Interval Finite Element Methods: New Directions

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Abstract

FEM Techniques are Important. Finite Element Methods are actively used to solve many practical problems in numerous areas of science and engineering. Many physical and engineering phenomena are described by partial differential equations, and FEM is, in most cases, the best way to solve these equations.

It Is Important to Take Uncertainty into Account. In many problems, e.g., in fundamental physics, we know the exact equations, we know the exact values of the parameters of these equations, and all we need is solve these equations as fast and as accurately as possible. These are the cases when the traditional FEM techniques directly lead to practically useful results. Of course, since we approximate the actual continuous domain with a finite collection of finite elements, the FEM solution is only an approximation to the actual continuous filed, but as we increase the number of finite elements and make them smaller and smaller, the FEM results become more and more accurate, and so when the elements are small enough, we get the desired solution with a very high accuracy.

There are other many other application problems, however, where we only know the approximate equations, or where we know the equations, but we only know the approximate values of the corresponding parameters. For example, in many civil engineering problems, we do not know the exact values of the corresponding Young modulus, we only know the bounds for this value coming from the fact that we know the material, and we know the bound for this type of material. In such problems, even if we use extremely small finite elements so that the discretization error is negligible, the resulting FEM solution may still be very different from the actual behavior of an

analyzed system – because of the uncertainty in the parameters and/or equations.

In such situations, to make the FEM results practically useful, we must be able to estimate how different the actual solution can be from these FEM results. In other words, we need to be able to estimate how the uncertainty in the parameters of the system can affect the results of applying the FEM techniques.

Traditional Approach: Stochastic FEM. This question is of paramount importance in science and engineering, and, of course, there has already been a lot of research aiming to answer this question. Most of this research is based on the assumption that we know the exact probability distributions corresponding to all the imprecisely known parameters. In this stochastic FEM case, we can, in principle, apply Monte-Carlo simulations: simulate all the parameters according to their known distributions, apply FEM for the system with the simulated values of the corresponding parameters, and then perform the statistical analysis of the FEM results – and thus, get the probability distribution for these results.

Interval FEM. This stochastic FEM approach works well in many practical situations. In many practical situations, however, we do not know the probabilities of different values of imprecisely known parameters. For example, in civil engineering, we often only know the lower and upper bounds on the Young module, but the probabilities of different values within the corresponding interval may depend on the manufacturing process and may be thus drastically different from one building to another. In situations which require reliable estimates, e.g., when we analyze the stability of a building, it is not enough to select one possible distribution and confirm that the building is stable under this distribution; to get a reliable result, we must make sure that the building remains stable for all possible distributions on the given interval.

There is an area of applied mathematics and computer science specifically tailored towards situations when the only information about an unknown quantity x is the interval $\mathbf{x}$ of possible values – the area of interval computations. Lately, there has been a lot of progress in applying interval computation techniques to FEM situations with interval uncertainty. These methods have led to interesting practical applications to building stability and similar problems.

First Challenge: Need to Combine Intervals and Probabilities. However, there are still practical problems for which the interval FEM methods are not fully adequate. Indeed, as of now, there are two types of methods to handle uncertainty in FEM problems: stochastic FEM methods take care of the situations when we know the exact probability distribution of all imprecise parameters, and interval FEM methods handle situations when we have no information about any of the probability distributions – we only know the intervals of possible value of these parameters. In other words, at present, we only know how to handle uncertainty in two extreme situations:

when we have full information about the probabilities and we have no information whatsoever about the probabilities. Many practical situations are in between these two extremes: we do not have full information about the probabilities, but we do have partial information about these probabilities. For example, we may have interval bounds for some of the parameters, but we may know the probability distribution for other parameters. For example, we may know only intervals of possible values of the manufacturing-related parameters, but, when we have good records, we may know probabilities of different values of, say, weather-related parameters.

In such situations, we can, of course, simple ignore the information about the probabilities and only take into account the interval of possible values of the parameters. In this way, we will get some information about the possible bounds on the difference between the computed FEM solution and the actual field – but we will thus ignore the information about the probabilities. It is therefore desirable to extend the interval and stochastic FEM techniques to the case when we have a combination of interval and probabilistic uncertainty.

How can we do that? For example, in an important case when we have interval uncertainty for some parameters and probabilistic uncertainty for some other parameters, we can apply Monte-Carlo techniques to simulate parameters with known probability distributions. For each such simulation, we can then use interval FEM techniques to take into account the corresponding interval uncertainty. As a result of applying interval FEM techniques, we get the interval bounds for the resulting FEM inaccuracy. By repeating this simulation several times, we get several bounds – and hence, the resulting bounds distribution. By using this bounds distribution, we can now supplement the interval FEM information that the FEM inaccuracy Δy is bounded by a certain value Δ, with the information that with probability 90%, we can get a narrower bound that bounds Δy in at least 90% of the case, yet narrower bound which holds in at least 80% of the cases, etc.

Second Challenge: Higher Order FEM. Another direction is extension of interval FEM techniques to higher order methods – hp-FEM – which have shown to be superior to traditional methods in many practical problems, superior both in terms of higher accuracy and in terms of smaller computation time.

Third Challenge: using Interval Computations to Prove FEM Results. Finally, it is desirable to use of interval computation techniques – techniques which provide guaranteed bounds for functions on continuous domains – in proving results about FEM methods, results which should be valid for all possible values of the corresponding parameters. We will describe our preliminary results in this direction.