Experiments with Range Computations using Extrapolation

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Abstract. The *natural interval extension* (NIE) used widely in interval analysis has the first order convergence property, i.e., the excess width of the range enclosures obtained with the NIE goes down at least *linearly* with the domain width. Here, we show how range approximations of *higher convergence orders* can be obtained from the sequence of range enclosures generated with the NIE and uniform subdivision. We combine the well-known *Richardson Extrapolation Process* (Sidi, 2003) with *Brezenski's error control method* (Brezenski, 1983) to generate non-validated range approximations to the true range. We demonstrate the proposed method for accelerating the convergence orders on several multidimensional examples, varying from one to six dimensions. These numerical experiments also show that considerable computational savings can be obtained with the proposed procedure. However, the theoretical basis of the proposed method remains to be investigated.

Keywords: Extrapolation, NIE, REP, Acceleration of Sequences.

1. Introduction

A major focus of interval analysis (Moore, 1979) is developing interval algorithms which produce sharp bounds on the solutions. The *natural interval extension* (NIE) is the simplest tool that is widely used in interval analysis to compute the range enclosures of functions. The range enclosures obtained using NIE can be tightened further with the help of the uniform subdivision method (Moore, 1979). These range enclosures possesses the property of first order convergence, i.e., the excess width of the computed range enclosures goes down at least linearly with the domain width.

In this paper, we propose a new method to accelerate the convergence rate of the range enclosures, obtained with the NIE and uniform subdivision, using an *extrapolation* process, such as the Richardson extrapolation process (REP). In the proposed method, we first obtain the range enclosures with the NIE and uniform subdivision, for a sequence of geometrically increasing subdivision factors. Then, we construct two separate sequences of lower and upper bounds from the obtained range enclosures. Next, we extrapolate these sequences to their respective limits (which are the range infimum and range supremum) using the REP. This produces the Romberg Tables for the range infimum and supremum. To these Romberg Tables, we next apply Brezenski's error control criterion and generate the so-called Brezenski's tables of intervals containing the range infimum and supremum. Finally, from the Brezenski's tables, we construct a table of intervals approximating the range. The sequences of the range approximating intervals in this table converges columnwise increasingly faster than the sequence of range enclosures obtained with the existing method of NIE and uniform subdivision.

The outline of the paper is as follows. In section 2, we discuss the basics of the sequence transformation, the REP and Brezenski's error control criterion. In section 3, we review the NIE and uniform subdivision. In section 4, we present the proposed algorithms. In section 5, we demonstrate the effectiveness of the proposed method on several multidimensional examples, varying from one to six dimensions. Finally, in section 6 we draw the conclusions of the work.

2. Extrapolation Process - Sequence Transformation

Extrapolation methods (equivalently, *convergence acceleration methods* or *sequence transformations*) are popularly used for accelerating the convergence process of sequences (Brezenski and Zaglia, 2002). Extrapolation methods basically transform the original sequence into another one which converges to the limit more quickly (when the limit exists).

Let (S_n) be a sequence of (real or complex) numbers which converges to the limit S and (T_n) be another sequence obtained by transforming the sequence (S_n) using some suitable transformation method T.

In order to obtain a higher rate of convergence, the new transformed sequence (T_n) must exhibit the following properties:

1. (T_n) must converge.
2. (T_n) must converge to the same limit as (S_n).
3. (T_n) must converge to S faster than (S_n), that is

$$\lim_{n\to\infty} (T_n - S) \,/\, (S_n - S) = 0$$

If the new sequence (T_n) possesses property (3), we say that the transformation T *accelerates the convergence* of the sequence (S_n) or that the sequence (T_n) *converges faster* than (S_n).

These properties, in general, do not hold for all converging sequences (S_n). We can obtain the new transformed sequence (T_n) possessing the above mentioned properties only if the sequence (S_n) to be accelerated belongs to the kernel $\mathfrak{K}_T$ of the transformation used (the kernel $\mathfrak{K}_T$ is the set of sequences for which there exists an S such that $\forall n \geq N$, $T_n = S$, cf. (Brezenski and Zaglia, 2002)).

For instance, amongst the wide range of transformation methods available, the well-known Aitken's Δ^2 transformation process is given by

$$T_n = S_n - \frac{(S_{n+1} - S_n)^2}{S_{n+2} - 2S_{n+1} + S_n},\ n = 0, 1, \ldots \tag{1}$$

For the Aitken's process, the kernel $\mathfrak{K}_T$ is the set of sequences of the form

$$S_n = S + a\lambda^n \tag{2}$$

where, a and λ are scalars with $a \neq 0$ and $\lambda \neq 1$. Usually, S is the limit of the sequence (S_n), but this is not always the case. In the Aitken's process, S is the *limit* of (S_n) if $|\lambda| < 1$, and is called the *anti-limit* if $|\lambda| > 1$. It can be shown (Brezenski and Zaglia, 2002) that the Aitken's process accelerates the convergence of all sequences for which there exists a $\lambda \in [-1\ ,+1)$ such that

$$\lim_{n\to\infty} \frac{(S_{n+1} - S)}{(S_n - S)} = \lambda$$

A sequence transformation $T : (S_n) \to (T_n)$ is said to be an extrapolation method if it is such that $\forall n \geq N$, $T_n = S$ if and only if $(S_n) \in \mathfrak{K}_T$. Thus, any sequence transformation can be viewed as an extrapolation method.

Amongst the various extrapolation methods (Sidi, 2003), perhaps the most popular and widely used method is the REP. Let $K \in \mathbb{N}$, $\rho \geq 2$, and $\{S_n\}$ be the sequence to be accelerated. The REP can be given as

$$T_0^{(j)} = S_j,\ j = 0, 1, \ldots, K \tag{3}$$

$$T_k^{(j)} = T_{k-1}^{(j)} + \frac{\left(T_{k-1}^{(j)} - T_{k-1}^{(j-1)}\right)}{(\rho^k - 1)},\ \begin{cases} k = 1, 2, \ldots, K, \\ j = k, \ldots, K. \end{cases} \tag{4}$$

which is similar to the Aitken's Δ^2 process for the first extrapolated column $k = 1$ as given in (1).

The sequences $\left\{T_k^{(j)}\right\}$ computed using (4) can be arranged in a two-dimensional array called the Romberg Table, denoted $[T]^k$, cf. Table 1. The arrows in the table show the flow of computations. The k^{th} column of the Romberg Table is referred to as the $(k-1)^{th}$ *extrapolated* column. Details of the REP are in (Sidi, 2003).

Table 1. The Romberg Table, $[T]^K$ with $K = 5$ (i.e., with 5 extrapolated columns)

$T_0^{(0)}$										
	$\searrow$									
$T_0^{(1)}$	$\rightarrow$	$T_1^{(1)}$								
	$\searrow$		$\searrow$							
$T_0^{(2)}$	$\rightarrow$	$T_1^{(2)}$	$\rightarrow$	$T_2^{(2)}$						
	$\searrow$		$\searrow$		$\searrow$					
$T_0^{(3)}$	$\rightarrow$	$T_1^{(3)}$	$\rightarrow$	$T_2^{(3)}$	$\rightarrow$	$T_3^{(3)}$				
	$\searrow$		$\searrow$		$\searrow$		$\searrow$			
$T_0^{(4)}$	$\rightarrow$	$T_1^{(4)}$	$\rightarrow$	$T_2^{(4)}$	$\rightarrow$	$T_3^{(4)}$	$\rightarrow$	$T_4^{(4)}$		
	$\searrow$		$\searrow$		$\searrow$		$\searrow$		$\searrow$	
$T_0^{(5)}$	$\rightarrow$	$T_1^{(5)}$	$\rightarrow$	$T_2^{(5)}$	$\rightarrow$	$T_3^{(5)}$	$\rightarrow$	$T_4^{(5)}$	$\rightarrow$	$T_5^{(5)}$

2.1. Brezenski's Error Control criterion

Brezenski's theorem on error control (Brezenski, 1983) explains how to construct a sequence of intervals containing the unknown limit of the sequence under consideration.

Let $\{S_n\}$ be the sequence under consideration. Let S be the limit of the sequence $\{S_n\}$. Let $\{T_n\}$ and $\{V_n\}$ be two other sequences obtained by applying REP to $\{S_n\}$. Suppose the sequence $\{T_n\}$ converges faster than $\{S_n\}$, and $\{V_n\}$ converges faster than $\{T_n\}$, both to the same limit S. Thus, $\{S_n\}$, $\{T_n\}$, and $\{V_n\}$ can be successive columns of the Romberg Table 1.

Let $b \in R$ (called as the Brezenski's factor). Define

$$V_n(b) = V_n - b(V_n - T_n), \quad n \in N$$

and construct the interval

$$J_n(b) = [\min(V_n(b),\ V_n(-b)), \max(V_n(b),\ V_n(-b))] \tag{5}$$

THEOREM 1. *(*Brezenski, 1983*) If $T_n - S = o(S_n - S)$ and if $V_n - S = o(T_n - S)$ then $\forall b \neq 0, \exists N : \forall n \geq N,\ S \in J_n(b)$. Moreover $V_n(\pm b) - S = o(S_n - S)$.*

REMARK 2.1. *Brezenski has pointed out a fundamental practical point in (*Brezenski, 1983*): "Under some assumptions, the theorem given above says that for all n greater than N, S belongs to some interval. However, such a N is not known without adding supplementary assumptions. Such an N has been attained if the interval at the step $n+1$ is contained in the interval obtained at the step n, whatever $n \geq N$ may be. This is a good test for having attained this N".*

REMARK 2.2. *As pointed out in Theorem 1, the Brezenski's sequence of intervals $V_n(\pm b)$ (so, also $J_n(b)$) can have a rate of convergence faster than $\{S_n\}$, at the most of $\{T_n\}$, but not faster than $\{T_n\}$. Hence, we lose the benefit of extrapolation by one column.*

REMARK 2.3. *The value of Brezenski's factor b decides the two factors in constructing the Brezenski's sequence of intervals $J_n(b)$ in (5). One is the width of the sequence of intervals $J_n(b)$, and the other is the value of N referred to in Theorem 1. Larger the value of b, wider is the interval $J_n(b)$, but smaller is N. Whereas, smaller the value of b, tighter is the interval $J_n(b)$, but larger is N. In general, the suggested value of b is between 0 and 1, cf. (*Brezenski, 1983*).*

3. THE NIE AND UNIFORM SUBDIVISION

Consider the interval vector (also called as a box) $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_l)^T$ with components $\mathbf{x}_j = \left[\underline{x}_j, \overline{x}_j\right]$. Denote the range of a function $f : \mathbb{R}^n \to \mathbb{R}$ over the box $\mathbf{x}$ as

$$f^{range}(\mathbf{x}) = \{f(x) \,|x \in \mathbf{x}\}$$

Let $f(\mathbf{x})$ denote the natural interval extension (NIE) of f, and $e(\mathbf{x})$ be the error interval associated with the range enclosure obtained with $f(\mathbf{x})$. Then, we can express $f(\mathbf{x})$ as

$$f(\mathbf{x}) = \left[\underline{f(\mathbf{x})},\ \overline{f(\mathbf{x})}\right] = f^{range}(\mathbf{x}) + e(\mathbf{x}) \tag{6}$$

Suppose we uniformly subdivide the interval vector $\mathbf{x}$ using the subdivision factor N, as follows (wid $\mathbf{x}$ denotes the width of the box $\mathbf{x}$):

$$\mathbf{x}_{i,j} = [\underline{\mathbf{x}}_i + (j-1)\ \text{wid}\ \mathbf{x}_i/N,\ \underline{\mathbf{x}}_i + j\ \text{wid}\ \mathbf{x}_i/N],\ j = 1, 2, \ldots, N \tag{7}$$

$$\mathbf{x}_i = \bigcup_{j=1}^{N} \mathbf{x}_{i,j} \tag{8}$$

$$\mathbf{x} = \bigcup_{j_i=1}^{N} \left(\mathbf{x}_{1,j_1},\ \mathbf{x}_{2,j_2}, \ldots, \mathbf{x}_{l,j_l}\right) \tag{9}$$

Let $e_{(N)}(\mathbf{x})$ be the error interval associated with N partitions of the interval vector $\mathbf{x}$, expressed as

$$e_{(N)}(\mathbf{x}) = \bigcup_{j_i=1}^{N} e\left(\mathbf{x}_{1,j_1},\ \mathbf{x}_{2,j_2}, \ldots, \mathbf{x}_{l,j_l}\right) \tag{10}$$

Define $f_{(N)}(\mathbf{x})$ as

$$f_{(N)}(\mathbf{x}) = \bigcup_{j_i=1}^{N} f\left(\mathbf{x}_{1,j_1},\ \mathbf{x}_{2,j_2}, \ldots, \mathbf{x}_{l,j_l}\right) = f^{range}(\mathbf{x}) + e_{(N)}(\mathbf{x}) \tag{11}$$

Then, Moore (Moore, 1979) has shown that there exists a constant σ such that the *excess width* is given by

$$\text{wid}\ e_{(N)}(\mathbf{x}) \le \frac{\sigma}{N}\ \text{wid}\ \mathbf{x} \tag{12}$$

or

$$\text{wid}\ e_{(N)}(\mathbf{x}) = \frac{\sigma}{N}\ \text{wid}\ \mathbf{x} + O\left(\text{wid}\ \mathbf{x}^2\right) \tag{13}$$

From (11) and (13),

$$\underline{f_{(N)}(\mathbf{x})} = \underline{f^{range}(\mathbf{x})} + \frac{\sigma}{N}\ \text{wid}\ \mathbf{x} + O\left(\text{wid}\ \mathbf{x}^2\right) \tag{14}$$

$$\overline{f_{(N)}(\mathbf{x})} = \overline{f^{range}(\mathbf{x})} + \frac{\sigma}{N} \text{ wid } \mathbf{x} + O\left(\text{wid } \mathbf{x}^2\right) \tag{15}$$

Comparing (14) with (2), for the infimum of the range we have

$$S_n \leftrightarrow \underline{f_{(N)}(\mathbf{x})}, \quad S \leftrightarrow \underline{f^{range}(\mathbf{x})} \quad \lambda^n \leftrightarrow \frac{1}{N^n} \quad a \leftrightarrow \sigma \tag{16}$$

Similarly, from (2) and (15), for the supremum of the range we have

$$S_n \leftrightarrow \overline{f_{(N)}(\mathbf{x})}, \quad S \leftrightarrow \overline{f^{range}(\mathbf{x})} \quad \lambda^n \leftrightarrow \frac{1}{N^n} \quad a \leftrightarrow \sigma \tag{17}$$

REMARK 3.1. *Using NIE and different subdivision factors N, we can thus construct two separate sequences converging to two different limits. One is the sequence of lower bounds on the range enclosures converging to the range infimum, and the other is the sequence of upper bounds on the range enclosures converging to the range supremum. In our work, we shall construct these two separate sequences of lower and upper bounds of the range enclosure and extrapolate them to their respective limits (we do not directly apply extrapolation to the sequence of intervals enclosing the range).*

4. The Proposed Method

Based on Remark 3.1, we first construct two separate sequences of lower and upper bounds on the infimums and supremums of the range enclosures, and then obtain the Romberg tables for the infimum and supremum by extrapolating these sequences separately to their respective limits.

The algorithm *Sequence_infsup* accepts as inputs the initial box $\mathbf{x}$, the function f, and number K of extrapolated columns required in the Romberg Table. It returns the sequences of infimums $\left\{A_0^{(j)}\right\}_{j=0}^{K}$ and the sequences of supremums $\left\{B_0^{(j)}\right\}_{j=0}^{K}$. The sequences of lower and upper bounds are generated for a geometrically increasing uniform subdivision factor $N_j = 2^j, \quad j = 0, 1, \ldots, K$.

ALGORITHM SEQUENCES OF LOWER AND UPPER BOUNDS:
$\left[\left\{A_0^{(j)}\right\}_{j=0}^{K}, \ \left\{B_0^{(j)}\right\}_{j=0}^{K}\right] = \text{Sequence\_infsup}(\mathbf{x},\ f,\ K)$

Inputs: Initial box $\mathbf{x}$, function f, the number K of extrapolated columns in Romberg table.

Output: The sequence of infimums $\left\{A_0^{(j)}\right\}_{j=0}^{K}$

and supremums $\left\{B_0^{(j)}\right\}_{j=0}^{K}$.

BEGIN Algorithm

1. Set
$$A_0^{(0)} = \underline{f(\mathbf{x})}, \quad B_0^{(0)} = \overline{f(\mathbf{x})}$$
2. FOR $j = 1, 2, \ldots, K$

   a) Compute the number of elements in the uniform subdivision partition as $N_j = 2^j$.
   b) Using N_j, partition the initial box $\mathbf{x}$ uniformly as per (7), (8), and (9)
   c) In this subdivision partition of $\mathbf{x}$, obtain the range enclosure $f_{(N_j)}(\mathbf{x})$ as per (11)
$$f_{(N_j)}(\mathbf{x}) = \left[\underline{f_{(N_j)}(\mathbf{x})}, \ \overline{f_{(N_j)}(\mathbf{x})}\right] = \bigcup_{j_i=1}^{N_j} f\left(\mathbf{x}_{1,j_1}, \ \mathbf{x}_{2,j_2}, \ldots, \mathbf{x}_{l,j_l}\right)$$
   d) Set
$$A_0^{(j)} \longleftarrow \underline{f_{(N_j)}(\mathbf{x})}, \ B_0^{(j)} \longleftarrow \overline{f_{(N_j)}(\mathbf{x})}$$

3. RETURN $\left\{A_0^{(j)}\right\}_{j=0}^{K}$ and $\left\{B_0^{(j)}\right\}_{j=0}^{K}$.

END Algorithm

4.1. Romberg Table for the infimum and supremum with the REP

Having constructed the sequences of lower and upper bounds on the range enclosure, we can now apply the REP and obtain the respective Romberg tables by executing the algorithm *Romberg_inf* and *Romberg_sup.*

ALGORITHM ROMBERG TABLE FOR THE INFIMUM:

$$[A]^K = \text{Romberg\_inf}\left(\left\{A_0^{(j)}\right\}_{j=0}^{K}, \ K\right)$$

Inputs: The sequence of lower bounds $\left\{A_0^{(j)}\right\}_{j=0}^{K}$, and the number K of columns required in the Romberg Table.

Output: The Romberg Table $[A]^K$ containing the extrapolated sequences.

BEGIN Algorithm

1. Set $T_0^{(j)} = A_0^{(j)},\ j = 0, 1, \ldots, K.$

2. Construct the Romberg Table for the range infimum, using the REP in (4):

$$A_k^{(j)} = A_{k-1}^{(j)} + \frac{A_{k-1}^{(j)} - A_{k-1}^{(j-1)}}{(2^k - 1)}, \quad \begin{cases} k = 1, 2, \ldots, K, \\ j = k, \ldots, K. \end{cases}$$

$$[A]^K = \left\{ A_k^{(j)},\ k = 0, 1, \ldots, K,\ j = k, \ldots, K \right\}$$

3. RETURN the Romberg Table $[A]^K$

END Algorithm

We can have a similar algorithm *Romberg_sup* based on $\left\{ B_0^{(j)} \right\}_{j=0}^{K}$ to generate the Romberg Table $[B]^K$ of extrapolated sequences for the range supremum (the description is omitted here).

4.2. Brezenski's Error control and approximated bounds

As there is no guarantee that the extrapolation on the lower bound sequence will again result in a lower bound on the range infimum, it is necessary to have an error estimate for the entries in the Romberg Table (the same also holds true for the upper bound sequence). Among the many error estimation methods (Brezenski and Zaglia, 2002; Sidi, 2003; Walz, 1996), we adopt the error control criterion proposed by Brezenski (Brezenski and Zaglia, 2002) to generate intervals which asymptotically contain the true range.

Based on Theorem 1 and Remark 2.1, we can have an algorithm to construct the so-called Brezenski's Table of intervals for the range infimum and Brezenski's Table of intervals for the range supremum, and from these, the final Table range approximations with higher order convergence rate.

ALGORITHM RANGE APPROXIMATOR:

$[Range\_approx]^K$ =Range_Approx$\left([A]^K \text{ and } [B]^K\right)$

Input: The Romberg Tables $[A]^K$ and $[B]^K$, and a value for Brezenski's factor $b \in \mathbb{R}$.

Output: The Table $[Range\_approx]^K$ containing the range approximating intervals.

BEGIN Algorithm

1. Set $k = 0$.

2. From the Romberg Table $[A]^K$, construct Brezenski's table $[C]^K$ of intervals for the infimum as follows (cf. equation 5):

$$V_{k+2}^{(j)}(b) = A_{k+2}^{(j)} - b\left(A_{k+2}^{(j)} - A_{k+1}^{(j)}\right), \quad \begin{cases} k = 0, 1, \ldots, K-2, \\ j = k+2, \ldots, K. \end{cases}$$

$$C_{k+2}^{(j)} = \left[\min\left(V_{k+2}^{(j)}(+b),\ V_{k+2}^{(j)}(-b)\right),\ \max\left(V_{k+2}^{(j)}(+b),\ V_{k+2}^{(j)}(-b)\right)\right],$$

$$k = 0, 1, \ldots, K-2,\ j = k+2, \ldots, K.$$

3. Similarly, from the Romberg Table $[B]^K$, construct Brezenski's Table $[D]^K$ of intervals for supremums.

4. Check for nestedness[1] of the intervals in Table $[C]^K$. For each nested interval, find its infimum. Form the Table $[C_L]^K$ with these infimums as the corresponding entries. Do likewise for the intervals in $[D]^K$, using the supremum of each nested interval to form the Table $[D_U]^K$.

5. Construct intervals whose lower and upper endpoints are the corresponding entries of $[C_L]^K$ and $[D_U]^K$, respectively. Construct a Table of range approximations $[Range\_approx]^K$ based on these intervals.

6. RETURN $[Range\_approx]^K$.

END Algorithm

5. Numerical Experiments

We test and compare the performance of the proposed technique on several multidimensional examples. The examples considered and the test results are listed in the Appendix.

The range overestimation of the intervals in $[Range\_approx]^K$, and the order of convergence for the same are shown in Tables 2 to 12. The range overestimation of the intervals are computed as

$$\text{Range Overestimation} = \text{wid}\ [Range\_approx]^K - \text{wid}\ f^{range}(\mathbf{x}).$$

In Tables 2 to 12, the 'a' part of the table shows the range overestimation of the intervals in $[Range\_approx]^K$. The first column ($k = 0$) shows the range overestimation of the range enclosures obtained with the NIE and uniform subdivision, whereas the second column ($k = 1$), gives the range overestimation for the first extrapolated column. The subsequent columns in the tables show the range overestimations for extrapolated columns $k = 2, \ldots, K$.

[1] Nestedness is checked columnwise, for consecutive intervals in each column.

The 'b' part of the table shows the order of convergence of the same intervals. A star entry in the tables signifies that the computed quotient value is erratic, because the numerical zero (i.e. zero within machine precision) is reached for the corresponding range overestimations.

The comparison of the number of subdivisions required and the number of boxes generated to achieve the desired accuracy $\varepsilon = 1e - 11$, with the proposed method and with the existing NIE and uniform subdivision method are shown in Tables 13 and 14, respectively[2].

5.1. Discussion

Based on the results in Tables 2 to 12, we make the following observations.

- From the quotient entries in the 'b' part of Tables 2 to 12, we observe that sequences converge columnwise with the order $O\left(\frac{1}{N^K}\right)$. Thus, it seems beneficial to apply extrapolation and accelerate the rate of convergence of the range enclosures obtained with the NIE and the uniform subdivision.
- With the proposed technique, the number of subdivisions required to achieve the desired accuracy are significantly reduced compared to the existing method.
- In all the examples, the intervals in Table $[Range\_approx]^K$ enclose the true function range (these Tables are omitted here for want of space, but are available from the authors).

6. CONCLUSIONS

Summarizing the results of the numerical tests, we see that the proposed technique based on extrapolation works well, and generates range approximating intervals of high accuracy. We see that the number of uniform subdivisions required by the proposed method is significantly less compared to the existing NIE and uniform subdivision method. We also obtain, the significant reductions in the number of generated boxes to achieve the desired accuracy.

However, it should be pointed out that we have also come across several examples where the REP was unsuccessful. For instance, this happened in the example

$$f(x) = 1 - 5x + \frac{1}{3}x^3,\ x \in [2,3]\,.$$

The reason for the same is not yet clear. The range approximating intervals generated by this technique are non-validated intervals, and a technique to rigorously validate the same remains to be found. The theoretical proof for the proposed method is also to be investigated.

[2] In some examples, we have subdivided more than necessary, just to further illustrate that a much higher accuracy can usually be achieved with just one or two additional elements in the Romberg Tables.

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Appendix

EXAMPLE 1. *The 1-dimensional example of Makino and Berz (*Makino, 1998*, Example 1).*

$$f(x) = 1/x + x, \qquad x \in [1.9,\ 2.1]\,.$$

Table 2.

Table 2a

Range overestimation for the function given in Example 1

N	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
2	0.050					
4	0.025					
8	0.013	0.013				
16	$6.29E-3$	$6.29E-3$	$8.07E-5$			
32	$3.14E-3$	$3.14E-3$	$1.99E-5$	$2.49E-7$		
64	$1.67E-3$	$1.67E-3$	$4.98E-6$	$3.12E-8$	$4.01E-10$	
128	$7.87E-4$	$7.87E-4$	$1.24E-6$	$3.91E-9$	$2.49E-11$	$3.10E-13$
256	$3.94E-4$	$3.94E-4$	$3.10E-7$	$4.89E-10$	$1.55E-12$	$9.77E-15$

Table 2b

Quotients of the above entries of Table 2a

N	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
2	1.9963					
4	1.9972					
8	1.9983	1.9983				
16	1.9991	1.9991	4.036			
32	1.9995	1.9995	4.017	7.980		
64	1.9997	1.9997	4.008	7.990	16.104	
128	1.9998	1.9998	4.004	7.995	16.045	31.773
	$O\left(\frac{1}{N}\right)$	$O\left(\frac{1}{N}\right)$	$O\left(\frac{1}{N^2}\right)$	$O\left(\frac{1}{N^3}\right)$	$O\left(\frac{1}{N^4}\right)$	$O\left(\frac{1}{N^5}\right)$

Comments: In the above Table 2a, for the uniform subdivision factor $N = 256$, the second extrapolated column ($k = 2$) gives a reduction in the overestimation by 1271 times (from $3.94E-4$ to $3.10E-7$), whereas in the 5^{th} extrapolated column ($k = 5$) the reduction is $4.03e + 10$ times (from $3.94E-4$ to $9.77E-15$). The rate of convergence of excess width is given in Table 2b. Here, we see that the excess width obtained with the NIE (given in column 2) goes down linearly with $O\left(\frac{1}{N}\right)$. The rate of convergence is accelerated in the subsequent extrapolated columns from $O\left(\frac{1}{N}\right)$ in column 3 to $O\left(\frac{1}{N^5}\right)$ in column 7.

EXAMPLE 2. *The 1-dimensional example of Cornelius and Lohner (*Cornelius and Lohner, 1984, *Example 2).*

$$f(x) = \frac{x+2}{\sqrt{x}}, \qquad x \in [1,3]\,.$$

Table 3.

Table 3a

Range overestimation for the function given in Example 2

N	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
4	0.88						
8	0.43	0.43					
16	0.22	0.22					
32	0.11	0.11	$8.24E-4$				
64	0.05	0.05	$1.88E-4$	$5.78E-6$			
128	0.03	0.03	$4.51E-5$	$6.98E-7$	$3.42E-8$		
256	0.01	0.01	$1.10E-5$	$8.58E-8$	$1.87E-9$	$7.90E-11$	
512	$6.67E-3$	$6.67E-3$	$2.73E-6$	$1.06E-8$	$1.10E-10$	$2.36E-12$	$1.43E-13$
1024	$3.33E-3$	$3.33E-3$	$6.78E-7$	$1.32E-9$	$6.64E-12$	$7.19E-14$	$2.22E-15$

Table 3b

Quotients of the above entries of Table 3a

N	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
4	2.031						
8	2.014	2.014					
16	2.006	2.006					
32	2.003	2.003	4.38				
64	2.001	2.001	4.18	8.29			
128	2.0008	2.0008	4.09	8.14	18.25		
256	2.0004	2.0004	4.04	8.07	17.07	33.52	
512	2.0002	2.0002	4.02	8.03	16.52	32.77	64
	$O\left(\frac{1}{N}\right)$	$O\left(\frac{1}{N}\right)$	$O\left(\frac{1}{N^2}\right)$	$O\left(\frac{1}{N^3}\right)$	$O\left(\frac{1}{N^4}\right)$	$O\left(\frac{1}{N^5}\right)$	$O\left(\frac{1}{N^6}\right)$

Comments: In the above Table 3a, for the uniform subdivision factor $N = 1024$, the second extrapolated column ($k = 2$) gives a reduction in the overestimation by 4912 times (from $3.33E-3$ to $6.78E-7$), whereas in the 6^{th} extrapolated column ($k = 6$) the reduction is $1.50e+12$ times (from $3.33E-3$ to $2.22E-15$). The rate of convergence of excess width is given in Table 3b. Here, we see that the excess width obtained with the NIE (given in column 2) goes down linearly with $O\left(\frac{1}{N}\right)$. The rate of convergence is accelerated in the subsequent extrapolated columns from $O\left(\frac{1}{N}\right)$ in column 3 to $O\left(\frac{1}{N^6}\right)$ in column 8.

EXAMPLE 3. *The 1-dimensional example of Costabile et. al. (*Costabile and Serra, 1996*, Example 1).*

$$f(x) = \sin(x)\cos(x), \qquad x \in [0, 0.5]\,.$$

Table 4.

Table 4a

Range overestimation for the function given in Example 3

N	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
2	0.044	0.044					
4	0.025	0.025					
8	0.014	0.014					
16	$6.98E-3$	$6.98E-3$	$4.18E-4$				
32	$3.54E-3$	$3.54E-3$	$1.04E-4$	$9.73E-7$			
64	$1.78E-3$	$1.78E-3$	$2.58E-5$	$1.34E-7$	$4.34E-9$		
128	$8.95E-4$	$8.95E-4$	$6.43E-6$	$1.75E-8$	$2.66E-10$	$1.72E-12$	
256	$4.48E-4$	$4.48E-4$	$1.61E-6$	$2.24E-9$	$1.65E-11$	$5.47E-14$	$1.06E-15$

Table 4b

Quotients of the above entries of Table 4a

N	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
2	1.73						
4	1.87						
8	1.94	1.94					
16	1.97	1.97	4.032				
32	1.99	1.99	4.017	7.26			
64	1.99	1.99	4.008	7.65	16.29		
128	2.00	2.00	4.004	7.83	16.15	31.42	–
	$O\left(\frac{1}{N}\right)$	$O\left(\frac{1}{N}\right)$	$O\left(\frac{1}{N^2}\right)$	$O\left(\frac{1}{N^3}\right)$	$O\left(\frac{1}{N^4}\right)$	$O\left(\frac{1}{N^5}\right)$	

Comments: In the above Table $4a$, for the uniform subdivision factor$N = 256$, the second extrapolated column ($k = 2$) gives a reduction in the overestimation by 278 times (from $4.48E-4$ to $1.61E-6$), whereas in the 6^{th} extrapolated column ($k = 6$) the reduction is $4.23e+11$ times (from $4.48E-4$ to $1.06E-15$). The rate of convergence of excess width is given in Table $4b$. Here, we see that the excess width obtained with the NIE (given in column 2) goes down linearly with $O\left(\frac{1}{N}\right)$. The rate of convergence is accelerated in the subsequent extrapolated columns from $O\left(\frac{1}{N}\right)$ in column 3 to $O\left(\frac{1}{N^5}\right)$ in column 7.

EXAMPLE 4. *The 2-dimensional example of Asaithambi et al. (*Asaithambi and Moore, , *Example 2).*

$$f(x) = x_1\,(1-x_1)\left(1-\frac{5}{8}x_2+\frac{3}{2}x_2^2-x_2^3\right), \qquad x_1\in[-1,1]\,,\; x_2\in[0,1]\,.$$

Table 5.

Table 5a

Range overestimation for the function given in Example 4

N	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$
16	0.30	0.30			
32	0.09	0.09			
64	0.016	0.016			
128	$8.02E-3$	$8.02E-3$			
256	$3.96E-3$	$3.96E-3$	$1.05E-4$		
512	$1.97E-3$	$1.97E-3$	$2.60E-5$	$8.94E-8$	
1024	$9.80E-4$	$9.80E-4$	$6.47E-6$	$1.12E-8$	0.0
2048	$4.89E-4$	$4.89E-4$	$1.61E-6$	$1.40E-9$	0.0
4096	$2.44E-4$	$2.44E-4$	$4.03E-7$	$1.75E-10$	0.0

Table 5b

Quotients of the above entries of Table 5a

N	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$
16	3.428	3.428			
32	5.381	5.381			
64	2.052	2.052			
128	2.026	2.026			
256	2.013	2.013	4.038		
512	2.007	2.007	4.019	8.0	∞
1024	2.003	2.003	4.010	8.0	∞
2048	2.002	2.002	4.005	8.0	∞
	$O\left(\frac{1}{N}\right)$	$O\left(\frac{1}{N}\right)$	$O\left(\frac{1}{N^2}\right)$	$O\left(\frac{1}{N^3}\right)$	

Comments: In the above Table 5a, for the uniform subdivision factor $N = 4096$, the second extrapolated column ($k = 2$) gives a reduction in the overestimation by 606 times (from $2.44E-4$ to $4.03E-7$), whereas in the 4^{th} extrapolated column ($k = 4$) we obtain the true function range. The rate of convergence of excess width is given in Table 5b. Here, we see that the excess width obtained with the NIE (given in column 2) goes down linearly with $O\left(\frac{1}{N}\right)$. The rate of convergence is accelerated in the subsequent extrapolated columns from $O\left(\frac{1}{N}\right)$ in column 3 to $O\left(\frac{1}{N^3}\right)$ in column 5.

EXAMPLE 5. *The 2-dimensional three-hump camel back function example of Asaithambi et al. (*Asaithambi and Moore, , *Example 4).*

$$f(x) = 2x_1^2 - 1.05x_1^4 + \frac{1}{6}x_1^6 - x_1x_2 + x_2^2, \qquad x_1 \in [-2,4]\,,\ x_2 \in [-2,4]\,.$$

Table 6.

Table 6a

Range overestimation for the function given in Example 5

N	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$
128	13.45	13.45			
256	6.581	6.581			
512	3.143	3.143			
1024	1.572	1.572			
2048	0.787	0.787	$6.19E-3$		
4096	0.394	0.394	$4.39E-4$	$4.24E-7$	
8192	0.197	0.197	$1.10E-4$	$5.29E-8$	$4.23E-5$
16384	0.098	0.098	$2.74E-5$	$6.61E-9$	$1.06E-11$
32768	0.049	0.049	$6.86E-6$	$8.26E-10$	$6.97E-13$

Table 6b

Quotients of the above entries of Table 6a

N	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$
128	2.043	2.043			
256	2.094	2.094			
512	1.999	1.999			
1024	1.997	1.997			
2048	1.998	1.998	14.13		
4096	1.999	1.999	3.998	8.017	
8192	1.999	1.999	3.999	8.009	3998812
16384	1.999	1.999	3.999	8.004	15.18
	$O\left(\frac{1}{N}\right)$	$O\left(\frac{1}{N}\right)$	$O\left(\frac{1}{N^2}\right)$	$O\left(\frac{1}{N^3}\right)$	$O\left(\frac{1}{N^4}\right)$

Comments: In the above Table 6a, for the uniform subdivision factor $N = 32768$, the second extrapolated column ($k = 2$) gives a reduction in the overestimation by 7143 times (from 0.049 to $6.86E-6$), whereas in the 4^{th} extrapolated column ($k = 4$) the reduction is $7.03e + 10$ times (from 0.049 to $6.97E - 13$). The rate of convergence of excess width is given in Table 6b. Here, we see that the excess width obtained with the NIE (given in column 2) goes down linearly with $O\left(\frac{1}{N}\right)$. The rate of convergence is accelerated in the subsequent extrapolated columns from $O\left(\frac{1}{N}\right)$ in column 3 to $O\left(\frac{1}{N^4}\right)$ in column 6.

EXAMPLE 6. *The 2-dimensional exponential function of Moore (*Moore, 1979*, pp. 45).*

$$f(x) = x_1 \exp\left(x_1 + x_1^2\right) - x_2^2, \qquad x_1 \in [1,2]\,,\ x_2 \in [0,1]\,.$$

Table 7.

Table7a

Range overestimation for the function given in Example 6

N	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
4	0.51						
8	0.25	0.25					
16	0.13	0.13	0.015				
32	0.06	0.06	$4.29E-3$				
64	0.03	0.03	$1.10E-3$	$1.50E-5$			
128	0.02	0.02	$2.78E-4$	$9.34E-7$	$3.25E-7$		
256	$7.81E-3$	$7.81E-3$	$6.96E-5$	$5.83E-8$	$2.03E-8$	$1.58E-10$	
512	$3.91E-3$	$3.91E-3$	$1.74E-5$	$3.64E-9$	$1.27E-9$	$2.46E-12$	$8.34E-13$
1024	$1.95E-3$	$1.95E-3$	$4.35E-6$	$2.27E-10$	$7.91E-11$	$4.44E-14$	$1.91E-14$

Table 7b

Quotients of the above entries of Table 7a

N	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
4	2.0278						
8	2.0151	2.0151					
16	2.0085	2.0085	3.528				
32	2.0044	2.0044	3.888				
64	2.0022	2.0022	3.972	16.112			
128	2.0011	2.0011	3.993	16.028	16.037		
256	2.0005	2.0005	3.998	16.007	16.009	64.14	
512	2.0002	2.0002	3.999	16.001	16.001	55.37	*
	$O\left(\frac{1}{N}\right)$	$O\left(\frac{1}{N}\right)$	$O\left(\frac{1}{N^2}\right)$	$O\left(\frac{1}{N^3}\right)$	$O\left(\frac{1}{N^4}\right)$	$O\left(\frac{1}{N^5}\right)$	

Comments: In the above Table 7a, for the uniform subdivision factor $N = 1024$, the second extrapolated column ($k = 2$) gives a reduction in the overestimation by 448 times (from $1.95E-3$ to $4.35E-6$), whereas in the 6^{th} extrapolated column ($k = 6$) the reduction is$1.02e+11$ times (from $1.95E-3$ to $1.91e-14$). The rate of convergence of excess width is given in Table 7b. Here, we see that the excess width obtained with the NIE (given in column 2) goes down linearly with $O\left(\frac{1}{N}\right)$. The rate of convergence is accelerated in the subsequent extrapolated columns from $O\left(\frac{1}{N}\right)$ in column 3 to $O\left(\frac{1}{N^5}\right)$ in column 7.

EXAMPLE 7. *The 3-dimensional function of Makino (*Makino and Berz, 2003*, pp. 403).*

$$f(x,y,z) = \frac{4\tan(3y)}{3x + x\sqrt{\frac{6x}{-7(x-8)}}} - 120 - 2x - 7z(1+2y) - \sinh(0.5 + \frac{6y}{8y+7}) + \frac{(3y+13)^2}{3z}$$
$$-20z(2z-5) + \frac{5x\tanh(0.9z)}{\sqrt{5y}} - 20y\sin(3z),$$
$$x_1 \in [1.75, 2.25]\,,\ x_2 \in [0.75, 1.25]\,,\ x_3 \in [0.75, 1.25]\,.$$

Table 8.

Table 8a

Range overestimation for the function given in Example 7

N	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
8	16.5	16.5					
16	8.16	8.16					
32	4.06	4.06					
64	2.03	2.03	$8.71e-3$	$7.54e-4$			
128	1.01	1.01	$2.43e-3$	$9.32e-5$	$5.68e-7$	$1.06e-7$	
256	0.51	0.51	$6.39e-4$	$1.16e-5$	$3.63e-8$	$3.32e-9$	$2.07e-11$
512	0.25	0.25	$1.64e-4$	$1.44e-6$	$2.29e-9$	$1.04e-10$	$4.67e-13$

Table 8b

Quotients of the above entries of Table 8a

N	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
8	2.035						
16	2.024						
32	2.017	2.017					
64	2.009	2.009					
128	2.005	2.005	3.580	8.092			
256	2.003	2.003	3.802	8.061	15.647	31.9162	
512	2.0007	2.0007	3.903	8.034	15.851	31.9161	44.202
	$O\left(\frac{1}{N}\right)$	$O\left(\frac{1}{N}\right)$	$O\left(\frac{1}{N^2}\right)$	$O\left(\frac{1}{N^3}\right)$	$O\left(\frac{1}{N^4}\right)$	$O\left(\frac{1}{N^5}\right)$	$O\left(\frac{1}{N^6}\right)$

Comments: In the above Table 8a, for the uniform subdivision factor $N = 512$, the second extrapolated column ($k = 2$) gives a reduction in the overestimation by 1524.4 times (from 0.25 to $1.64e-4$), whereas in the 6^{th} extrapolated column ($k = 6$) the reduction is $5.35e+11$ times (from 0.25 to $4.67e-13$). The rate of convergence of excess width is given in Table 8b. Here, we see that the excess width obtained with the NIE (given in column 2) goes down linearly with $O\left(\frac{1}{N}\right)$. The rate of convergence is accelerated in the subsequent extrapolated columns from $O\left(\frac{1}{N}\right)$ in column 3 to $O\left(\frac{1}{N^6}\right)$ in column 8.

EXAMPLE 8. *The 4-dimensional trigonometric function of More et al. (*More and Hillstrom, , *Example 26).*

$$f(x) = \sum_{i=1}^{4} f_i(x)^2\,,\;\; f_i(x) = 4 - \sum_{j=1}^{4} \cos x_j + i\,(1-\cos x_i) - \sin x_i, \qquad x_i \in [0.75, 2.75]^4\,.$$

Table 9.

Table 9a

Range overestimation for the function given in Example 8

N	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
16	0.70		0.15			
32	0.37	0.37	0.04	$2.17E-3$		
64	0.19	0.19	$9.66E-3$	$2.56E-4$		
128	0.10	0.10	$2.44E-3$	$3.08E-5$		
256	0.05	0.05	$6.12E-4$	$3.76E-6$	$7.75E-8$	$1.35E-8$
512	0.02	0.02	$1.53E-4$	$4.63E-7$	$6.08E-9$	$4.22E-10$
1024	0.01	0.01	$3.84E-5$	$5.75E-8$	$4.19E-10$	$1.34E-11$
2048	$6.07E-3$	$6.07E-3$	$9.60E-6$	$7.17E-9$	$2.76E-11$	$6.42E-13$
4096	$3.04E-3$	$3.04E-3$	$2.40E-6$	$8.95E-10$	$1.98E-12$	$2.50E-13$

Table 9b

Quotients of the above entries of Table 9a

N	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
16	1.897		3.827			
32	1.949	1.949	3.920	8.465		
64	1.974	1.974	3.962	8.334		
128	1.987	1.987	3.981	8.195		
256	1.993	1.993	3.990	8.105	12.75	31.92
512	1.996	1.996	3.995	8.054	14.52	31.59
1024	1.998	1.998	3.997	8.027	15.18	*
2048	1.999	1.999	3.998	8.012	13.91	*
	$O\left(\frac{1}{N}\right)$	$O\left(\frac{1}{N}\right)$	$O\left(\frac{1}{N^2}\right)$	$O\left(\frac{1}{N^3}\right)$	$O\left(\frac{1}{N^4}\right)$	$O\left(\frac{1}{N^5}\right)$

Comments: In the above Table 9a, for the uniform subdivision factor $N = 4096$, the second extrapolated column ($k = 2$) gives a reduction in the overestimation by 1266 times (from $3.04E-3$ to $2.40E-6$), whereas in the 5^{th} extrapolated column ($k = 5$) the reduction is $1.22e+10$ times (from $3.04E-3$ to $2.50E-13$). The rate of convergence of excess width is given in Table 9b. Here, we see that the excess width obtained with the NIE (given in column 2) goes down linearly with $O\left(\frac{1}{N}\right)$. The rate of convergence is accelerated in the subsequent extrapolated columns from $O\left(\frac{1}{N}\right)$ in column 3 to $O\left(\frac{1}{N^5}\right)$ in column 7.

EXAMPLE 9. *The 5-dimensional Griewank function of Ratz and Csendes (*Ratz and Csendes, 1995*, pp. 205).*

$$f(x) = \sum_{i=1}^{5} \frac{x_i^2}{400} - \prod_{i=1}^{5} \cos\left(\frac{x_i}{\sqrt{i}}\right) + 1, \qquad x_i \in [-601, -599]^5 .$$

Table 10.

Table 10a

Range overestimation for the function given in Example 9

N	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$
4	0.14				
8	0.08				
16	0.04	0.04			
32	0.02	0.02			
64	$9.72E-3$	$9.72E-3$	$4.31E-5$		
128	$4.86E-3$	$4.86E-3$	$7.68E-6$	$4.103E-6$	$5.06E-8$
256	$2.43E-3$	$2.43E-3$	$1.53E-6$	$5.16E-7$	$1.47E-9$
512	$1.22E-3$	$1.22E-3$	$3.35E-7$	$6.46E-8$	$4.64E-11$

Table 10b

Quotients of the above entries of Table 10a

N	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$
4	1.926				
8	1.952				
16	1.986	1.986			
32	1.995	1.995			
64	1.998	1.998	5.61		
128	1.999	1.999	5.01	7.95	34.33
256	1.999	1.999	4.58	7.99	31.76
	$O\left(\frac{1}{N}\right)$	$O\left(\frac{1}{N}\right)$	$O\left(\frac{1}{N^2}\right)$	$O\left(\frac{1}{N^3}\right)$	$O\left(\frac{1}{N^4}\right)$

Comments: In the above Table 10a, for the uniform subdivision factor $N = 512$, the second extrapolated column ($k = 2$) gives a reduction in the overestimation by 3642 times (from $1.22E-3$ to $3.35E-7$), whereas in the 4^{th} extrapolated column ($k = 4$) he reduction is $2.63e+7$ times (from $1.22E-3$ to $4.64E-11$). The rate of convergence of excess width is given in Table 10b. Here, we see that the excess width obtained with the NIE (given in column 2) goes down linearly with $O\left(\frac{1}{N}\right)$. The rate of convergence is accelerated in the subsequent extrapolated columns from $O\left(\frac{1}{N}\right)$ in column 3 to $O\left(\frac{1}{N^4}\right)$ in column 6.

EXAMPLE 10. *The 6-dimensional trigonometric function of More et al. (*More and Hillstrom, , *Example 26).*

$$f(x) = \sum_{i=1}^{6} f_i(x)^2, \; f_i(x) = 6 - \sum_{j=1}^{6} \cos x_j + i(1 - \cos x_i) - \sin x_i, \qquad x_i \in [0.75, 2.75]^6 .$$

Table 11.

Table 11a

Range overestimation for the function given in Example 10

N	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
4	5.56					
8	3.43					
16	1.88		0.33			
32	0.98	0.98	0.08	$1.21E-3$		
64	0.50	0.50	0.02	$1.12E-4$	$6.53E-6$	
128	0.25	0.25	$5.29E-3$	$1.11E-5$	$2.11E-6$	
256	0.13	0.13	$1.32E-3$	$1.20E-6$	$1.87E-7$	$1.87E-8$
512	0.06	0.06	$3.31E-4$	$1.37E-7$	$1.34E-8$	$5.82E-10$
1024	0.03	0.03	$8.28E-5$	$1.63E-8$	$8.90E-10$	$1.92E-11$

Table 11b

Quotients of the above entries of Table 11a

N	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
4	1.622					
8	1.822					
16	1.914		3.957			
32	1.957	1.957	3.984	10.74		
64	1.979	1.979	3.994	10.07	03.09	
128	1.989	1.989	3.997	09.31	11.32	
256	1.994	1.994	3.999	08.74	13.96	32.06
512	1.997	1.997	3.999	08.40	15.03	30.32
	$O\left(\frac{1}{N}\right)$	$O\left(\frac{1}{N}\right)$	$O\left(\frac{1}{N^2}\right)$	$O\left(\frac{1}{N^3}\right)$	$O\left(\frac{1}{N^4}\right)$	$O\left(\frac{1}{N^5}\right)$

Comments: In the above Table 11a, for the uniform subdivision factor $N = 1024$, the second extrapolated column ($k = 2$) gives a reduction in the overestimation by 362 times (from 0.03 to $8.28E-5$), whereas in the 5^{th} extrapolated column ($k = 5$) the reduction is $1.56e+9$ times (from 0.03 to $1.92E-11$). The rate of convergence of excess width is given in Table 11b. Here, we see that the excess width obtained with the NIE (given in column 2) goes down linearly with $O\left(\frac{1}{N}\right)$. The rate of convergence is accelerated in the subsequent extrapolated columns from $O\left(\frac{1}{N}\right)$ in column 3 to $O\left(\frac{1}{N^5}\right)$ in column 7.

EXAMPLE 11. *The 3-dimensional non-rational example of Horowitz (*Horowitz, 1993*, pp. 129). The magnitude function for the non-rational system is*

$$f(x) = -10\log_{10}\{1 + x_2(x_2 + 2\cos 2x_1)\}, \quad x_1 \in [1,2],\ x_2 \in [0.4, 0.6],\ x_3 \in [0.01, 0.02].$$

Table 12.

Table 12a

Range overestimation for the function given in Example 11

N	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$
4	0.959				
8	0.457	0.457			
16	0.223	0.223	0.014		
32	0.110	0.110	$2.85E-3$	$1.69E-4$	
64	0.055	0.055	$6.53E-4$	$1.98E-5$	$2.06E-6$
128	0.027	0.027	$1.56E-4$	$2.40E-6$	$1.02E-7$
256	0.013	0.013	$3.83E-5$	$2.96E-7$	$5.72E-9$
512	6.824	6.824	$9.47E-6$	$3.67E-8$	$3.38E-10$
1024	$3.41E-3$	$3.41E-3$	$2.35E-6$	$4.57E-9$	$2.08E-11$

Table 12b

Quotients of the above entries of Table 12a

N	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$
4	2.1019				
8	2.0469	2.0469			
16	2.0226	2.0226	4.792		
32	2.0111	2.0111	4.366	8.534	
64	2.0055	2.0055	4.177	8.256	20.11
128	2.0027	2.0027	4.087	8.125	17.89
256	2.0013	2.0013	4.043	8.062	16.90
512	2.0006	2.0006	4.021	8.031	16.27
	$O\left(\frac{1}{N}\right)$	$O\left(\frac{1}{N}\right)$	$O\left(\frac{1}{N^2}\right)$	$O\left(\frac{1}{N^3}\right)$	$O\left(\frac{1}{N^4}\right)$

Comments: In the above Table 12a, for the uniform subdivision factor $N = 1024$, the second extrapolated column ($k = 2$) gives a reduction in the overestimation by 1451 times (from $3.41E-3$ to $2.35E-6$), whereas in the 4^{th} extrapolated column ($k = 4$) the reduction is $1.64e+8$ times (from $3.41E-3$ to $2.08E-11$). The rate of convergence of excess width is given in Table 12b. Here, we see that the excess width obtained with the NIE (given in column 2) goes down linearly with $O\left(\frac{1}{N}\right)$. The rate of convergence is accelerated in the subsequent extrapolated columns from $O\left(\frac{1}{N}\right)$ in column 3 to $O\left(\frac{1}{N^4}\right)$ in column 6.

Table 13. Comparison of the number of uniform subdivisions required to achieve a range accuracy of $1e-11$ with the existing and proposed methods

Example Named	l	No. of Subdivisions Required	
		Existing method	Proposed method
1	1	37	7
2	1	37	8
3	1	37	7
4	2	37	10
5	2	51	14
6	2	41	9
7	3	46	8
8	4	44	10
9	5	41	9
10	6	51	10
11	3	41	10

Table 14. Comparison of the number of boxes processed to achieve a range accuracy of $1e-11$ with the existing and proposed methods

Example Named	l	No. of Subboxes Generated	
		Existing method	Proposed method
1	1	$1.37e+11$	128
2	1	$1.37e+11$	256
3	1	$1.37e+11$	128
4	2	$2.75e+11$	2048
5	2	$4.50e+15$	32768
6	2	$4.40e+12$	1024
7	3	$2.11e+14$	768
8	4	$7.04e+13$	4096
9	5	$1.10e+13$	2560
10	6	$1.35e+16$	6144
11	3	$6.60e+12$	3072