Reliable Dynamic Analysis of Transportation Systems

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Abstract: In transportation engineering, dynamic analysis is an essential procedure for designing reliable systems. However, in current procedures of dynamic analysis for transportation systems, the possible presence of uncertainty in the system's mechanical properties and/or applied forces is not considered. In this work, a new method is developed for the dynamic analysis of continuous uncertain systems subjected to uncertain loads induced by passage of moving vehicles. First, an interval formulation is used to quantify the uncertainty present in the system's mechanical characteristics and/or magnitude of dynamic force. Then, having the interval parameters, the bounds on modal responses of the continuous system are obtained leading to determination of the upper-bounds of total response that may be used for design purposes. An example problem that illustrates the behavior of the method and a comparison with Monte-Carlo simulations are presented.

Keywords: Transportation, Dynamics, Interval, Uncertainty

1. Introduction

In design of transportation facilities, the performance of the system must be guaranteed over its lifetime. Moreover, dynamic analysis is a fundamental procedure for designing reliable systems that are subjected to dynamic forces induced by passage of moving vehicles.

However, in current procedures for dynamic analysis of transportation systems, the possible existence of uncertainty in either mechanical properties of the system or the characteristics of forcing function is generally not considered. These uncertainties can be attributed to physical imperfections, modeling inaccuracies and system complexities.

Although, in a design process, uncertainty is accounted for by a combination of load amplification and strength reduction factors that are based on probabilistic models of historic data, consideration of the effects of uncertainty has been removed from current dynamic analysis of transportation systems.

In this work, a new method is developed to perform dynamic analysis of a continuous system subjected to a moving load in the presence of uncertainty in the system's mechanical properties as well as uncertainty in the magnitude of dynamic loads. An interval formulation is used to represent the presence of uncertainty.

Using interval calculation procedures, the upper bounds of system's response are obtained which can be used for reliable design purposes. It is shown that this method can achieve the bounds of dynamic response without Monte-Carlo simulation procedure.

2. Deterministic Dynamic Analysis

The partial differential equation of motion for a flexural beam subjected to a load moving with constant velocity (Figure 1) is:

$$EI\frac{\partial^4 u(x,t)}{\partial x^4} + \overline{m}\frac{\partial^2 u(x,t)}{\partial t^2} = P_\circ \delta(x - vt) \qquad (0 \le t \le \frac{L}{v}) \qquad (1)$$

where, E is modulus of elasticity, I is the moment of inertia, u is the displacement, t is time, $\overline{m}$ is mass per unit length, $P_\circ$ is the magnitude of load, v is the velocity of the load and δ is the Dirac-delta function.

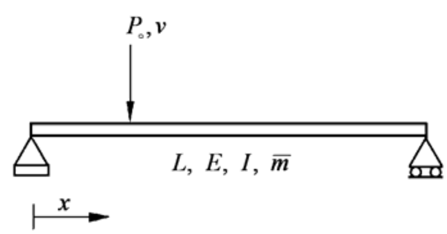


Figure 1. Simply-supported beam with moving load.

Considering free vibration of the system and assuming a harmonic solution of the form: $u(x,t) = \varphi(x)e^{i\omega t}$, in which $\varphi(x)$ a spatial function and ω is the circular natural frequency, the linear eigenvalue problem is:

$$\frac{d^2}{dx^2}\left(EI\,\frac{d^2\varphi(x)}{dx^2}\right)=\omega^2\overline{m}\,\varphi(x) \tag{2}$$

Applying boundary conditions for the simply-supported flexural Bernoulli beam, ($\varphi(0)=\varphi''(0)=\varphi(L)=\varphi''(L)=0$), the solution to the characteristic equation for natural circular frequencies and corresponding mass-orthonormalized eigenfunctions (mode shapes) are:

$$\omega_n = n^2\pi^2\sqrt{\frac{EI}{\overline{m}L^4}} \tag{3}$$

$$\varphi_n(x)=\sqrt{\frac{2}{\overline{m}L}}\sin(\frac{n\pi x}{L}) \tag{4}$$

where, n is the mode number.

The solution for the forced vibration may be expressed as:

$$u(x,t)=\sum_{n=1}^{\infty} y_n(t)\varphi_n(x) \tag{5}$$

Where, $y_n(t)$ are the modal coordinates.

Substituting Eq. (5) in the governing equation, Eq. (1), premultiplying by $\varphi_n(x)$, integrating over the domain, decoupling and adding modal damping ratio (ζ_n), the modal equation becomes:

$$\ddot{y}_n(t)+2\zeta_n\omega_n\dot{y}_n(t)+\omega_n^2 y_n(t)=\int_0^L \varphi_n(x)P_\circ\delta(x-vt)dx \tag{6}$$

or:

$$\ddot{y}_n(t)+2\zeta_n\omega_n\dot{y}_n(t)+\omega_n^2 y_n(t)=\Gamma_n\sin(\frac{n\pi v}{L}t)=P_\circ\sqrt{\frac{2}{\overline{m}L}}\sin(\frac{n\pi v}{L}t) \tag{7}$$

where, $\Gamma_n = P_\circ\sqrt{2/\overline{m}L}$ is the modal participation factor.

Defining a scaled generalized modal coordinate:

$$d_n(t) = \frac{y_n(t)}{\Gamma_n} \tag{8}$$

Eq. (7) is rewritten in terms of the scaled modal coordinate, $d_n(t)$, as:

$$\ddot{d}_n(t) + 2\zeta_n \omega_n \dot{d}_n(t) + \omega_n^2 d_n(t) = \sin(\frac{n\pi v}{L} t) \qquad (0 \le t \le \frac{L}{v}) \tag{9}$$

For each decoupled generalized modal equation, the maximum modal coordinate is obtained from the response spectrum (maximum ratio of dynamic to static response) for modal frequency and assumed modal damping ratio (Figure 2).

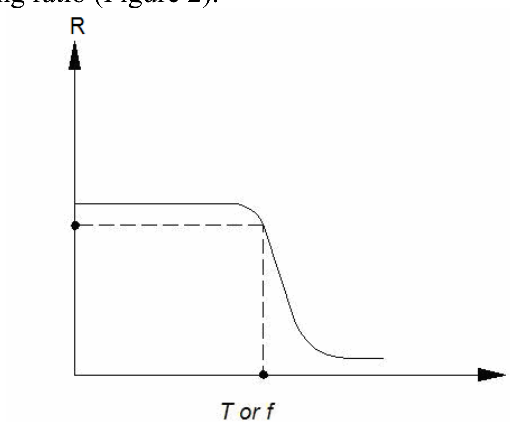


Figure 2. A generic response spectrum.

Then, the maximum modal displacement response is obtained as the multiplication of the maximum modal coordinate, modal participation factor, and mode shape as:

$$u_{n,\max} = (d_{n,\max})(\Gamma_n)(\varphi_n(x)) = (d_{n,\max})(\frac{2P_\circ}{\overline{m}L})\sin(\frac{n\pi x}{L}) \tag{10}$$

Finally, the total displacement response is obtained using superposition of modal maxima. The superposition can be performed by considering Square Root of Sum of Squares (SRSS) of modal maxima as (Rosenblueth 1962):

$$u_{\max} = \sqrt{\sum_{n=1}^{\infty} u^2{}_{n,\max}} \tag{11}$$

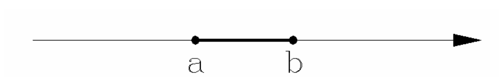
For practical purposes, the infinite series must be truncated. For systems with different patterns of load and boundary conditions, the same procedures can be used.

3. Interval Variables

The concept of interval numbers has been originally applied in the error analysis associated with digital computing. Quantification of the uncertainties introduced by truncation of real numbers in numerical methods was the primary application of interval methods (Moore 1966).

A real interval is a closed set defined by extreme values as (Figure 3):

$$\tilde{Z} = [z^l, z^u] = \{z \in \Re \mid z^l \le z \le z^u\} \tag{12}$$



$$\tilde{x} = [a,b]$$

Figure 3. An interval variable.

In this work, the symbol (~) represents an interval quantity. One interpretation of an interval number is a random variable whose probability density function is unknown but non-zero only in the range of interval.

Another interpretation of an interval number includes intervals of confidence for α-cuts of fuzzy sets. The interval representation transforms the point values in the deterministic system to inclusive set values in the system with bounded uncertainty.

4. Interval Dynamic Analysis

The partial differential equation of motion for a flexural beam subjected to a load moving with constant velocity with interval uncertainty in modulus of elasticity and magnitude of load is:

$$\tilde{E}I \frac{\partial^4 u(x,t)}{\partial x^4} + \overline{m} \frac{\partial^2 u(x,t)}{\partial t^2} = \tilde{P}_\circ \delta(x - vt) \qquad (0 \le t \le \frac{L}{v}) \tag{13}$$

where, $\tilde{E} = [E^l, E^u]$ and $\tilde{P}_\circ = [P_\circ^l, P_\circ^u]$.

Then, the interval eigenvalue problem becomes:

$$\frac{d^2}{dx^2}\left(\tilde{E}I\,\frac{d^2\varphi(x)}{dx^2}\right) = \tilde{\omega}^2\overline{m}\,\varphi(x) \tag{14}$$

Applying boundary conditions, the solution for natural circular frequencies and corresponding mode shapes are:

$$\tilde{\omega}_n = n^2\pi^2\sqrt{\frac{\tilde{E}I}{\overline{m}L^4}} \tag{15}$$

$$\varphi_n(x) = \sqrt{\frac{2}{\overline{m}L}}\sin\left(\frac{n\pi x}{L}\right) \tag{16}$$

Eq. (15) can be rewritten as:

$$\tilde{\omega}_n = n^2\pi^2([\sqrt{E^l}, \sqrt{E^u}])\sqrt{\frac{I}{\overline{m}L^4}} \tag{17}$$

This shows that the lower bound of modulus of elasticity (or in general stiffness) yields the lower bound of natural circular frequency and similarly, the upper bound of modulus of elasticity yields the upper bound of natural circular frequency. This leads to an evident realization of monotonic behavior of natural circular frequencies due to variation in stiffness in continuous dynamic systems.

In discrete systems, because of the complexity of the eigenvalue problem, this realization is not straightforward. Modares and Mullen (2004) proved this monotonic behavior of natural frequencies in discrete systems using monotonicity of eigenvalues for symmetric matrices subjected to non-negative definite perturbations.

The interval modal coordinate is determined using the excitation response spectrum evaluated for the corresponding interval of natural circular frequency and assumed modal damping ratio (Figure 4).

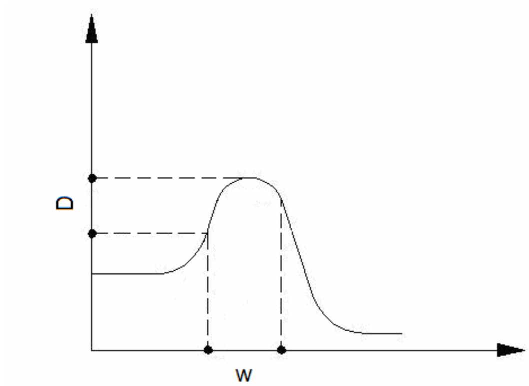


Figure 4. Determination of $\tilde{d}_n$ corresponding to a $\tilde{\omega}_n$ for a generic response spectrum

Having the interval modal coordinate, the maximum (upperbound) modal coordinate $d_{n,\max}$ is determined as:

$$d_{n,\max} = \max(\tilde{d}_n) \tag{18}$$

The interval modal participation factor is:

$$\tilde{\Gamma}_n = \tilde{P}_\circ \sqrt{\frac{2}{\overline{m}L}} \tag{19}$$

Therefore, the maximum modal coordinate is:

$$\Gamma_{n,\max} = \max(\tilde{\Gamma}_n) = (P_\circ^{\,u})\sqrt{\frac{2}{\overline{m}L}} \tag{20}$$

Then, the maximum modal displacement response is obtained as the multiplication of maximum modal coordinate, maximum modal participation factor and mode shape as:

$$u_{n,\max} = (d_{n,\max})(\Gamma_{n,\max})(\varphi_n(x)) = (d_{n,\max})(\frac{2P_\circ^{\,u}}{\overline{m}L})\sin(\frac{n\pi x}{L}) \tag{21}$$

Finally, the total displacement response is obtained using superposition of modal maxima. Using SRSS, the total response is:

$$u_{\max} = \sqrt{\sum_{n=1}^{\infty} u^2{}_{n,\max}} \tag{22}$$

5. Numerical Example

The example obtains the bounds on dynamic mid-span displacement for a continuous flexural simply-supported beam with interval uncertainty in the modulus of elasticity and magnitude of moving load.

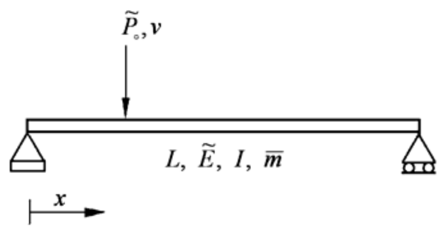


Figure 5. Flexural beam with uncertainty in modulus of elasticity and magnitude of moving load.

The beam's length is $L = 200\,ft$, mass is $\overline{m} = 11 kips/g$ per foot, the moment of inertia is $I = 700\,ft^4$, assumed modal damping ratio $\zeta = 1\%$, and uncertain modulus of elasticity is $E = ([0.9,1.1])576000 kips/ft^2$. The moving load's velocity is $v = 55 mph$, and its parametric uncertain magnitude of load is $\tilde{P}_\circ = [0.9,1.1]P_\circ$.

5.1. Solution

The problem is solved using the present method and the results are compared with Monte-Carlo simulation solution using bounded uniformly distributed random variables in 10000 simulations.

The solution for bounds on modal natural circular frequencies is summarized in table (1).

	Lower Bound *Present Method*	Lower Bound *Monte-Carlo Simulation*	Upper Bound *Monte-Carlo Simulation*	Upper Bound *Present Method*
$\frac{\omega_n}{(n^2)}$	1.41717	1.41718	1.56673	1.56675

Table1. Bounds on Natural Circular frequencies

The response spectrum for the first (fundamental) mode is obtained and shown in figure (6).

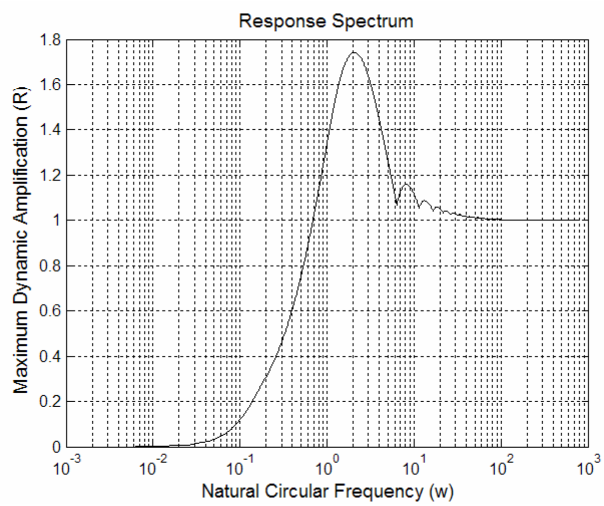


Figure 6. Response spectrum for fundamental mode of the example problem.

The upperbounds the mid-span displacement response for the fundamental mode is summarized in table (2).

	Upper Bound *Monte-Carlo Simulation*	Upper Bound *Present Method*
$\frac{u_1}{P_\circ}$	8.06557e-004	8.12128e-004

Table2. Upper bounds of displacement response

The first-mode beam response is depicted in figure (7).

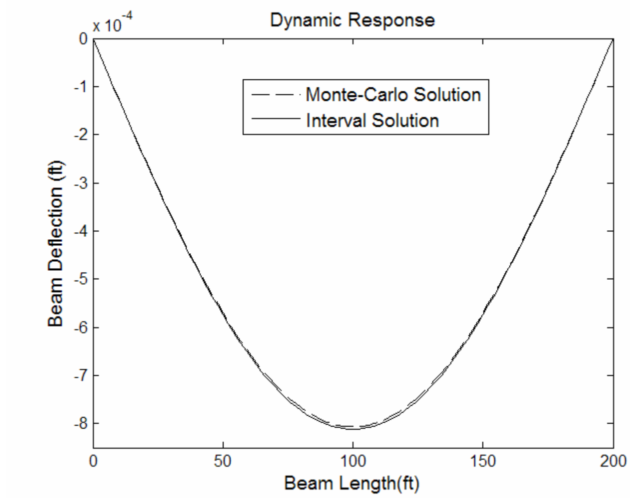


Figure 7. Beam deflection for the fundamental mode response of the example problem.

The results show that using the proposed method, the system's physics is preserved and also, the obtained sharp solutions are upper-bounds to solutions obtained by methods that produce inner-bound results such as Monte-Carlo simulation.

6. Conclusions

A new method for dynamic analysis of transportation systems with uncertainty in the mechanical characteristics of the system as well as the properties of the moving load is developed.

This computationally efficient method shows that implementation of interval analysis in a continuous dynamic system preserves the problem's physics and the yields sharp and robust results. This may be attributed to completeness of the closed-form solution in continuous dynamic systems.

The results show that obtaining bounds does not require expensive stochastic procedures such as Monte-Carlo simulations.

The simplicity of the proposed method makes it attractive to introduce uncertainty in analysis of continuous dynamic systems.

References

Biot, M. A. Vibrations of Buildings during Earthquake, *Ph.D. Thesis* Aeronautics Department, Calif. Inst. of Tech., Pasadena, California, U.S.A. 1932.

Clough, R.W. and J. Penzien. Dynamics of Structures. McGraw-Hill, New York 1993.

Modares, M. and R. L. Mullen. Free Vibration of Structures with Interval Uncertainty. *9th ASCE Specialty Conference on Probabilistic Mechanics and Structural Reliability* 2004.

Moore, R. E. Interval Analysis. Prentice Hall, Englewood, NJ 1966.

Muhanna, R. L. and R. L. Mullen. Uncertainty in Mechanics Problems-Interval-Based Approach. *Journal of Engineering Mechanics* June-2001, pp.557-566 2001.

Neumaier, A. Interval Methods for Systems of Equations. Cambridge University Press, Cambridge 1990.

Rosenblueth, E. and J. I. Bustamente. Distribution of Structural Response to Earthquakes *Proc. ASCE*, Vol. 88, No. EM 3 p. 75 1962.

Veletsos, A. S. and N. M. Newmark, N. M. Natural Frequencies of Continuous Flexural Members. *Trans. ASCE*, Vol. 122, p. 249 1957.