# A method for outer interval solution of parametrized systems of linear equations

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**Abstract.** Consider the systems of linear interval equations whose coefficients are affine-linear functions of interval parameters. Such systems, called parametrized systems of linear interval equations, are encountered in many practical problems, e.g in structure mechanics. A direct method for computing a tight enclosure for the solution set is proposed in this paper. It is proved that for systems with real matrix and interval right-hand vector the method generates the hull of the solution set. For such systems an explicit formula for the hull is also given. Finally some numerical examples are provided to demonstrate the usefulness of the method in structure mechanics.

**Keywords:** parametrization, intervals, linear equations, truss structures

## 1. Introduction

A system of linear interval equations

$$[A]x = [b] \tag{1}$$

with coefficient matrix $[A] \in \mathbb{IR}^{n\times n}$ and right-hand vector $[b] \in \mathbb{IR}^n$ is defined as a family of linear equations

$$Ax = b,\ (A \in [A],\, b \in [b]). \tag{2}$$

The solution set of (1) is given by

$$\sum\Big([A],\, [b]\Big) = \{x \mid Ax = b,\ A \in [A],\ b \in [b]\}. \tag{3}$$

When computing inner and outer bounds for the solution set (3) it is implicitly assumed $A$ and $b$ to vary independently within $[A]$ and $[b]$. In practice there might be further constraints on matrices within $[A]$ and $[b]$. Taking into account these contraints leads to the parametrized systems of linear interval equations. Consider the family of linear algebraic systems of the following type

$$A(p)x = b(p), \tag{4}$$

with

$$A_{ij}(p) = \omega(i,\, j)^{\mathrm{T}} p \tag{5a}$$

$$b_j(p) = \omega(0,\, j)^{\mathrm{T}} p \tag{5b}$$

and $p \in [p] \in \mathbb{IR}^k$ [6]. Such systems are encountered in many practical applications, e.g. in structure mechanics [3], [7].

The family of systems (4) is usually written in the form

$$A([p])x = b([p]) \tag{6}$$

and is called parametrized system of linear interval equations.

The (united) solution set of the system (6) is defined as

$$\sum\Big(A([p]),\, b([p])\Big) = \{\, x \mid A(p)x = b(p),\, p \in [p]\} \tag{7}$$

If the solution set is bounded then the interval hull for it exists. In order to guarantee that the solution set is bounded matrix $A([p])$ must be regular (for all $p \in [p]$ $A(p)$ is regular). In practice it is usually required that the matrix $A([p])$ is an H-matrix.

In this paper a direct method for computing a tight enclosure for (7) is proposed. The method is based on the following inclusion

$$\Diamond\Big(\sum\left(A([p]),\, b([p])\right)\Big) \subseteq \tilde{x} + \langle [D]\rangle |Z| [-1,\, 1] \tag{8}$$

where

$$[Z]_i = \sum_{j=1}^{n} R_{ij}\Big(\omega(0,\, j) - \sum_{k=1}^{n} \tilde{x}_k \omega(j,\, k)\Big)^{\mathrm{T}}[p], \tag{9}$$

$$[D]_{ij} = \left(\sum_{k=1}^{n} R_{ik}\omega(k,\, j)\right)^{\mathrm{T}}[p], \tag{10}$$

$R = \mathrm{mid}\Big(A([p])\Big)^{-1}$ and $\tilde{x} = R\check{b}$.

It is proved that for systems with $\mathrm{rad}\Big(A([p])\Big) = 0$ the inclusion in (8) is an equality. For such systems an explicit formula for the hull of the solution set (7) is also given.

Finally some numerical examples of truss structures are provided to demonstrate the usefulness of the method in structure mechanics.

## 2. Basic notion

By $\mathbb{IR}$, $\mathbb{IR}^n$, $\mathbb{IR}^{n\times n}$ denote the set of real compact intervals, respectively interval vectors with $n$ components and the set of interval $n \times n$ matrices.

For interval $[a] = [\underline{a},\, \overline{a}] = \{x \mid \underline{a} \le x \le \overline{a}\}$ define the midpoint

$$\check{a} = \mathrm{mid}([a]) = (\underline{a} + \overline{a})/2$$

the radius

$$\mathrm{rad}([a]) = (\overline{a} - \underline{a})/2$$

and minimal absolute value (mignitude)

$$\langle [x]\rangle = \mathrm{mig}([x]) = \min\{|x| \mid x \in [x]\}.$$

An interval matrix $[A] \in \mathbb{IR}^{n\times n}$ is interpreted as a set of real $n \times n$ matrices

$$[A] = \{A \in \mathbb{R}^{n\times n} \mid A_{ij} \in [A]_{ij},\ i, j = 1, \dots, n\}$$

An $n \times 1$ matrix is just an interval vector. In analogy to one-dimensional case certain real matrices are related to each interval matrix. Middle matrix $\mathrm{mid}([A])$ and the radius $\mathrm{rad}([A])$ are computed componentwise. For square interval matrices an Ostrowsky matrix $\langle[A]\rangle$ is defined with entries

$$\langle[A]\rangle_{ij} = \mathrm{mig}([A]_{ij}),\ i \neq j$$
$$\langle[A]\rangle_{ij} = -|[A]_{ij}|,\ i = j.$$

A square matrix $[A] \in \mathbb{IR}^{n\times n}$ is called regular if all $A \in [A]$ are nonsingular. If $\check{A}[A]$ is regular then $[A]$ is strongly regular.

An interval matrix $[A]$ is an H-matrix iff there exist a vector $u > 0$ such that

$$\langle[A]\rangle u > 0.$$

If $S$ is a bounded set of real matrices then $\inf S$ and $\sup S$ exist, and the *hull* of S,

$$\diamondsuit S = [\inf S, \sup S] = \diamondsuit S = \bigcap\{[Y] \mid [Y] \in \mathbb{IR},\ [Y] \supseteq S\}$$

is the tightest interval matrix enclosing $S$.

## 3. Minimal enclosure

In case of parametrized systems with real matrices, $\mathrm{rad}\Big(A([p])\Big) = 0$, the hull of the solution set (7) is given by an explicit formula.

THEOREM 1. *Let $A([p])x = b([p])$, $[p] \in \mathbb{IR}^k$ $R = mid\Big(A([p])\Big)$ and $\tilde{x} = R \cdot mid\Big(b([p])\Big)$. If $rad\Big(A([p])\Big) = 0$ then*

$$\diamondsuit\Big(\sum(A([p]),\ b([p]))\Big) = \tilde{x} + [Z]',$$

*where*

$$[Z]'_i = \sum_{j=1}^{n}\Big(R_{ij}\cdot\omega(0,\ j)\Big)^T[-rad([p]),\ rad([p])]. \tag{11}$$

*Proof.* Since $\mathrm{rad}([A]) = 0$, hence $A([p]) = A$, $\check{A} = A$, $R = A^{-1}$ $(RA = I)$ and $\tilde{x} = A^{-1}\check{b}$ $(A\tilde{x} = \check{b})$. Then one has

$$\begin{aligned}\diamondsuit\Big(\sum(A([p]),\ b([p]))\Big) &= \diamondsuit\Big(\sum(A,\ b([p]))\Big) = \\ &= \tilde{x} + \diamondsuit\Big(\sum(A,\ b([p]) - A\tilde{x})\Big) = \\ &= \tilde{x} + \diamondsuit\Big(\sum(A,\ b([p]) - \check{b})\Big) = \\ &= \tilde{x} + \diamondsuit\Big(\{R(b(p) - \check{b}),\ p \in [p]\}\Big).\end{aligned}$$

$$\diamondsuit\Big(\{R(b(p)-\check{b}),\ p\in[p]\}\Big)_i = \diamondsuit\Big\{\sum_{j=1}^{n} R_{ij}(b(p)-\check{b})_j,\ p\in[p]\Big\} =$$

$$= \diamondsuit\Big\{\sum_{j=1}^{n} R_{ij}(\omega(0,\,j)^{\mathrm{T}}\cdot p - \omega(0,\,j)^{\mathrm{T}}\cdot \check{p}),\ p\in[p]\Big\} =$$

$$= \diamondsuit\Big\{\sum_{j=1}^{n} (R_{ij}\cdot\omega(0,\,j))^{\mathrm{T}}(p-\check{p}),\ p\in[p]\Big\} =$$

$$= \Big(\sum_{j=1}^{n} (R_{ij}\cdot\omega(0,\,j))^{\mathrm{T}}\Big)([p]-\check{p}) =$$

$$= \Big(\sum_{j=1}^{n} (R_{ij}\cdot\omega(0,\,j))^{\mathrm{T}}\Big)[-\mathrm{rad}([p]),\,\mathrm{rad}([p])].$$

The equality before the last one holds since every component $p_i$ occurs at most once in the preceding expression. □

## 4. Main result

Most of the methods for enclosing the solution set of parametized systems of equations are iterative [1], [2], [5], [6]. However, each iteration enlarges the enclosure because of the roundings has to be made in arithmetic operations. The method based on the formula (8) has polynomial complexity and computes the enclosure of the solution set (7) in one step, and hence has a great advantage over the iterative methods. In what follows the theoretical background for the method is presented.

THEOREM 2 (Neumaier [4]). *Let $[A]\in\mathbb{IR}^{n\times n}$. If $[A]$ is an H-matrix then for all $[b]\in\mathbb{IR}^n$ holds*

$$\diamondsuit \sum\Big([A],\,[b]\Big) \subseteq \langle[A]\rangle^{-1}[b][-1,\,1].$$

THEOREM 3. *Let $A([p])x = b([p])$ with $[p]\in\mathrm{IR}^k$, $R\in\mathbb{R}^{n\times n}$, and $\tilde{x}\in\mathbb{R}^n$. If $[D]$ given by formula (10) is an H-matrix then*

$$\diamondsuit\Big(\sum\Big(A([p]),\,b([p])\Big)\Big) \subseteq \tilde{x} + \langle[D]\rangle^{-1}|[Z]|[-1,\,1], \qquad (12)$$

*where $[Z]$ is defined by formula (9).*

*Proof.* Vector $x\in\sum(A([p]),\,b([p]))$ iff there exists such $p\in[p]$ that $A(p)x=b(p)$. Since $[D]$ is an H-matrix then both sides of this equality can be multipled by $A(p)^{-1}$. Hence

$$x = A(p)^{-1}b(p) = \tilde{x} + A(p)^{-1}(b(p)-A(p)\tilde{x}) =$$
$$= \tilde{x} + (R\cdot A(p))^{-1}(R(b(p)-A(p)\tilde{x})).$$

Since $R \cdot A(p) \in [D]$, $R(b(p) - A(p)\tilde{x}) \in [Z]$ then the following relation holds

$$(R \cdot A(p))^{-1}(R(b(p) - A(p)\tilde{x})) \in \Diamond\Big(\sum([D],\, [Z])\Big),$$

and hence

$$x \in \tilde{x} + \Diamond\Big(\sum([D],\, [Z])\Big). \tag{13}$$

Matrix $[D]$ is an H-matrix then by theorem 2

$$\Diamond\Big(\sum([D],\, [Z])\Big) \subseteq \langle[D]\rangle^{-1}|[Z]|[-1,\, 1]. \tag{14}$$

Equations (13) and (14) gives the thesis of the theorem. □

It is recomended to choose

$$R = \mathrm{mid}(A([p]))^{-1}$$

and

$$\tilde{x} = \mathrm{mid}(A([p]))^{-1} \cdot \mathrm{mid}(b([p]))$$

so that $[D]$ and $[Z]$ are of small norms (see theorem 4.1.10 [4]).

THEOREM 4. *Let $A([p])x = b([p])$, $[p] \in IR^k$ and $R = \check{A}^{-1}$, $\tilde{x} = R\check{b}.i$ If $rad(A([p])) = 0$ then*

$$\Diamond\Big(\sum\,(A([p]),\, b([p]))\Big) = \tilde{x} + \langle[D]\rangle^{-1}|[Z]|[-1,\, 1].$$

*where $[D]$ and $[Z]$ are given respectively by formula (10) and (9).*

*Proof.* To prove the theorem it suffices to show that

$$\tilde{x} + \langle[D]\rangle^{-1}|[Z]|[-1,\, 1] = \tilde{x} + [Z]'$$

where $[Z]'$ is given by (11).

Since $\mathrm{rad}(A([p])) = 0$, hence $R = \check{A}^{-1} = A^{-1}$ and then matrix $[D]$ and vector $[Z]$ takes the simpler form. $[D] = I$ and

$$[Z]_i = \Diamond\{R(b(p) - A\tilde{x}),\, p \in [p]\}_i = \Diamond\{R(b(p) - \check{b}),\, p \in [p]\}_i = [Z]'_i.$$

Hence

$$\tilde{x} + \langle[D]\rangle^{-1}|[Z]|[-1,\, 1] = \tilde{x} + |Z'|[-1,\, 1]. \tag{15}$$

Let now $\alpha = \sum_{j=1}^{n}(R_{ij} \cdot \omega(0,\, j))$. Then by symmetry of the interval $[-\mathrm{rad}([p]),\, \mathrm{rad}([p])]$ holds

$$\{|Z|'[-1,\, 1]\}_i = |\sum_{k=1}^{n}\alpha_k[-\mathrm{rad}([p]_k),\, \mathrm{rad}([p]_k)]|[-1,\, 1] =$$

$$|[-\sum_{k=1}^{n}\alpha_k\mathrm{rad}([p]_k),\, \sum_{k=1}^{n}\alpha_k\mathrm{rad}([p]_k)]|[-1,\, 1] =$$

$$|\sum_{k=1}^{n}\alpha_k \mathrm{rad}([p]_k)|[-1,\,1] = [-\sum_{k=1}^{n}\alpha_k \mathrm{rad}([p]_k),\,\sum_{k=1}^{n}\alpha_k \mathrm{rad}([p]_k)] = \sum_{k=1}^{n}\alpha_k[-\mathrm{rad}([p]_k),\,\mathrm{rad}([p]_k)] = [Z]'_i$$

This and equation (15) gives the thesis of the theorem. □

Table I. Algorithm

| |
|---|
| $R := \mathrm{mid}(A([p]))^{-1}$; |
| $\tilde{x} := R \cdot \mathrm{mid}(b([p]))$; |
| $[Z]_i = \sum_{j=1}^{n} R_{ij}\Big(\omega(0,\,j) - \sum_{k=1}^{n}\tilde{x}_k\omega(j,\,k)\Big)^{\mathrm{T}}[p]$ |
| $[D]_{ij} := \Big(\sum_{\nu=1}^{n} R_{i\nu}\omega(\nu,\,j)\Big)^{\mathrm{T}}[p]$; |
| $\mathrm{outer} := \tilde{x} + [-1,\,1]\langle[D]\rangle^{-1}\|[Z]\|$ |

## 5. Examples

**Example 1.** Baltimore bridge (1870).
For the plane truss structure (all bars, loads and displacements are in the same x-y plane) shown in Figure 1 subjected to downward forces of 80 [kN] at node № 11, 120 [kN] at node № 12 and 80 [kN] at node № 15, the displacements of the nodes are computed. Young's modulus E= $2.1 \times 10^{11}$ [Pa] and cross-section area A= 0.004 [$m^2$]. The lenghts of the bar elements are shown in the figure (unity equals 1 [m]). Assume the stiffness of some of the bar elements (denoted in the figure with thick lines) to be uncertain by ±5%. To compute the displacement, the parametrized system of linear interval equations must be sovled. The results are in table II and III.

**Example 2.** Plane truss with uncertain stiffness of 8 bar elements.

The plane truss shown in Figure 2 is subjected to downward forces of 30 [kN] at nodes № 2, 3 and 4. All bar elements have the same Young's modulus E=$7 \cdot 10^{10}$ and cross-section area A=0.003 [$m^2$]. Assume the stiffness of 8 bar elements to be uncertain by ±5%. The resultimg intevals vectors are presented is tables IV and V.

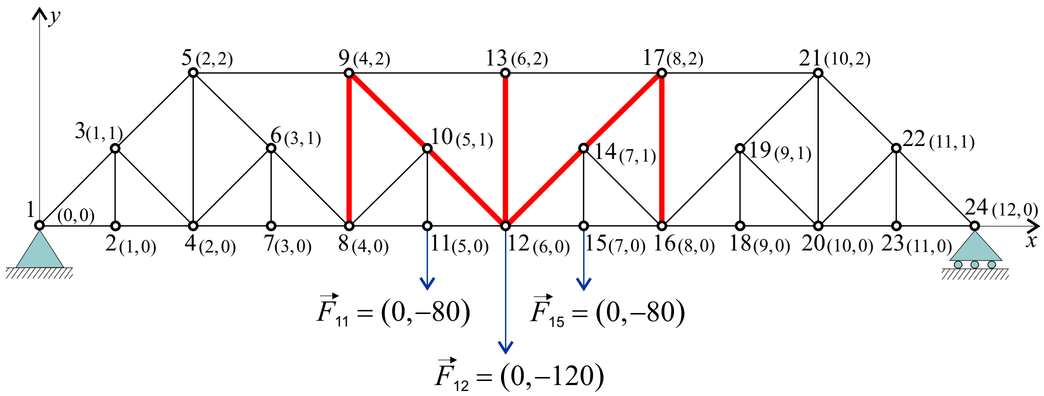


*Figure 1.* Scheme of the Baltimore bridge

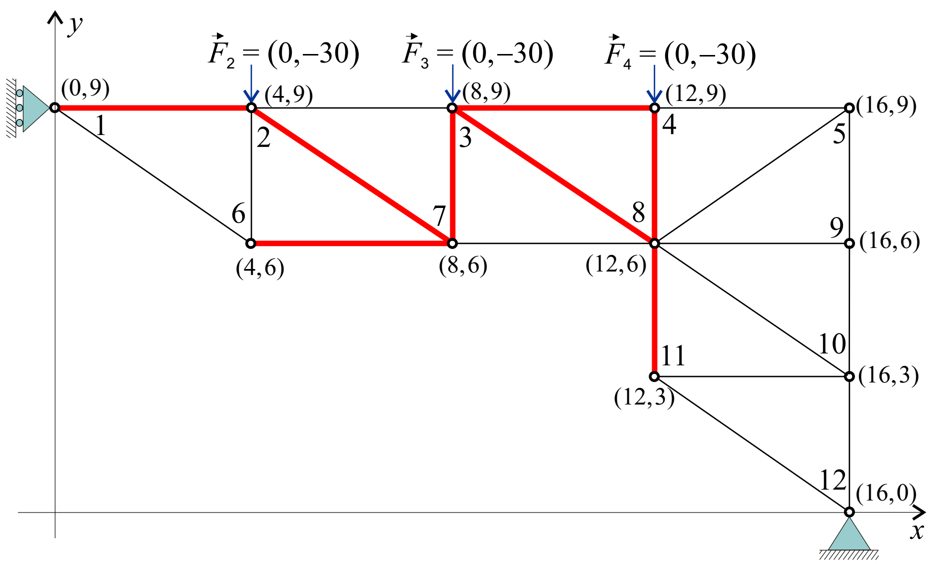


*Figure 2.* Truss structure with uncertain stiffnesses of 8 bar elements

**Example 3.** Plane truss with uncertain displacements of the supports.

The plane truss shown in Figure 3 has two supports: partial (sliding) support along y axis at node № 1 and full support at node № 12. Allow the movements of the supports at node 1 by $\Delta_{1-2} = 0.2$ [m] along y axis, at node 12 by 0.3 [m] along x axis and by 0.4 [m] along y axis. Now assume all movements to be uncertain by ±5% Uncetrain displacements of the support cause interval parameters appear only in the right-hand vector. In this case the method should give the hull. The results in tables VI and VII prove this to be true.

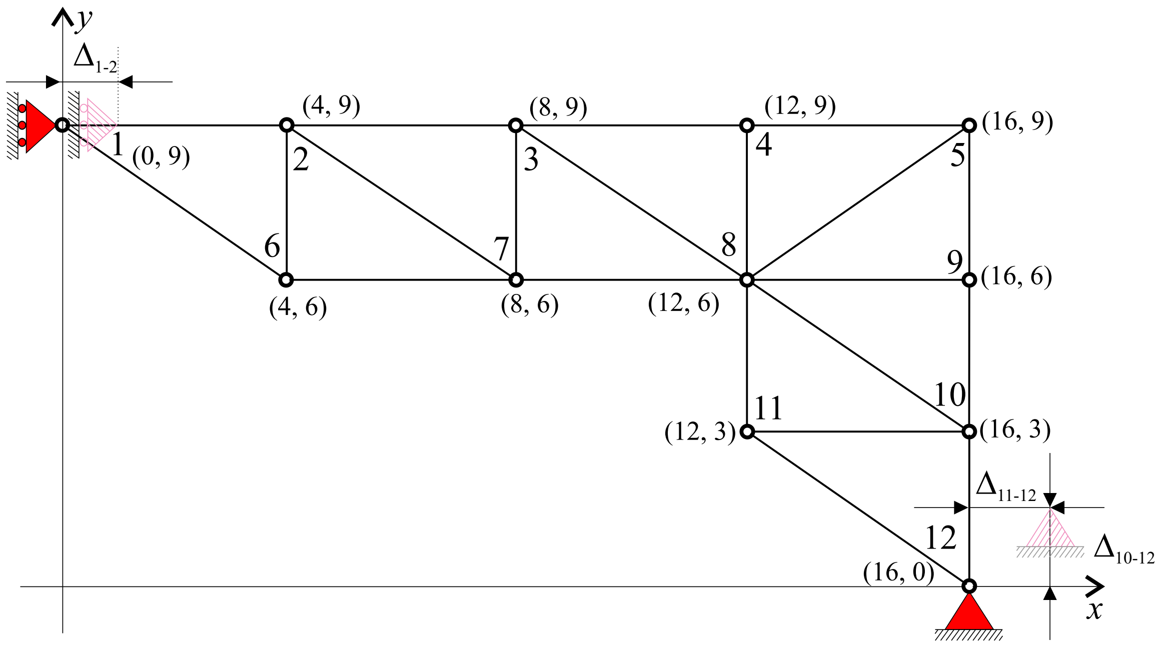


*Figure 3.* Truss structure with uncertain displacements of the supports

## 6. Results

The results produced by the method described in section 4 are presented in tables below. Column № 2 contains exact solution of non-interval system, comulmn № 3 contains the inner estimation obtained using the method of random sampling of parameter intervals (RSPI), column № 4 contains the results of the proposed method. Columns № 3 and 5 contain the relative error of the resulting intervals (in percent).

Table II. Example 1 ($x$-coords.)

| | $\mathbf{d}_0$ $[\times 10^{-5}]$ | RSPI $[\times 10^{-5}]$ | $r[*]/\mathbf{d}_0$ [%] | Method $[\times 10^{-5}]$ | $r[*]/\mathbf{d}_0$ [%] |
|---|---|---|---|---|---|
| $d_2^x$ | 16.67 | 16.67 | 0 | 16.67 | 0 |
| $d_3^x$ | 190.24 | [189.72, 190.76] | 0.3 | [189.34, 191.13] | 0.5 |
| $d_4^x$ | 33.33 | 33.33 | 0 | 33.33 | 0 |
| $d_5^x$ | 300 | [298.96, 301.05] | 0.3 | [298.21, 301.79] | 0.6 |
| $d_6^x$ | 190.24 | [189.72, 190.76] | 0.3 | [189.34, 191.13] | 0.5 |
| $d_7^x$ | 50 | 50 | 0 | 50 | 0 |
| $d_8^x$ | 66.67 | 66.67 | 0 | 66.67 | 0 |
| $d_9^x$ | 233.33 | [232.29, 234.38] | 0.4 | [231.54, 235.12] | 0.8 |
| $d_{10}^x$ | 172.01 | [170.6, 173.38] | 0.8 | [169.84, 174.17] | 1.3 |
| $d_{11}^x$ | 104.76 | 104.76 | 0 | 104.76 | 0 |
| $d_{12}^x$ | 142.86 | 142.86 | 0 | 142.86 | 0 |
| $d_{13}^x$ | 142.86 | [141.81, 143.9] | 0.7 | [141.07, 144.65] | 1.3 |
| $d_{14}^x$ | 113.71 | [112.38, 114.93] | 1.1 | [111.54, 115.88] | 1.9 |
| $d_{15}^x$ | 180.95 | 180.95 | 0 | 180.95 | 0 |
| $d_{16}^x$ | 219.05 | 219.05 | 0 | 219.05 | 0 |
| $d_{17}^x$ | 52.38 | [51.34, 53.43] | 2 | [50.59, 54.17] | 3.4 |
| $d_{18}^x$ | 235.71 | 235.71 | 0 | 235.71 | 0 |
| $d_{19}^x$ | 95.48 | [94.96, 96] | 0.5 | [94.58, 96.37] | 0.9 |
| $d_{20}^x$ | 252.38 | 252.38 | 0 | 252.38 | 0 |
| $d_{21}^x$ | −14.29 | [−15.33, −13.24] | 7.3 | [−16.08, −12.5] | 12.5 |
| $d_{22}^x$ | 95.48 | [94.96, 96] | 0.5 | [94.58, 96.37] | 0.9 |
| $d_{23}^x$ | 269.05 | 269.05 | 0 | 269.05 | 0 |
| $d_{24}^x$ | 285.71 | 285.71 | 0 | 285.71 | 0 |

Table III. Example 1 ($y$-coords.)

| | $\mathbf{d}_0$ $[\times 10^{-5}]$ | RSPI $[\times 10^{-5}]$ | $r[*]/\mathbf{d}_0$ [%] | Method $[\times 10^{-5}]$ | $r[*]/\mathbf{d}_0$ [%] |
|---|---|---|---|---|---|
| $d_2^y$ | −237.38 | [−237.9, −236.86] | 0.2 | [−238.27, −236.48] | 0.4 |
| $d_3^y$ | −237.38 | [−237.9, −236.86] | 0.2 | [−238.27, −236.48] | 0.4 |
| $d_4^y$ | −394.28 | [−395.33, −393.24] | 0.3 | [−396.07, −392.49] | 0.5 |
| $d_5^y$ | −394.28 | [−395.33, −393.24] | 0.3 | [−396.07, −392.49] | 0.5 |
| $d_6^y$ | −551.18 | [−552.75, −549.62] | 0.3 | [−553.87, −548.5] | 0.5 |
| $d_7^y$ | −551.18 | [−552.75, −549.62] | 0.3 | [−553.87, −548.5] | 0.5 |
| $d_8^y$ | −721.9 | [−723.99, −719.81] | 0.3 | [−725.47, −718.32] | 0.5 |
| $d_9^y$ | −745.7 | [−747.96, −743.69] | 0.3 | [−749.81, −741.6] | 0.5 |
| $d_{10}^y$ | −840.7 | [−842.68, −838.87] | 0.2 | [−843.9, −837.5] | 0.4 |
| $d_{11}^y$ | −850.23 | [−852.2, −848.39] | 0.2 | [−853.43, −847.03] | 0.4 |
| $d_{12}^y$ | −890.06 | [−893.29, −887.27] | 0.3 | [−895.42, −884.69] | 0.6 |
| $d_{13}^y$ | −890.06 | [−893.29, −887.27] | 0.3 | [−895.99, −884.12] | 0.7 |
| $d_{14}^y$ | −840.7 | [−842.64, −838.75] | 0.2 | [−843.9, −837.5] | 0.4 |
| $d_{15}^y$ | −850.23 | [−852.17, −848.27] | 0.2 | [−853.43, −847.03] | 0.4 |
| $d_{16}^y$ | −721.9 | [−723.98, −719.8] | 0.3 | [−725.47, −718.32] | 0.5 |
| $d_{17}^y$ | −745.7 | [−748.02, −743.3] | 0.3 | [−749.81, −741.6] | 0.6 |
| $d_{18}^y$ | −551.18 | [−552.75, −549.61] | 0.3 | [−553.87, −548.5] | 0.5 |
| $d_{19}^y$ | −551.18 | [−552.75, −549.61] | 0.3 | [−553.87, −548.5] | 0.5 |
| $d_{20}^y$ | −394.28 | [−395.32, −393.23] | 0.3 | [−396.07, −392.49] | 0.5 |
| $d_{21}^y$ | −394.28 | [−395.32, −393.23] | 0.3 | [−396.07, −392.49] | 0.5 |
| $d_{22}^y$ | −237.38 | [−237.9, −236.85] | 0.2 | [−238.27, −236.48] | 0.4 |
| $d_{23}^y$ | −237.38 | [−237.9, −236.85] | 0.2 | [−238.27, −236.48] | 0.4 |

Table IV. Example 2 (x-coords.)

| | $\mathbf{d}_0$ [$\times 10^{-5}$] | RSPI [$\times 10^{-5}$] | r[*]/$\mathbf{d}_0$ [%] | Method [$\times 10^{-5}$] | r[*]/$\mathbf{d}_0$ [%] |
|---|---|---|---|---|---|
| $d_2^x$ | −152.38 | [−160.4, −145.13] | 5 | [−160.4, −144.36] | 5.3 |
| $d_3^x$ | −228.57 | [−236.59, −221.32] | 3.3 | [−236.59, −220.55] | 3.5 |
| $d_4^x$ | −152.38 | [−163.76, −141.34] | 7.4 | [−165.26, −139.51] | 8.5 |
| $d_5^x$ | −76.19 | [−87.56, −65.15] | 14.7 | [−89.06, −63.32] | 16.9 |
| $d_6^x$ | 427.38 | [419, 435.56] | 1.9 | [416.48, 438.28] | 2.5 |
| $d_7^x$ | 427.38 | [419, 435.56] | 1.9 | [417.52, 437.24] | 2.3 |
| $d_8^x$ | 351.19 | [342.81, 359.37] | 2.4 | [341.33, 361.05] | 2.8 |
| $d_9^x$ | 351.19 | [342.81, 359.37] | 2.4 | [341.33, 361.05] | 2.8 |
| $d_{10}^x$ | 267.86 | [262.19, 273.57] | 2.1 | [261, 274.7] | 2.5 |
| $d_{11}^x$ | 115.48 | [109.81, 121.19] | 4.9 | [108.63, 122.32] | 5.9 |

Table V. Example 2 (y-coords.)

| | $\mathbf{d}_0$ [$\times 10^{-4}$] | RSPI [$\times 10^{-4}$] | r[*]/$\mathbf{d}_0$ [%] | Method [$\times 10^{-4}$] | r[*]/$\mathbf{d}_0$ [%] |
|---|---|---|---|---|---|
| $d_1^y$ | −308.25 | [−312.45, −304.18] | 1.3 | [−315.09, −301.41] | 2.2 |
| $d_2^y$ | −251.27 | [−255.54, −247.22] | 1.6 | [−258.08, −244.46] | 2.7 |
| $d_3^y$ | −149.84 | [−152.8, −147.16] | 1.9 | [−153.43, −146.25] | 2.4 |
| $d_4^y$ | −37.14 | [−37.97, −36.34] | 2.2 | [−38.19, −36.1] | 2.8 |
| $d_5^y$ | 4.29 | 4.29 | 0 | 4.29 | 0 |
| $d_6^y$ | −251.27 | [−255.53, −247.22] | 1.7 | [−258.08, −244.46] | 2.7 |
| $d_7^y$ | −154.13 | [−157.19, −151.38] | 1.9 | [−158.32, −149.93] | 2.7 |
| $d_8^y$ | −32.86 | [−33.48, −32.24] | 1.9 | [−33.6, −32.11] | 2.3 |
| $d_9^y$ | 0 | 0 | 0 | 0 | 0 |
| $d_{10}^y$ | −4.29 | −4.29 | 0 | −4.29 | 0 |
| $d_{11}^y$ | 24.29 | [−25.04, −23.52] | 3.1 | [−25.2, −23.37] | 3.8 |

Table VI. Example 3 (x-coords.)

| | $\mathbf{d}_0$ [$\times 10^{-3}$] | Method (hull) [$\times 10^{-3}$] | r[*]/$d_0$ [%] |
|---|---|---|---|
| $d_2^x$ | 20 | [19, 21] | 5 |
| $d_3^x$ | 20 | [19, 21] | 5 |
| $d_4^x$ | 20 | [19, 21] | 5 |
| $d_5^x$ | 20 | [19, 21] | 5 |
| $d_6^x$ | 13.33 | [11.67, 15] | 12.5 |
| $d_7^x$ | 13.33 | [11.67, 15] | 12.5 |
| $d_8^x$ | 13.33 | [11.67, 15] | 12.5 |
| $d_9^x$ | 13.33 | [11.67, 15] | 12.5 |
| $d_{10}^x$ | 6.67 | [4.33, 9] | 35 |
| $d_{11}^x$ | 6.67 | [4.33, 9] | 35 |

Table VII. Example 3 (y-coords.)

| | $\mathbf{d}_0$ [$\times 10^{-3}$] | Method (hull) [$\times 10^{-3}$] | r[*]/$d_0$ [%] |
|---|---|---|---|
| $d_1^y$ | 35.56 | [28.44, 42.67] | 20 |
| $d_2^y$ | 26.67 | [21.33, 32] | 20 |
| $d_3^y$ | 17.78 | [14.22, 21.33] | 20 |
| $d_4^y$ | 8.89 | [7.11, 10.67] | 20 |
| $d_5^y$ | 0 | 0 | – |
| $d_6^y$ | 26.67 | [21.33, 32] | 20 |
| $d_7^y$ | 17.78 | [14.22, 21.33] | 20 |
| $d_8^y$ | 8.89 | [7.11, 10.67] | 20 |
| $d_9^y$ | 0 | 0 | – |
| $d_{10}^y$ | 0 | 0 | – |
| $d_{11}^y$ | 8.89 | [7.11, 10.67] | 20 |

## 7. Conclusions

The problem of solving parametrized systems of linear interval equations is very important in practical applications. Well known classical methods, such as interval version of Gauss Elimination or Preconditioned Interval Gauss-Seidel iteration fail since they compute enclosure for the solution set (3) which is generally much larger then solution set (7). A direct method for solving paraterized systems of linear interval equations based on the inclusion (8) was proposed and checked to be usefull in structure mechanics. The method produced tight enclosure for the solutions set of parametrized systems for all exemplary truss structures.

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