Buckling Analysis of Structures with Uncertain Properties and Loads Using an Interval Finite Element Method

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Abstract. In order to ensure the safety of a structure, one must provide for adequate strength of structural elements. In addition, one must prevent large unstable deformations such as buckling. In most analyses of buckling, structural properties and applied loads are considered certain. This approach ignores the fact that imperfections and unknown changes in properties, albeit small, are required for onset of buckling. In this paper, we extend the interval finite element methods developed by the authors to solve for the possible values of loads that will result in a structural stability failure. The analysis requires that interval axial element forces in each frame element in a structure be calculated. These values are calculated from a linear system of interval equations resulting from the static structural analysis. Using the calculated axial loads, a subsequent interval eigenvalue problem is solved for the buckling loads. For both solutions of the linear system of equations and the eigenvalue problem, the unique properties of the finite element methods result in sharp solutions. Several structural problems are presented as exemplars. The sharpness of the solution is demonstrated by comparing to combinatorial solutions.

1. Introduction

In order to ensure the safety of a structure, one must provide for adequate strength of structural elements. In addition, one must prevent large unstable deformations known as buckling. In determining adequate strength as well as adequate stability, the finite element method has become the standard of practice for predicting a structure's behavior.

In current practice, uncertainty in system parameters is not considered during the analysis. Uncertainty is accounted for in a design by a combination of load amplification and strength reduction factor.

Such factors are based on probabilistic models of historic data. Thus, consideration of the impact of changing uncertainty on a design has been removed from current analysis tools.

In order to mitigate this problem, the authors among others (REF IFEM) have developed an interval based finite element method (IFEM). IFEM allows a structural analyst to calculate the impact of uncertainty in parameters on the structure's predicted behavior. To our knowledge, IFEM has only been applied to analysis addressing the strength criterion. In this paper, we extend IFEM to linear stability analysis of structures.

The method presented in this paper requires that interval internal element axial forces in each element in a structure be calculated. These values are calculated from a linear system of interval equations resulting from the static structural analysis. Using the calculated internal forces, a subsequent interval eigenvalue problem is formulated. The solution of the interval eigenvalue problem is then used to calculate the bounds on the critical buckling load. For both the solution of the linear system of equations and the eigenvalue problem, the unique properties of the finite element method are employed to achieve sharp results.

In the following, a brief review of IFEM for calculation of internal element forces is presented. Section 3 describes the formulation of the interval linear stability problem. In section 4, a method for calculation of exact bounds on the resulting eigenvalue problem is then given. An example problem is presented in section 5. Observations and conclusions are given in section 6.

2. Review of Static Interval Finite Element Methods

The linear stability analysis of structures requires the element forces to be determined as the first step in the analysis. For problems with interval values for the stiffness or loads, one needs an interval solution to the underlying statics problem. For the solution of interval finite element (IFEM) problems, Muhanna and Mullen (2001) introduced an Element-by-Element interval finite element formulation, in which a guaranteed enclosure for the solution of interval linear systems of equations was achieved. This method accounts for the parametric representation of element properties and a very sharp enclosure for the solution set due to loading, material and geometric uncertainty in solid mechanics problems. Element matrices were formulated based on the physics, and Lagrange multiplier or penalty methods were applied to impose the necessary constraints for compatibility and equilibrium.

For example, a two-element finite element construct is shown in figure (1). In this example, (E) is Young's modulus, (A) the cross-sectional area, and (L) the length of each element. Subscripts here indicate element number. Nodal loads are denoted by (P), and nodal displacements are denoted by (u).

The conventional finite element formulation results in a global stiffness matrix as given in Eq. (1)

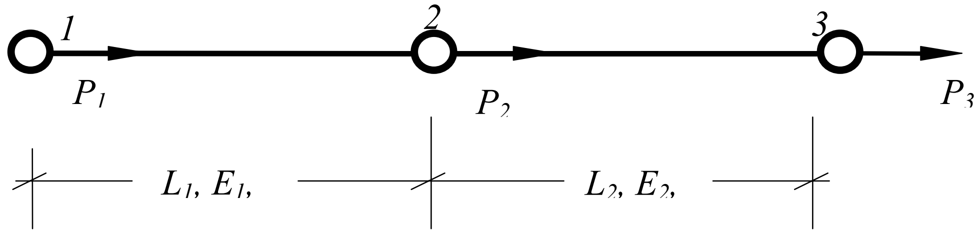

Figure (1) : Two connected linear truss elements.

$$\begin{pmatrix} \frac{E_1A_1}{L_1} & -\frac{E_1A_1}{L_1} & 0 \\ -\frac{E_1A_1}{L_1} & \frac{E_1A_1}{L_1}{}_1 + \frac{E_2A_2}{L_2} & -\frac{E_2A_2}{L_2} \\ 0 & -\frac{E_2A_2}{L_2} & \frac{E_2A_2}{L_2} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix} \tag{1}$$

When the parameters E, A, or L are interval quantities, the resulting interval matrix allows for independent interval values for elements of this matrix which is not physically possible. The element-by-element method generates a global stiffness matrix in the form shown in Eq. (2).

$$K = \mathrm{DS} = \begin{pmatrix} \chi_1 & 0 & 0 & 0 \\ 0 & \chi_1 & 0 & 0 \\ 0 & 0 & \chi_2 & 0 \\ 0 & 0 & 0 & \chi_2 \end{pmatrix} \begin{pmatrix} \frac{E_1A_1}{L_1} & -\frac{E_1A_1}{L_1} & 0 & 0 \\ -\frac{E_1A_1}{L_1} & \frac{E_1A_1}{L_1} & 0 & 0 \\ 0 & 0 & \frac{E_2A_2}{L_2} & -\frac{E_2A_2}{L_2} \\ 0 & 0 & -\frac{E_2A_2}{L_2} & \frac{E_2A_2}{L_2} \end{pmatrix} \tag{2}$$

where χ_i is the interval multiplier of the *i*th finite element obtained due to uncertainty in E_i, A_i, and l_i. Such a form (i.e.***DS***) allows for factoring out the interval multiplier, resulting with an exact inverse for (***DS***).

To ensure compatibility (unique displacements for all elements connecting to a node), one adds constraint conditions in the form of Eq. (3). The resulting system of linear interval equations becomes Eq. (3) and (4)

Equation (1) can be introduced in the following equivalent form:

$$\tilde{C}U = 0 \tag{3}$$

$$KU + \tilde{C}^T \lambda = P \tag{4}$$

If we express $\boldsymbol{K}$ $(n \times n)$ in the form $\boldsymbol{D\tilde{S}}$ and substitute in equation (4):

$$D\tilde{S}U = P - \tilde{C}^T \lambda \tag{5}$$

where $\boldsymbol{D}\,(n \times n)$ is interval diagonal matrix, where its diagonal entries are the positive interval multipliers associated with each element, and n is the multiplication of degrees of freedom per element and the number of elements in the structure. $\boldsymbol{\tilde{S}}$ $(n \times n)$ is a deterministic singular matrix (fixed point matrix). If we multiply equation (3) by $\boldsymbol{D\tilde{C}^T}$ and add the result to equation (5), we get:

$$D(\tilde{S}U + \tilde{C}^T \tilde{C}U) = (P - \tilde{C}^T \lambda) \tag{6}$$

or:

$$D(\widetilde{S}U + \widetilde{Q}U) = (P - \widetilde{C}^T \lambda)$$
$$D(\widetilde{S} + \widetilde{Q})U = (P - \widetilde{C}^T \lambda)$$
$$D\widetilde{R}U = (P - \widetilde{C}^T \lambda) \quad (7)$$

where $(\widetilde{R})$ is a deterministic positive definite matrix, and the displacement vector ***U*** can be obtained from equation (7) in the following form:

$$U = \widetilde{R}^{-1} D^{-1} (P - \widetilde{C}^T \lambda) \quad (8)$$

where ($\widetilde{R}^{-1} D^{-1}$) is an exact inverse of the interval matrix $(D\widetilde{R})$. Equation (8) can be presented in the form:

$$U = \widetilde{R}^{-1} M \;\; \delta \quad (9)$$

Matrix (***M***) has the dimensions (n × number of elements), and its derivation has been discussed in the previous works of the authors (Mullen and Muhanna 1999, Muhanna and Mullen 1999). The vector $\boldsymbol{\delta}$ is an interval vector that has the dimension of (number of elements × *1*) and its elements are the diagonal entries of $\boldsymbol{D}^{-1}$ with the difference that every interval value associated with an element is occurring only once.

If the interval vector $\boldsymbol{\lambda}$ can be determined exactly, the solution of Eq. (8) will represent an exact hull for the solution set of the general interval FE equilibrium equation.

More details on optimal implementation of the above concepts for static finite element solutions is presented in another paper in this proceedings. (Muhanna, Mullen and Zhang 2004).

3. Problem Definition

Deterministic Buckling Analysis*:*
As discussed in the previous section, the buckling analysis using the linear finite element method is carried out in two main steps. First, a parametric static analysis is performed using an arbitrary ordinate of applied load.

$$[Ke]\{u\} = \{P\} \quad (10)$$

The solution output includes the internal axial forces in terms of the load ordinate. Using these results, the geometric stiffness of the structure is developed which represents the pre-compression load's effects on the total stiffness of the structure (McGuire, Gallagher and Ziemian 2000).

Second, a generalized eigenvalue problem is performed between the elastic and geometric stiffness matrices of the structure in order to find the critical buckling loads in terms of the geometric and material characteristics of the structure.

$$([Ke] - \lambda[Kg])\{u\} = \{0\} \quad (11)$$

Buckling Analysis for Structures with Bounded Uncertainty*:*
For structures with bounded uncertainty present in the stiffness characteristics, the buckling analysis procedure requires the modifications on the following: First, the representation of stiffness characteristics must consider the presence of uncertainty using interval numbers. Second, the static analysis must be performed using the obtained interval stiffness matrix; hence, the calculated element axial forces are interval values. Third, using the obtained element interval axial forces, the interval geometric stiffness matrix can be established. Fourth, the interval eigenvalue problem must be solved in order to obtain the bounds on the critical buckling loads.

I. Interval Representation of Uncertainty

Interval Number*:*
A real interval is a closed set of the form:

$$\widetilde{Z} = [z^l, z^u] = \{z \in \Re \mid z^l \leq z \leq z^u\} \tag{12}$$

In this work, the symbol (~) represents an interval quantity.

Interval Formulation:
The structure's global stiffness can be viewed as a summation of the element contributions to the global stiffness matrix:

$$[Ke] = \sum_{i=1}^{n} [L_i][Ke_i][L_i]^T \tag{13}$$

where $[L_i]$ is the element Boolean connectivity matrix and $[K_i]$ is the element stiffness matrix in the global coordinate system. Considering the presence of uncertainty in the stiffness properties, the non-deterministic element stiffness matrix is expressed as:

$$[\widetilde{Ke}_i] = ([l_i, u_i])[Ke_i] \tag{14}$$

in which $[l_i, u_i]$ is an interval number that pre-multiplies the deterministic element stiffness matrix. Therefore, the structure's global stiffness matrix in the presence of any uncertainty is the linear summation of the contributions of non-deterministic interval element stiffness matrices:

$$[\widetilde{Ke}] = \sum_{i=1}^{n} ([l_i, u_i])[L_i][Ke_i][L_i]^T = \sum_{i=1}^{n} ([l_i, u_i])[\overline{Ke}_i] \tag{15}$$

in which, $[\overline{Ke}_i]$ is the deterministic element stiffness contribution to the global stiffness matrix.

II. Interval Geometric Stiffness Matrix

Using the obtained interval axial forces by IFEM (Section 2), the interval geometric stiffness matrix can be set up. The structure geometric stiffness can be viewed as a summation of the element contributions to the global geometric stiffness matrix:

$$[Kg] = \sum_{i=1}^{n} [L_i]((f_i)[\hat{K}g_i])[L_i]^T \qquad (16)$$

where, (f_i) is the element axial force and $[\hat{K}g_i]$ is the force independent matrix of geometric stiffness. Considering the axial force as an interval quantity, the interval structure's geometric stiffness matrix can be established as:

$$[\widetilde{K}g] = \sum_{i=1}^{n} (\widetilde{f}_i)[L_i][Kg_i][L_i]^T = \sum_{i=1}^{n} (\widetilde{f}_i)[\overline{K}g_i] \qquad (17)$$

where $[\overline{K}g_i] = [L_i][\hat{K}g_i][L_i]^T$ and $(\widetilde{f}_i = [f\min_i, f\max_i])$ is the element interval axial load.

III. Interval Eigenvalue Problem for Buckling Analysis

Hollot and Barlett (1987) studied the spectra of eigenvalues of an interval matrix family which are found to depend on the spectrum of its extreme sets. Dief (1991) presented a method for computing interval eigenvalues of an interval matrix based on an assumption of invariance properties of eigenvectors.

The concept of interval eigenvalue problem has been used in structures with interval uncertainty. Modares and Mullen (2004) have introduced a method for the solution of the interval eigenvalue problem which determines the exact bounds of the natural frequencies of a structure using IFEM formulation.

In order to obtain the bounds on the critical buckling loads, a generalized interval eigenvalue problem must be performed between the interval elastic and interval geometric stiffness matrices as:

$$(\sum_{i=1}^{n} ([l_i, u_i])[\overline{K}e_i])\{u\} = (\widetilde{\lambda})(\sum_{i=1}^{n} (\widetilde{f}_i)[\overline{K}g_i])\{u\} \qquad (18)$$

Interval Eigenvalue Problem Definition:
The eigenvalue problems for matrices containing interval values are known as the interval eigenvalue problems. If $[\widetilde{A}]$ is an interval matrix $(\widetilde{A} \in IR^{n\times n})$ and $[A]$ is a member of the interval matrix $(A \in \widetilde{A})$, the interval eigenvalue problem is shown as:

$$([A] - \lambda[I])\{u\} = 0, (A \in \widetilde{A}) \qquad (19)$$

The solution of interest to the real interval eigenvalue problem is defined as an inclusive set of real values $(\widetilde{\lambda})$ such that for any member of the interval matrix, the solution to its eigenvalue problem is a member of the solution set shown as:

$$\{\lambda \in \widetilde{\lambda} = [\lambda^l, \lambda^u] \mid \forall A \in \widetilde{A} : ([A] - \lambda[I])\{u\} = 0\} \qquad (20)$$

which is the enclosure of all possible solutions. A sharp enclosure is defined as the solution with the smallest radius as:

$$\{\exists \lambda \in \widetilde{\lambda} = [\lambda^l, \lambda^u] \mid \forall A \in \widetilde{A} : ([A] - \lambda [I])\{u\} = 0\} \tag{21}$$

4. Solution for Interval Eigenvalue Problem

The following concepts must be considered in order to bound the non-deterministic interval eigenvalue problem, Eq.(18).

The classical linear eigenpair problem for a symmetric matrix is:

$$Ax = \lambda x \tag{22}$$

with the solution of real eigenvalues ($\lambda_1 \le \lambda_2 \le ... \le \lambda_n$) and corresponding eigenvectors ($x_1, x_2, ..., x_n$). This equation can be transformed into a ratio of quadratics known as the *Rayleigh quotient*:

$$R(x) = \frac{x^T A x}{x^T x} \tag{23}$$

The Rayleigh quotient for a symmetric matrix is bounded between the smallest and the largest eigenvalues (Bellman 1960 and Strang 1976).

$$\lambda_1 \le R(x) = \frac{x^T A x}{x^T x} \le \lambda_n \tag{24}$$

Thus, the first eigenvalue (λ_1) can be obtained by performing an *unconstrained minimization* on the scalar-valued function of Rayleigh quotient:

$$\min_{x \in R^n} R(x) = \min_{x \in R^n} \left(\frac{x^T A x}{x^T x}\right) = \lambda_1 \tag{25}$$

For finding the next eigenvalues, the concept of maximin characterization can be used. This concept obtains the kth eigenvalue by imposing (*k-1*) constraints on the minimization of the Rayleigh quotient:

$$\lambda_k = \max[\min R(x)] \tag{26}$$

(subject to constrains $(x^T z_i = 0), i = 1,...k-1, k \ge 2$)

Bounding the Critical Buckling Loads:

Using the concepts of minimum and maximin characterizations of eigenvalues for symmetric matrices, the solution to the generalized interval eigenvalue problem for the critical buckling loads of a structure with uncertainty in the stiffness characteristics (Eq.(18)) for the first eigenvalue can be shown as:

$$\tilde{\lambda}_1 = \min_{x \in R^n}\left(\frac{x^T[\tilde{K}e]x}{x^T[\tilde{K}g]x}\right) = \min_{x \in R^n}\left(\frac{x^T(\sum_{i=1}^{n}([l_i, u_i])[\overline{K}e_i])x}{x^T(\sum_{i=1}^{n}([f\min_i, f\max_i])[\overline{K}g_i])x}\right) \quad (27)$$

for the next eigenvalues:

$$\tilde{\lambda}_k = \max[\min_{x.z_i=0, i=1,\dots,k-1} \frac{x^T[\tilde{K}e]x}{x^T[\tilde{K}g]x}] = \max[\min_{x.z_i=0, i=1,\dots,k-1}\left(\frac{x^T(\sum_{i=1}^{n}([l_i, u_i])[\overline{K}e_i])x}{x^T(\sum_{i=1}^{n}([f\min_i, f\max_i])[\overline{K}g_i])x}\right)] \quad (28)$$

Bounding Deterministic Eigenvalue Problems for the Critical Buckling Loads*:*

Since the matrices $[\overline{K}e_i]$ and $[\overline{K}g_i]$ are non-negative definite, the terms $(x^T(\overline{K}e_i)x)$ and $(x^T(\overline{K}g_i)x)$ are non-negative. Therefore, the upper bounds on the eigenvalues in Eqs.(18) and (19) are obtained by considering maximum values of interval coefficients of uncertainty for all elements in the elastic stiffness matrix and the lower values of axial force in the geometric stiffness matrix . Similarly, the lower bounds on the eigenvalues are obtained by considering minimum values of interval coefficients of uncertainty for all elements in the elastic stiffness matrix and the upper values of axial force in the geometric stiffness matrix.

Also, it can be observed that any other element stiffness selected from the interval sets will yield eigenvalues between the upper and lower bounds. Using these concepts, the deterministic eigenvalue problems corresponding to the maximum and minimum critical buckling loads are obtained as:

$$(\sum_{i=1}^{n}(u_i)[\overline{K}_i])\{u\} = (\lambda_{\max})\sum_{i=1}^{n}(f\min_i)[\overline{K}g_i]\{u\} \quad (29)$$

$$(\sum_{i=1}^{n}(l_i)[\overline{K}_i])\{u\} = (\lambda_{\min})\sum_{i=1}^{n}(f\max_i)[\overline{K}g_i]\{u\} \quad (30)$$

5. Example

The bounds on the critical buckling load for a 2D statically indeterminate truss with interval uncertainty present in the modulus of elasticity of each element are determined (Figure (2)).

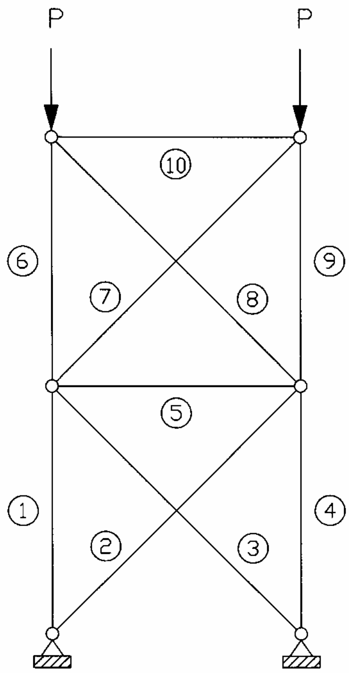

Figure (2): The structure of 2-D truss

The cross-sectional area A, the length for horizontal and vertical members L, the Young's moduli E for all elements are $\widetilde{E} = ([0.99, 1.01])E$.

The problem is solved using the method presented in this work. First, a static analysis on the structure with uncertainty is performed using IFEM, and the bounds on obtained element axial forces are obtained. Second, two deterministic eigenvalue problems are performed to obtain the bounds on the critical buckling load.

For comparison, a combinatorial analysis has performed which considers lower and upper values of uncertainty for each element i.e. solving ($2^n = 2^{10} = 1024$) deterministic problems.

The static analysis results obtained by IFEM and the brute force combination solution are summarized in Table (1).

	Lower Bound IFEM	Upper Bound IFEM	Lower Bound Combination Method	Upper Bound Combination Method
f_1	-0.7943	-0.7863	-0.7945	-0.7862
f_2	-0.3021	-0.2908	-0.3023	-0.2905
f_3	-0.3021	-0.2908	-0.3023	-0.2905
f_4	-0.7943	-0.7863	-0.7945	-0.7862
f_5	0.3887	0.4013	0.3882	0.4018
f_6	-0.8182	-0.8108	-0.8187	-0.8104
f_7	-0.2674	-0.2569	-0.2679	-0.2563
f_8	-0.2674	-0.2569	-0.2679	-0.2563
f_9	-0.8182	-0.8108	-0.8187	-0.8104
f_{10}	0.1817	0.1891	0.1812	0.1895

Table (1): Static Analysis of the example problem using IFEM and combination method

Second, a buckling analysis is performed using the method presented in this work. Also, the solution to a combinatorial buckling analysis is obtained, and the results for the fundamental critical buckling load is summarized in Table (2).

	Lower Bound Present Method	Lower Bound *with* Exact Forces	Lower Bound Combination Method	Relative Uncertainty Present Method	Relative Uncertainty Combination Method
P_{cr1} / AE	0.1080	0.1081	0.1093	0.021	0.010

Table (2): Buckling of the example problem using the present method and comparison with the combinatorial analysis results

In practice, the lowest buckling load is the only value of interest. As such, we have compared only the lower bound in Table 2. The example problem shows an overestimation of the width of the interval results of the proposed method compared to a combinatorial solution. The overestimation could be attributed to three possible sources: overestimation in the interval values of the static solution of internal forces, overestimation in the eigenvalue solution or overestimation from uncoupling of the element forces Eq. (10) and the critical load Eq. (11). The internal forces calculated by the interval method and the exact combinatorial results are correct in the first three digits. The eigen solution has been proved to be sharp. Therefore the uncoupling of the static solution from Eq. (10) and the stability equation (11), is the most likely cause of the overestimation of the width seen in the solution. This can be seen by examining a solution where the exact combinatorial internal forces are used in Eq. (11) to find the critical buckling load. In this calculation, the critical load in (0.1081), just slightly above the results from the proposed method of 0.1080 (See Table 2).

6. Discussion

In this paper, we have introduced a method for linear stability analysis of a structure with stiffness properties expressed as an interval quantity. To our knowledge, this is the first treatment of interval methods for structural stability. The conventional two step method consisting of solving the linear static problem for internal forces and subsequent solution of an eigen problem for the critical buckling load has been adapted from the non-interval approach. The method presented provides a lower bound for the minimum buckling load. Dependency of the interval internal forces and interval stiffness parameters have not been included in the method; this is the expected cause of loss of sharpness in the interval results.

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