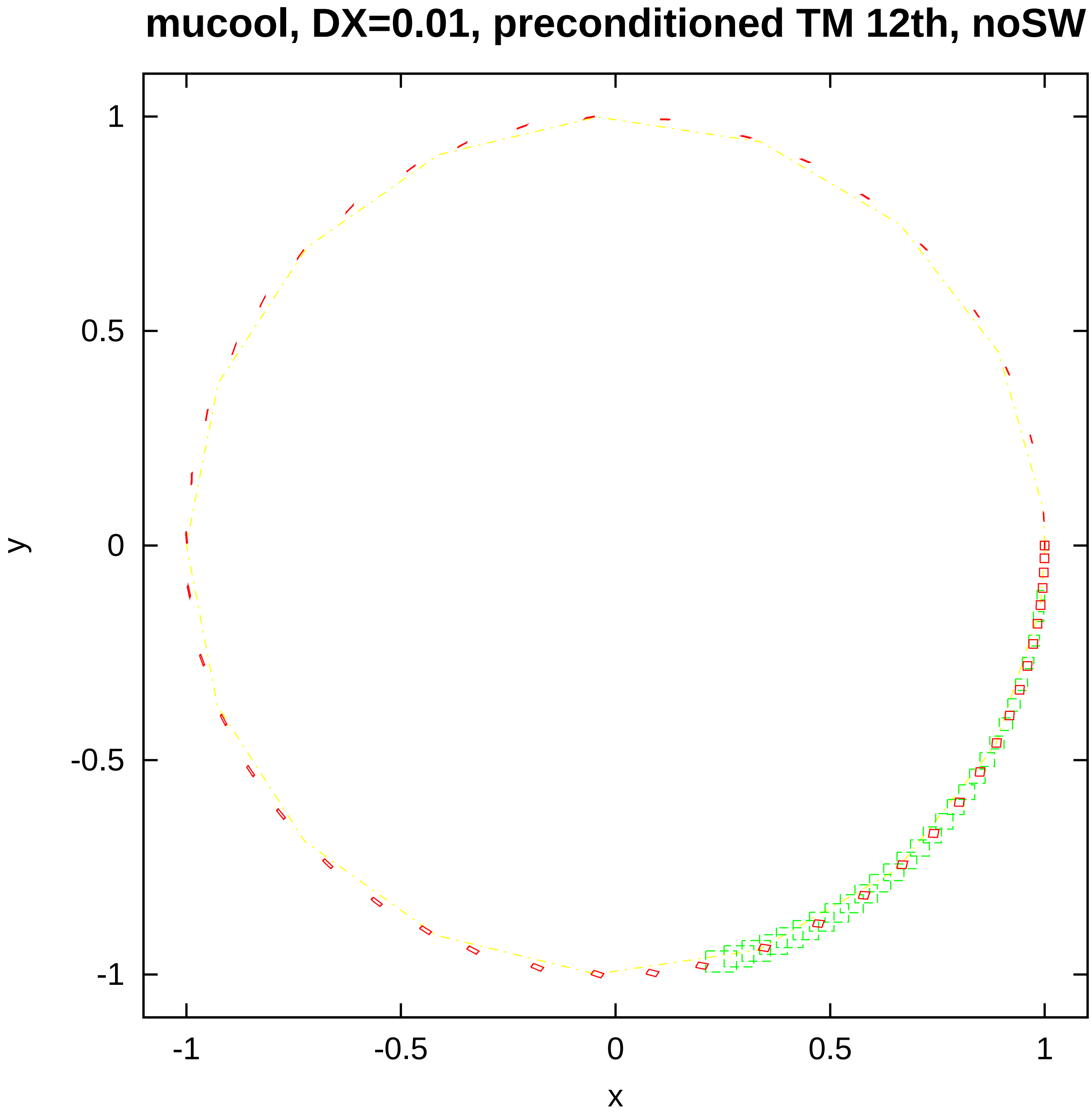
where ϕ is a characteristic angle for each particle.



Injection/Extraction
Vertical kicker
200 MHz rf 12 MV/m
Hydrogen Absorbers
Alternating Solenoid
Tilted for Bending B_y



a)
b)
Tilted Solenoids
RF Cavities
H_2 Absorber



mucool, DX=0.01, preconditioned TM 12th, noSW
1
0.5
0
-0.5
-1
y
-1
-0.5
0
0.5
1
x

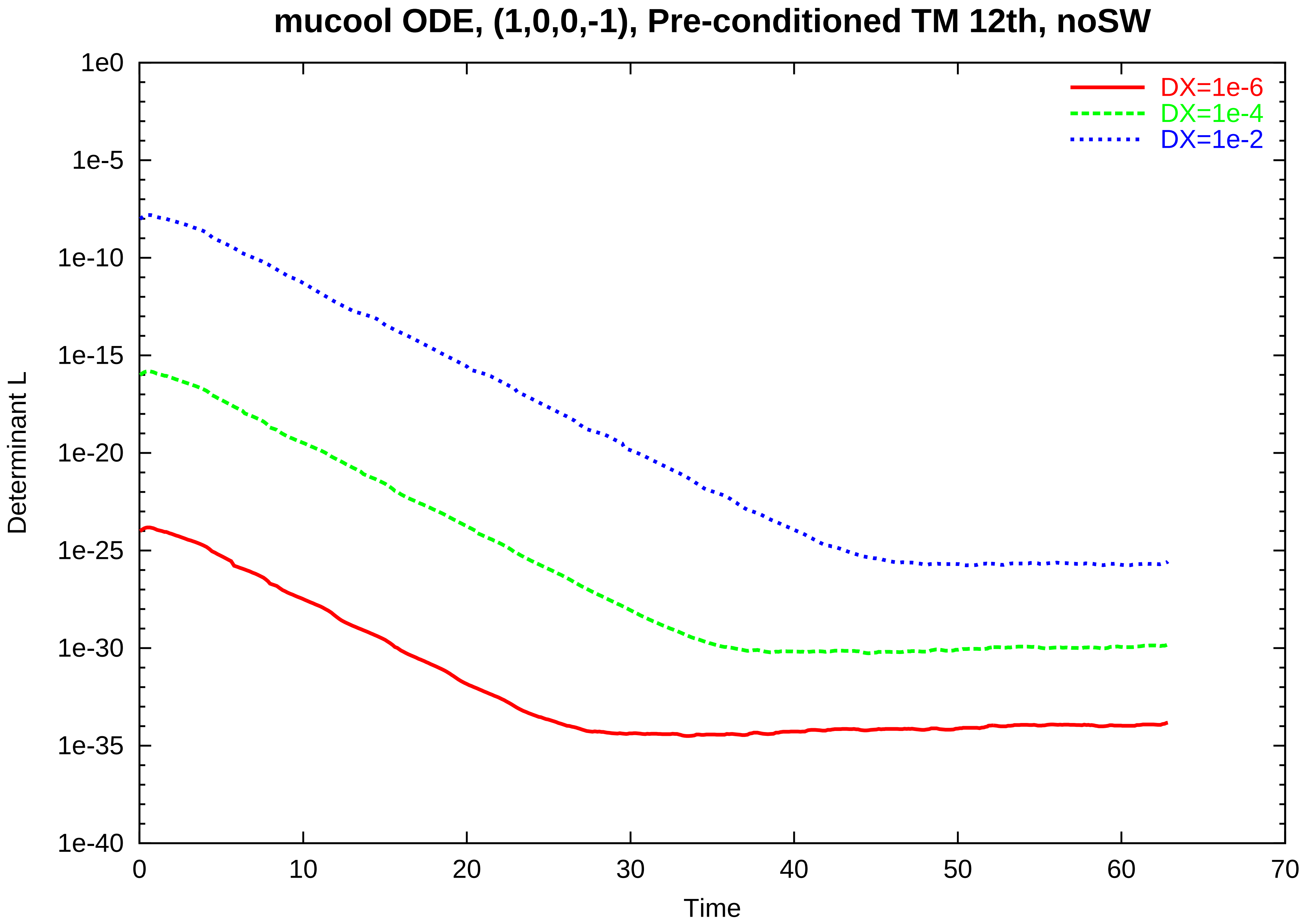
A Muon Cooling Ring - Results

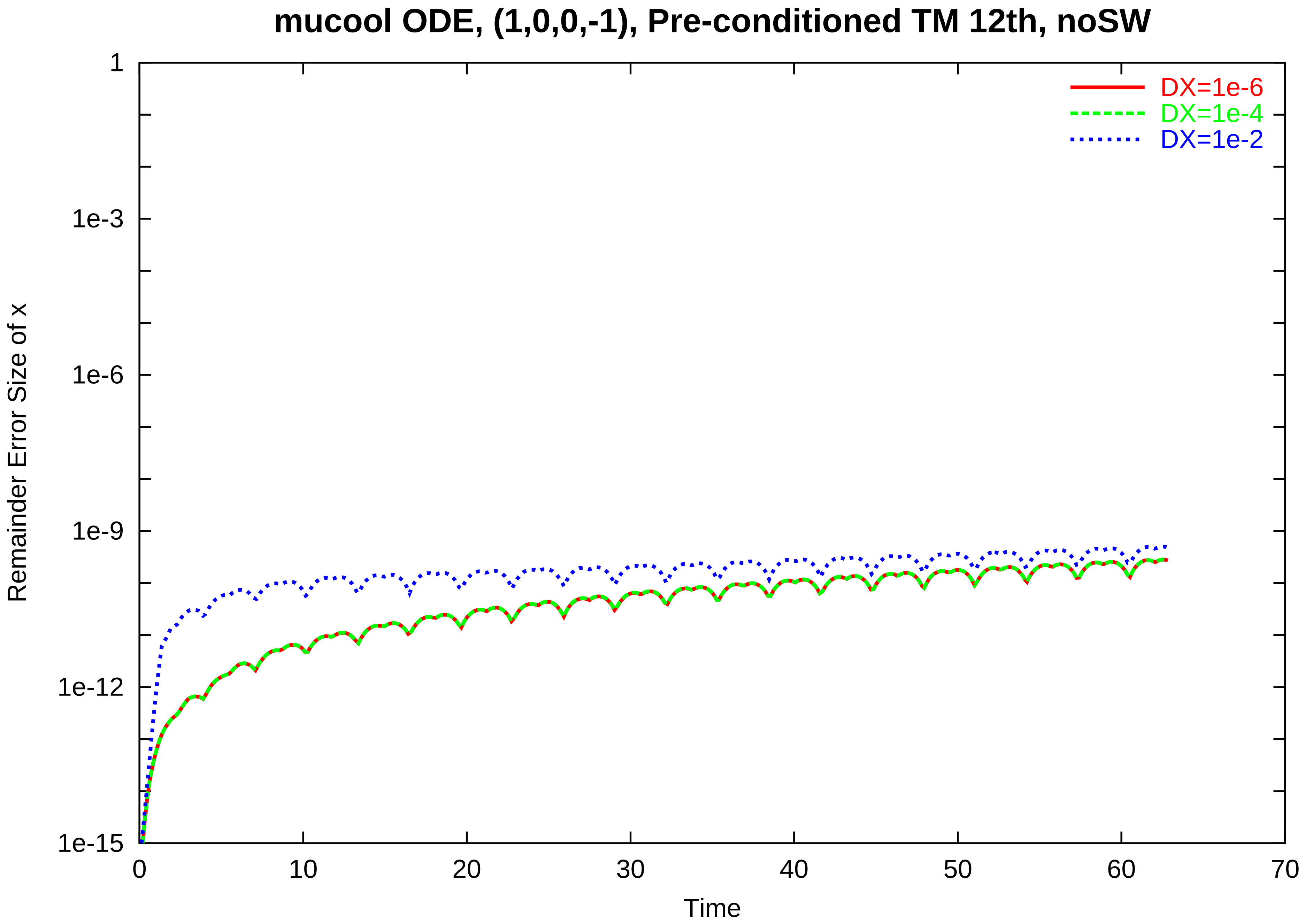
1. Need to treat a large box of $[-10^{-2}, 10^{-2}]^4$
2. Because of damping action towards the invariant limit cycle, the linear part of the motion is more and more ill-conditioned.

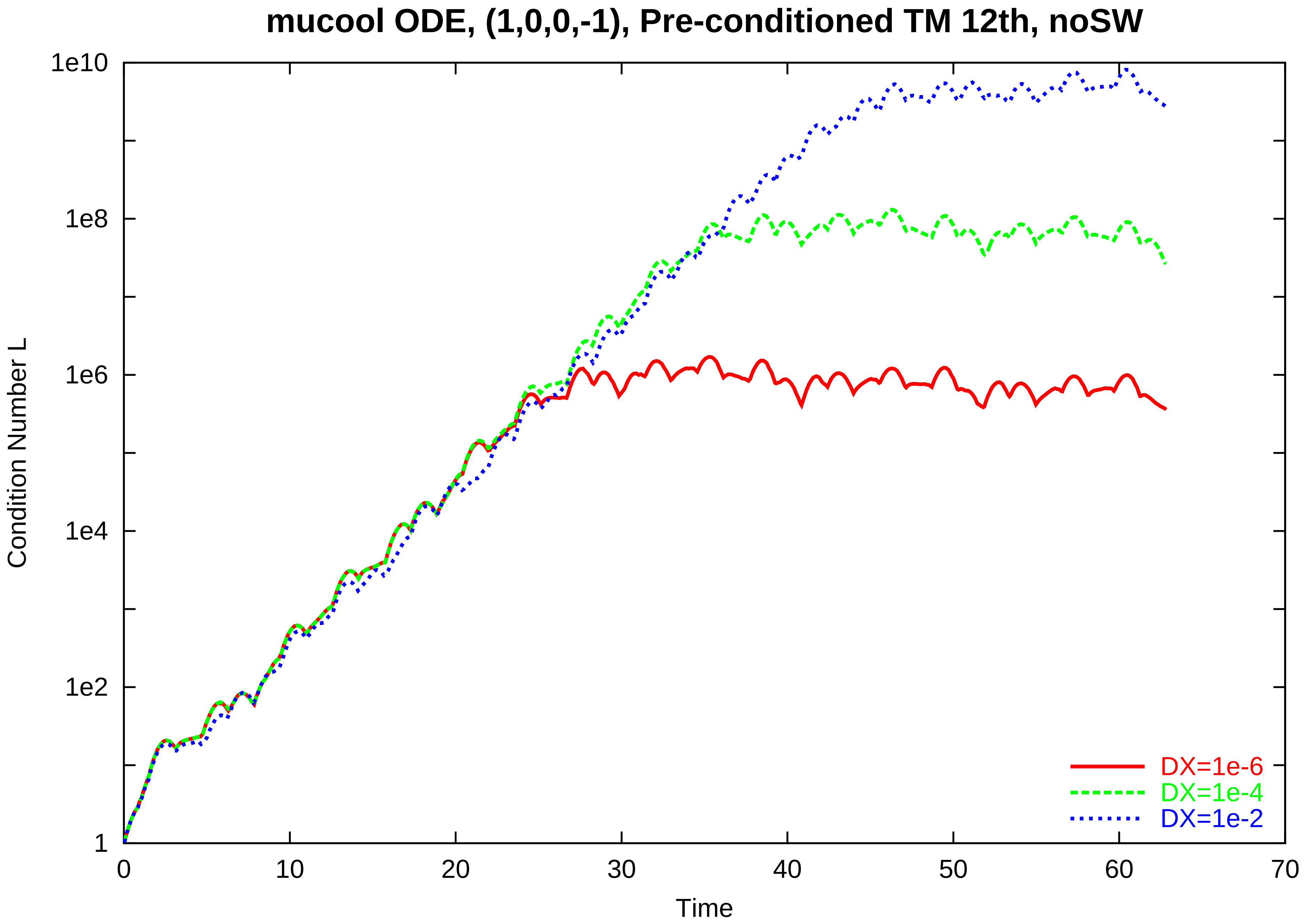
COSY easily integrates 10 cycles for $d = 10^{-2}$ with curvilinear preconditioning and QR preconditioning. AWA (method 4) behaves as follows:

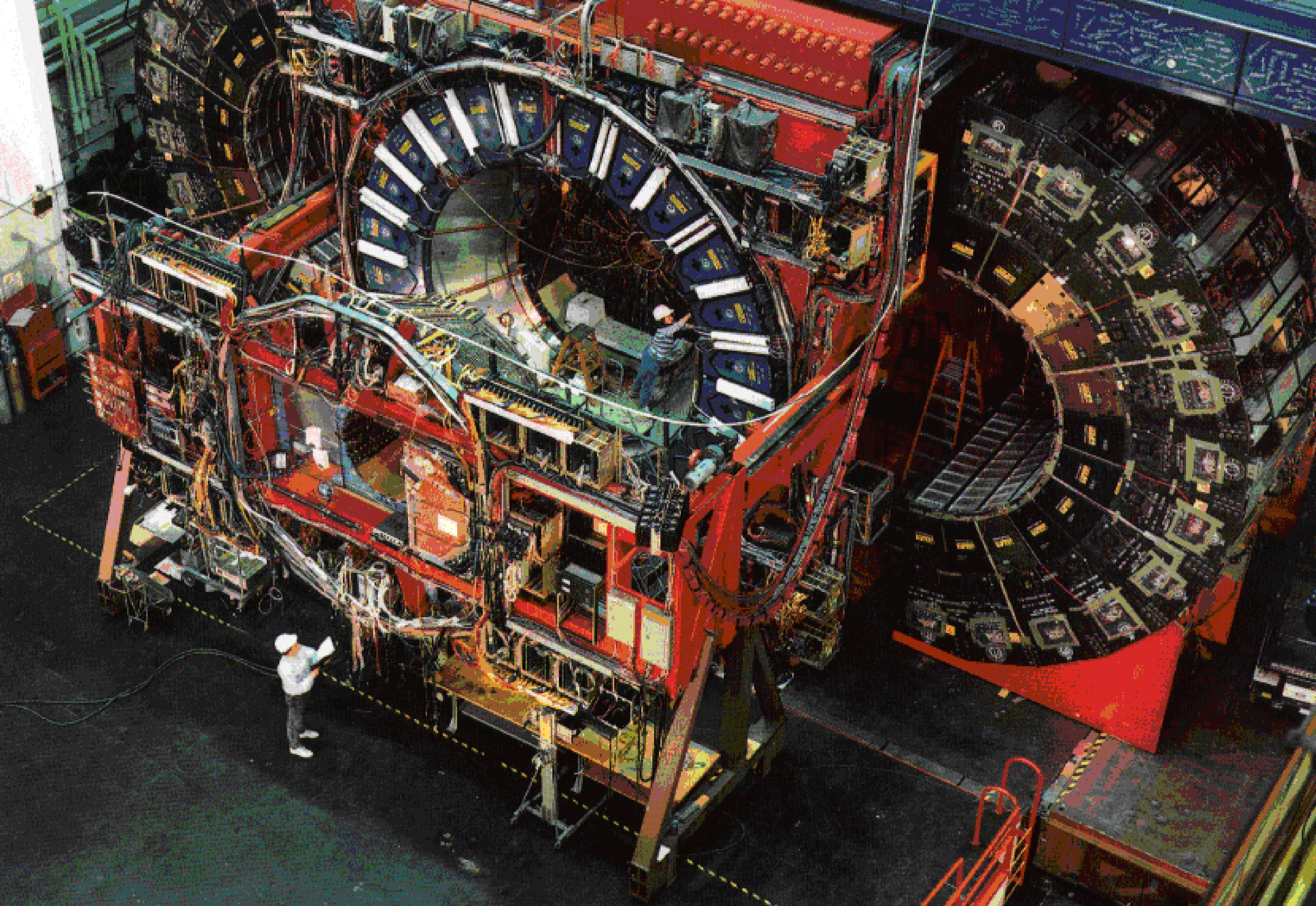
d	**Cycles**
10^{-2}	0.22
10^{-3}	1.25
10^{-4}	9.5

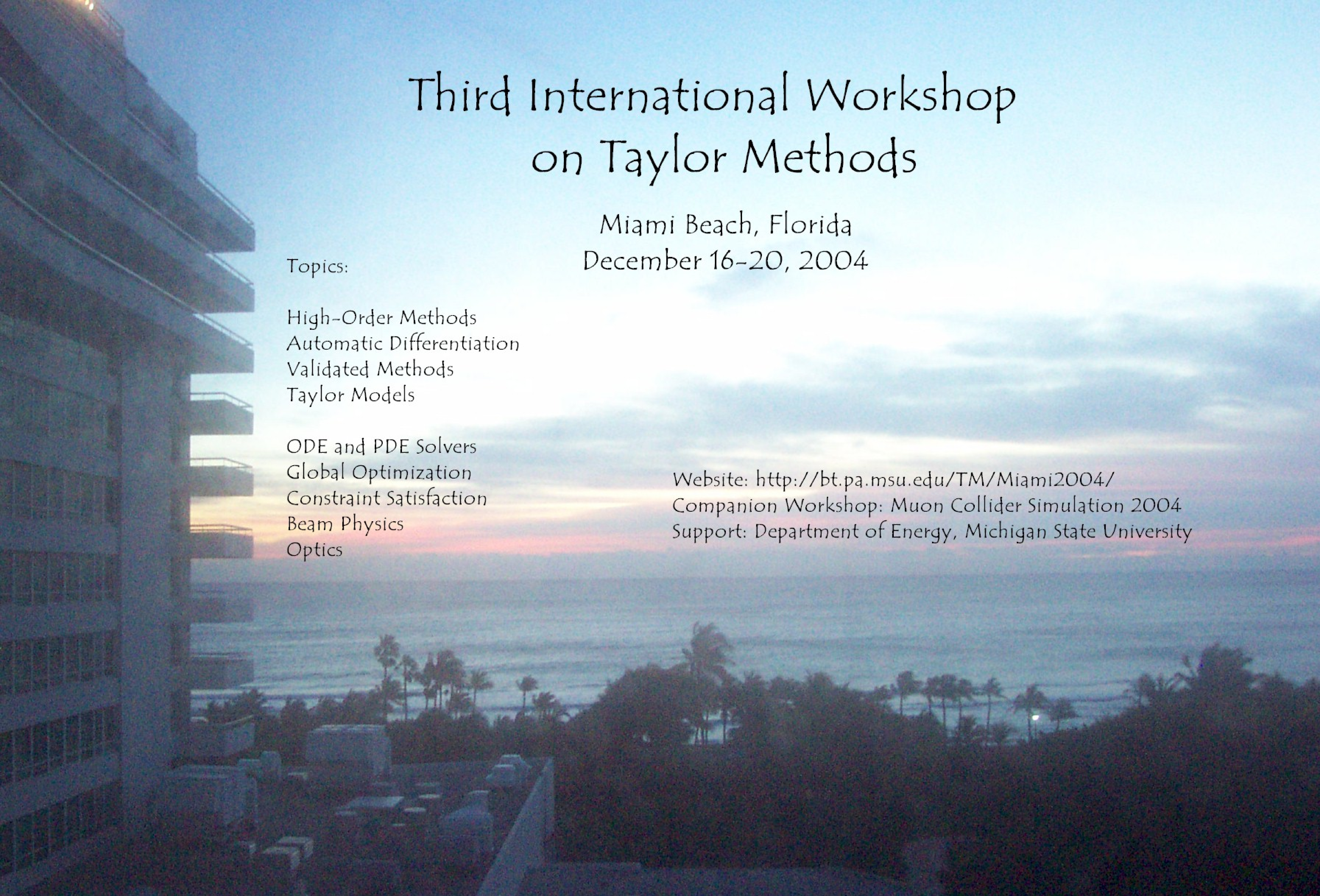
Thus, trying to simulate the system with AWA requires $> (10^2)^4 = 10^8$ subdivisions of the box that COSY can transport in one piece.


mucool ODE, (1,0,0,-1), Pre-conditioned TM 12th, noSW
Determinant L
Time
1e0
1e-5
1e-10
1e-15
1e-20
1e-25
1e-30
1e-35
1e-40
0
10
20
30
40
50
60
70
DX=1e-6
DX=1e-4
DX=1e-2


mucool ODE, (1,0,0,-1), Pre-conditioned TM 12th, noSW
DX=1e-6
DX=1e-4
DX=1e-2
Remainder Error Size of x
Time
1
1e-3
1e-6
1e-9
1e-12
1e-15
0
10
20
30
40
50
60
70


mucool ODE, (1,0,0,-1), Pre-conditioned TM 12th, noSW
Condition Number L
1e10
1e8
1e6
1e4
1e2
1
0
10
20
30
40
50
60
70
Time
DX=1e-6
DX=1e-4
DX=1e-2




Third International Workshop
on Taylor Methods
Miami Beach, Florida
December 16-20, 2004
Topics:
High-Order Methods
Automatic Differentiation
Validated Methods
Taylor Models
ODE and PDE Solvers
Global Optimization
Constraint Satisfaction
Beam Physics
Optics
Website: http://bt.pa.msu.edu/TM/Miami2004/
Companion Workshop: Muon Collider Simulation 2004
Support: Department of Energy, Michigan State University