Prediction of uncertain structural responses with fuzzy time series

BERND MÖLLER, UWE REUTER
Institut für Statik und Dynamik der Tragwerke, Technische Universität Dresden, 01062 Dresden, Germany,
email: moeller@rcs.urz.tu-dresden.de, uwe.reuter@tu-dresden.de

Abstract. In this paper mathematical methods for prediction of uncertain structural responses with the aid of fuzzy time series are presented. Uncertain measurments of structural loads and responses respectively at equally spaced discrete time points are modeled as fuzzy variables. Hence uncertain measurments over time are considered as time series with fuzzy data. The fuzzy variables are processed on the basis of generally applicable numerical methods for descriptive analysis as well as for stochastic analysis. Algorithms of stochastic analysis are used to forecast fuzzy time series. At this the new fuzzy-ARMA-process is introduced. Forecasts of fuzzy time series provides informationen about future structural responses.

The algorithm of analysis and forecast of fuzzy time series are presented in detail and demonstrated by way of numerical examples.

Keywords: Fuzzy time series; Fuzzy random processes; Fuzzy random variables; forecast

1. Introduction

The prediction of future structural responses is a challenging problem in civil engineering. The knowledge of unknown future impact and future system behavior enables the prediction of such important effects like damage behavior, development of safety level, development of durability or the expected life time of a system. The well established numerical structural analysis and safety assessment however presuppose the knowledge of adequate theoretical models.

As alternative fuzzy time series can be applied. They describe sequences of measurements consisting of imprecise data (Hareter, 2003). The uncertainty of the imprecise data is modeled as fuzziness (Möller and Beer, 2004). Time series with fuzzy data are regarded as realizations of a fuzzy random process, that can be viewed as a random process extended by the dimension fuzziness (Möller et al., 2005). In extension to a random process a fuzzy random process is defined as a sequence of fuzzy random variables. Therein, a fuzzy random variable is declared as set of uncertain realizations (fuzzy variables) in the space of the random elementary events. Each realization of a fuzzy random process then appears as a fuzzy function, which characterizes a sequence of fuzzy variables. In other words time series with fuzzy data can be interpreted as random realizations of an underlying fuzzy random process.

Methods for identification and quantification of the underlying fuzzy random process are presented. A new description of fuzzy variables by so called $l_\alpha r_\alpha$-discretization has been developed. This description enables prediction without the usually performed defuzzification and refuzzification of fuzzy data. The following types of fuzzy random processes are investigated: fuzzy-AR-processes, fuzzy-MA-processes, fuzzy-ARMA-processes, and fuzzy-white-noise-processes. Strategies for parameter estimation have been

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developed that are applicable for stationary and non-stationary fuzzy time series. After parameter estimation the underlying fuzzy random process is known and can be used for forecasting.

The developed theory is demonstrated by way of examples among others the heavy goods vehicle traffic over a bridge is forecasted. Furthermore, on the basis of measured settlements over a period of four years the future settlements for the next three years are predicted with a h-step-forecast.

2. Definition and description of fuzzy time series

Fuzzy time series are interpreted as random realizations of an underlying fuzzy random process. A fuzzy random process $(\tilde{\mathrm{X}}_\tau)_{\tau\in\mathbf{T}}$ is defined as a family of fuzzy random variables $\tilde{\mathrm{X}}_\tau$ with $\tau \in \mathbf{T}$. Thereby $\mathbf{T}$ denotes the space of equidistant points in time. In other words a fuzzy random process $(\tilde{\mathrm{X}}_\tau)_{\tau\in\mathbf{T}}$ is defined as the fuzzy result of the mapping

$$\tilde{\mathrm{X}}_\tau : \Omega \rightarrow \mathbf{F}(\mathbb{R}) \tag{1}$$

in which Ω denotes the space of the random elementary events ω and $\mathbf{F}(\mathbb{R})$ characterizes the set of all fuzzy numbers on $\mathbb{R}$. Fuzzy realizations $\tilde{\mathbf{X}}_\tau(\omega) = \tilde{x}_\tau$ with $\tau \in \mathbf{T}$ are assigned to each random elementary event $\omega \in \Omega$. Consequently the realizations of a fuzzy random process $(\tilde{\mathrm{X}}_\tau)_{\tau\in\mathbf{T}}$ form the fuzzy time series $(\tilde{x}_\tau)_{\tau\in\mathbf{T}}$. A realization $(\tilde{x}_\tau)_{\tau\in\mathbf{T}}$ of a fuzzy random process is plotted in Fig. 1.

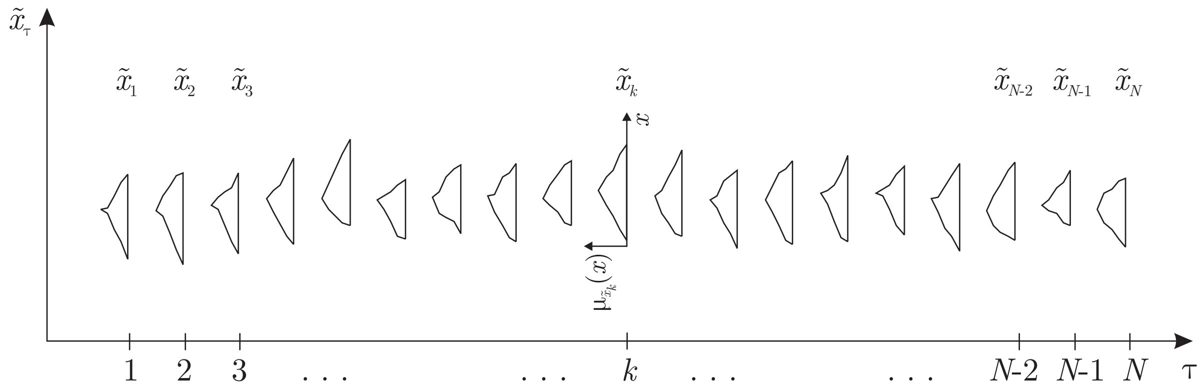

Figure 1. Fuzzy time series as realization of a fuzzy random process

At each specified point $\tau \in \mathbf{T}$ a fuzzy time series specifies a fuzzy variable $\tilde{x}_\tau$ in accordance with Eq. 1. A fuzzy variable $\tilde{x}$ is characterized by its membership function $\mu_{\tilde{x}}(x)$. A normalized membership function $\mu_{\tilde{x}}(x)$ is defined by the following equations.

$$0 \leq \mu_{\tilde{x}}(x) \leq 1 \quad \forall\, x \in \mathbb{R} \tag{2}$$

$$\exists\, x_l,\ x_r \ \text{ mit } \ \mu_{\tilde{x}}(x) = 1 \quad \forall\, x \in [x_l; x_r] \tag{3}$$

A fuzzy variable $\tilde{x}$ is referred to as convex if its membership function $\mu_{\tilde{x}}(x)$ monotonically decreases on each side of the maximum value, i.e., if

$$\mu_{\tilde{x}}(x_2) \geq \min\left[\mu(x_1);\ \mu(x_3)\right] \quad \forall\, x_1, x_2, x_3 \in \mathbb{R} \ \text{ mit } \ x_1 \leq x_2 \leq x_3 \tag{4}$$

applies.

A convex fuzzy variable $\tilde{x}$ is referred to as fuzzy number $\tilde{x}_Z$ if its membership function $\mu_{\tilde{x}}(x)$ is at least segmentally continuous and has the functional value $\mu_{\tilde{x}}(x) = 1$ at precisely one of the x values according to Eq. (5).

$$\begin{aligned} x_l = x_r \quad \text{with} \qquad x_l &= \ \min\left[x \in \mathbb{R} | \mu_{\tilde{x}_\tau}(x) = 1\right] \\ \text{and} \qquad x_r &= \ \max\left[x \in \mathbb{R} | \mu_{\tilde{x}_\tau}(x) = 1\right] \end{aligned} \tag{5}$$

In the case $x_l < x_r$ the fuzzy variable $\tilde{x}$ is a fuzzy interval $\tilde{x}_I$. The point x_l is referred to as the peak point of the fuzzy variable.

A convex fuzzy variable $\tilde{x}$ is characterized by a family of α-level sets X_α according to Eq. (6). Each α-level set X_α is a connected interval $[x_{\alpha l}, x_{\alpha r}]$.

$$\tilde{x} \ = \ (X_\alpha = [x_{\alpha l}, x_{\alpha r}] \,|\, \alpha \in [0, 1]) \tag{6}$$

The number of α-level sets is denoted by n. For $i = 1, 2, ..., n-1$ the following holds.

$$0 \leq \alpha_i \leq \alpha_{i+1} \leq 1 \tag{7}$$

$$\alpha_1 = 0 \quad \text{und} \quad \alpha_n = 1 \tag{8}$$

$$X_{\alpha_{i+1}} \subseteq X_{\alpha_i} \tag{9}$$

An example of a convex fuzzy variable $\tilde{x}$ characterized by $n = 4$ α-level sets X_α is shown in Fig. 2.

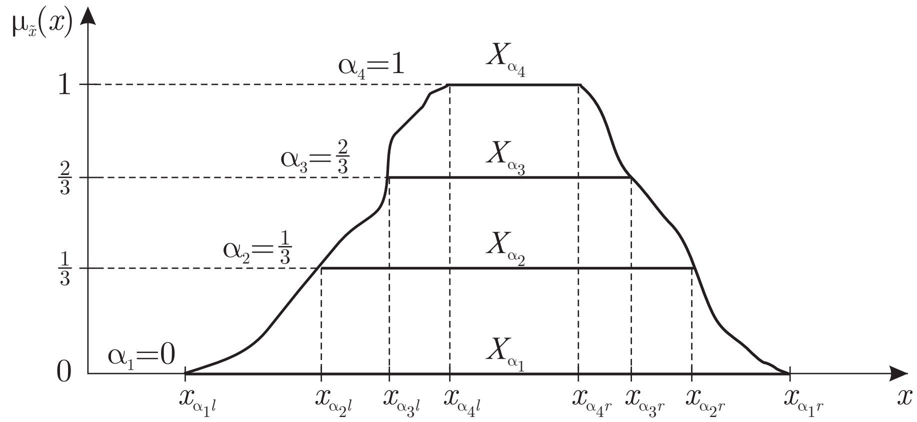

Figure 2. α-discretization of a convex fuzzy variable

In the following the new $l_\alpha r_\alpha$-discretization is presented. The interval boundaries $[x_{\alpha_i l}, x_{\alpha_i r}]$ of an α-level set X_{α_i} are expressed by Eqs. (10) and (11).

$$x_{\alpha_i l} = x_{\alpha_{i+1} l} - \Delta x_{\alpha_i l} \quad \text{with} \quad \Delta x_{\alpha_i l} = x_{\alpha_i lr} - x_{\alpha_i ll} \tag{10}$$

$$x_{\alpha_i r} = x_{\alpha_{i+1} r} + \Delta x_{\alpha_i r} \quad \text{with} \quad \Delta x_{\alpha_i r} = x_{\alpha_i rr} - x_{\alpha_i rl} \tag{11}$$

The counter $i = 1, 2, ..., n-1$ specifies α-level sets with $\alpha < 1$. For $i = 1$ the following equations hold, whereat the term $\Delta x_{\alpha_n l}$ is assigned to the peak point x_l.

$$x_{\alpha_n l} = \Delta x_{\alpha_n l} \quad \text{with} \quad \Delta x_{\alpha_n l} = x_l \tag{12}$$

$$x_{\alpha_n r} = x_{\alpha_n l} + \Delta x_{\alpha_n r} \quad \text{with} \quad \Delta x_{\alpha_n r} = x_r - x_l \tag{13}$$

The terms $\Delta x_{\alpha_i l}$ and $\Delta x_{\alpha_i r}$ are called $l_\alpha r_\alpha$-increments. The α-level sets have to fulfill Eq. (14).

$$X_{\alpha_k} \subseteq X_{\alpha_i} \quad \forall\, \alpha_i, \alpha_k \in [0;1] \quad \text{with} \quad \alpha_i \leq \alpha_k \tag{14}$$

With Eqs. (10) to (14) the $l_\alpha r_\alpha$-discretization is introduced. Fig. 3 illustrates the $l_\alpha r_\alpha$-discretization for $n = 4$.

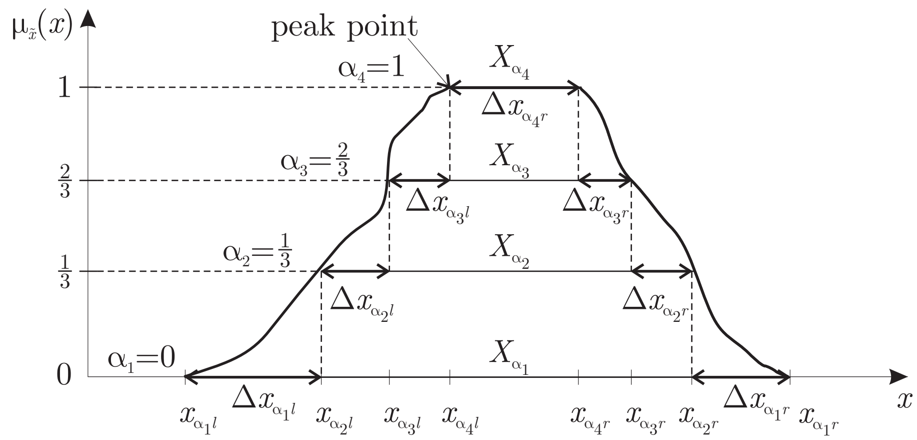

Figure 3. $l_\alpha r_\alpha$-Diskretisierung with 4 α-level sets

The $l_\alpha r_\alpha$-discretization enables an alternative, discrete representation of a fuzzy variable $\tilde{x}$ in the form of a column matrix introduced by Eq. (15), thereby Δx_1, Δx_2, ..., Δx_{2n} is a shortened form of $\Delta x_{\alpha_1 l}$, $\Delta x_{\alpha_2 l}$, ..., $\Delta x_{\alpha_1 r}$.

$$\tilde{x} = \begin{bmatrix} \Delta x_{\alpha_1 l} \\ \Delta x_{\alpha_2 l} \\ \vdots \\ \Delta x_{\alpha_n l} \\ \Delta x_{\alpha_n r} \\ \vdots \\ \Delta x_{\alpha_2 r} \\ \Delta x_{\alpha_1 r} \end{bmatrix} = \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \vdots \\ \Delta x_n \\ \Delta x_{n+1} \\ \vdots \\ \Delta x_{2n-1} \\ \Delta x_{2n} \end{bmatrix} \tag{15}$$

In context of time series with fuzzy data the following operators are introduced.

The multiplication of a real-valued $[2n, 2n]$ matrix $\underline{A}$ with a fuzzy variable $\tilde{x}$ represented by n α-levels is defined by the operator $\odot$ according to Eqs. (16) and (17). The arithmetic operation is equivalent to the matrix product and results the $l_\alpha r_\alpha$-increments Δz_j ($j = 1, 2, ..., 2n$) of the fuzzy result variable $\tilde{z}$.

$$\underline{A} \odot \tilde{x} = \tilde{z} \tag{16}$$

$$\begin{bmatrix} a_{1,1} & a_{2,2} & \cdots & a_{1,2n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{2n,1} & a_{2n,2} & \cdots & a_{2n,2n} \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \vdots \\ \Delta x_{2n} \end{bmatrix} = \begin{bmatrix} \Delta z_1 \\ \Delta z_2 \\ \vdots \\ \Delta z_{2n} \end{bmatrix} \tag{17}$$

The fuzzy result variable $\tilde{z}$ requires compliance with Eq. (14), so that Eq. (18) must be satisfied for $j = 1, 2, ..., n-1, n+2, ..., 2n$.

$$\Delta z_j = a_{j,1}\Delta x_1 + ... + a_{j,2n}\Delta x_{2n} \geq 0 \tag{18}$$

Furthermore a special fuzzy sum and subtraction respectively is required. The operators $\oplus$ and $\ominus$ respectively between two fuzzy variables $\tilde{x}$ and $\tilde{y}$ pursuant to Eq. (19) are introduced as the addition and subtraction respectively of the $l_\alpha r_\alpha$-increments according to Eq. (19)

$$\tilde{z} = \tilde{x} \oplus \tilde{y} \;\; \text{bzw.} \;\; \tilde{z} = \tilde{x} \ominus \tilde{y} \tag{19}$$

The fuzzy result variable $\tilde{z}$ requires compliance with Eq. (14), too. The corresponding conditions are shown in Eq. (20) in which the upper operators are applied for the fuzzy sum and the lower for the fuzzy difference.

$$\Delta z_j = \Delta x_j \pm \Delta y_j \geq 0 \quad \text{for} \quad j = 1, 2, ..., n-1, n+2, ..., 2n \tag{20}$$

Considering the priority rule ($\odot$ comes before $\oplus$) a combination of the introduced operators according to Eq. (21) is feasible.

$$\tilde{z} = \underline{A} \odot \tilde{x} \oplus ... \ominus ... \oplus ... \oplus \underline{B} \odot \tilde{y} \tag{21}$$

The fuzzy result variable $\tilde{z}$ also requires compliance with Eq. (14). But only the final $l_\alpha r_\alpha$-increments Δz_j must be nonnegative, negative intermediate results due the application of the associative law are allowed.

$$\Delta z_j \geq 0 \quad \text{for} \;\; j = 1, 2, ..., n-1, n+1, ..., 2n \tag{22}$$

The demand according to Eq. (22) also represents an boundary condition for the models introduced in the paper.

According to Eq. (1) a fuzzy variable $\tilde{x}_\tau$ is interpreted as a random realization of a fuzzy random variable $\tilde{\mathrm{X}}_\tau$. Under the assumption of convex fuzzy realizations $\tilde{\mathrm{X}}_\tau(\omega) = \tilde{x}$ a fuzzy random variable $\tilde{\mathrm{X}}_\tau$ is characterized by a family of random α-level sets X_α according to Eq. (23). At this the intervall boundaries $\mathrm{X}_{\alpha l}$ and $\mathrm{X}_{\alpha r}$ are real-valued random variables.

$$\tilde{\mathrm{X}}_\tau \;=\; (\mathrm{X}_\alpha = [\mathrm{X}_{\alpha l}, \mathrm{X}_{\alpha r}] \,|\, \alpha \in [0, 1]) \tag{23}$$

The $l_\alpha r_\alpha$-discretization enables a new definition of a fuzzy random variable $\tilde{\mathrm{X}}_\tau$ according to Eq. (24) for $i = 1, 2, ..., n-1$.

$$\begin{aligned} \tilde{\mathrm{X}}_\tau \;=\; \big(\mathrm{X}_{\alpha_i} &= [\mathrm{X}_{\alpha_{i+1}l} - \Delta\mathrm{X}_{\alpha_i l}; \mathrm{X}_{\alpha_{i+1}r} + \Delta\mathrm{X}_{\alpha_i r}] \,|\, \alpha_i \in [0,1); \\ \mathrm{X}_{\alpha_n} &= [\mathrm{X}_{\alpha_n l}; \mathrm{X}_{\alpha_n l} + \Delta\mathrm{X}_{\alpha_n r}] \qquad\qquad |\, \alpha_n = 1 \qquad \big) \end{aligned} \tag{24}$$

In this definition the terms $\Delta\mathrm{X}_{\alpha_i l}$ and $\Delta\mathrm{X}_{\alpha_i r}$ are correlated random variables and called random $l_\alpha r_\alpha$-increments. The $l_\alpha r_\alpha$-discretization enables an alternative, discrete representation of a fuzzy random variable $\tilde{\mathrm{X}}_\tau$ in the form of a column matrix introduced by Eq. (25), whereby the real-valued random variables $\Delta\mathrm{X}_1$, $\Delta\mathrm{X}_2$, ..., $\Delta\mathrm{X}_{2n}$ are shortened forms of the random $l_\alpha r_\alpha$-increments $\Delta\mathrm{X}_{\alpha_1 l}$, $\Delta\mathrm{X}_{\alpha_2 l}$, ..., $\Delta\mathrm{X}_{\alpha_1 r}$.

$$\tilde{\mathrm{X}}_\tau = \begin{bmatrix} \Delta\mathrm{X}_{\alpha_1 l} \\ \Delta\mathrm{X}_{\alpha_2 l} \\ \vdots \\ \Delta\mathrm{X}_{\alpha_n l} \\ \Delta\mathrm{X}_{\alpha_n r} \\ \vdots \\ \Delta\mathrm{X}_{\alpha_2 r} \\ \Delta\mathrm{X}_{\alpha_1 r} \end{bmatrix} = \begin{bmatrix} \Delta\mathrm{X}_1 \\ \Delta\mathrm{X}_2 \\ \vdots \\ \Delta\mathrm{X}_n \\ \Delta\mathrm{X}_{n+1} \\ \vdots \\ \Delta\mathrm{X}_{2n-1} \\ \Delta\mathrm{X}_{2n} \end{bmatrix} \tag{25}$$

According to Eq. (1) a fuzzy random process $(\tilde{\mathrm{X}}_\tau)_{\tau\in\mathbf{T}}$ is defined as a family of fuzzy random variables $\tilde{\mathrm{X}}_\tau$. For characterization of a fuzzy random process the first and second order moments of the process – like for random processes – are used. The first order moment is a fuzzy variable, that can be represented by $l_\alpha r_\alpha$-discretization. The $l_\alpha r_\alpha$-increments of the fuzzy expected value $E[\tilde{\mathrm{X}}_\tau] = \tilde{m}_{\tilde{\mathrm{X}}_\tau}$ of a fuzzy random process $(\tilde{\mathrm{X}}_\tau)_{\tau\in\mathbf{T}}$ are obtained according to Eq. (26).

$$\begin{aligned} E[\tilde{\mathrm{X}}_\tau] = \tilde{m}_{\tilde{\mathrm{X}}_\tau} \;&=\; \begin{bmatrix} \Delta m_{\alpha_1 l}(\tau) \\ \vdots \\ \Delta m_{\alpha_n l}(\tau) \\ \vdots \\ \Delta m_{\alpha_1 r}(\tau) \end{bmatrix} \\ &=\; \begin{bmatrix} \int\limits_0^\infty \Delta x_{\alpha_1 l}\, f_{\Delta\mathrm{X}_{\alpha_1 l}}(\Delta x_{\alpha_1 l}, \tau) d\Delta x_{\alpha_1 l} \\ \vdots \\ \int\limits_{-\infty}^\infty \Delta x_{\alpha_n l}\, f_{\Delta\mathrm{X}_{\alpha_n l}}(\Delta x_{\alpha_n l}, \tau) d\Delta x_{\alpha_n l} \\ \vdots \\ \int\limits_0^\infty \Delta x_{\alpha_1 r}\, f_{\Delta\mathrm{X}_{\alpha_1 r}}(\Delta x_{\alpha_1 r}, \tau) d\Delta x_{\alpha_1 r} \end{bmatrix} \end{aligned} \tag{26}$$

The functions $f_{\Delta\mathrm{X}_{\alpha_i l}}(\Delta x_{\alpha_i l}, \tau)$ and $f_{\Delta\mathrm{X}_{\alpha_i r}}(\Delta x_{\alpha_i r}, \tau)$ ($i = 1,\ 2,\ ...,\ n$) are probability density functions of the random $l_\alpha r_\alpha$-increments $\Delta\mathrm{X}_{\alpha_i l}(\tau)$ and $\Delta\mathrm{X}_{\alpha_i r}(\tau)$ of the fuzzy random variable $\tilde{\mathrm{X}}_\tau$ at time point τ.

Linear dependencies between two fuzzy random variables $\tilde{X}_{\tau_a}$ and $\tilde{X}_{\tau_b}$ of a fuzzy random process at time points τ_a and τ_b are quantified by the $l_\alpha r_\alpha$-covariance function ${}_{lr}K_{\tilde{X}_\tau}(\tau_a,\tau_b)$ according to Eq. (27).

$$
{}_{lr}K_{\tilde{X}_\tau}(\tau_a,\tau_b) = \begin{bmatrix} k^{\alpha_1 l}_{\alpha_1 l}(\tau_a,\tau_b) & k^{\alpha_2 l}_{\alpha_1 l}(\tau_a,\tau_b) & \cdots & k^{\alpha_1 r}_{\alpha_1 l}(\tau_a,\tau_b) \\ k^{\alpha_1 l}_{\alpha_2 l}(\tau_a,\tau_b) & k^{\alpha_2 l}_{\alpha_2 l}(\tau_a,\tau_b) & \cdots & k^{\alpha_1 r}_{\alpha_2 l}(\tau_a,\tau_b) \\ \vdots & \vdots & \ddots & \vdots \\ k^{\alpha_1 l}_{\alpha_1 r}(\tau_a,\tau_b) & k^{\alpha_2 l}_{\alpha_1 r}(\tau_a,\tau_b) & \cdots & k^{\alpha_1 r}_{\alpha_1 r}(\tau_a,\tau_b) \end{bmatrix} \tag{27}
$$

The elements of the $l_\alpha r_\alpha$-covariance function ${}_{lr}K_{\tilde{X}_\tau}(\tau_a,\tau_b)$ are defined by Eq. 28 where $i,j = 1,2,...,n$.

$$
\begin{aligned} k^{\alpha_i l}_{\alpha_j r}(\tau_a,\tau_b) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} & (\Delta x_{\alpha_i l} - \Delta m_{\alpha_i l}(\tau_a))(\Delta x_{\alpha_j r} - \Delta m_{\alpha_j r}(\tau_b))... \\ & ...f\left(\Delta x_{\alpha_i l}, \Delta x_{\alpha_j r}, \tau_a, \tau_b\right) d\Delta x_{\alpha_i l} d\Delta x_{\alpha_j r} \end{aligned} \tag{28}
$$

The $l_\alpha r_\alpha$-variance ${}_{lr}Var[\tilde{X}_\tau] = {}_{lr}\underline{\sigma}^2_{\tilde{X}_\tau}$ corresponds to the diagonal elements of the $l_\alpha r_\alpha$-covariance function ${}_{lr}K_{\tilde{X}_\tau}(\tau_a,\tau_b)$ with $\tau_a = \tau_b = \tau$.

A fuzzy random process is stationary if the $l_\alpha r_\alpha$-covariance function ${}_{lr}K_{\tilde{X}_\tau}(\tau_a,\tau_b)$ does not depend on τ_a and τ_b but just on the time lag $\Delta\tau = \tau_a - \tau_b$ and if the fuzzy expected value $E[\tilde{X}_\tau] = \tilde{m}_{\tilde{X}_\tau}$ is constant over time.

In the following a special case of fuzzy random processes is introduced. The known ARMA model is extended to time series with fuzzy data and results the fuzzy-ARMA-model. A fuzzy random process $(\tilde{X}_\tau)_{\tau\in\mathbf{T}}$ ist called fuzzy-ARMA$[p,q]$-process if it can be described by Eq. (29).

$$
\begin{aligned} \tilde{X}_\tau \;=\; & \underbrace{\underline{A}_1 \odot \tilde{X}_{\tau-1} \oplus ... \oplus \underline{A}_p \odot \tilde{X}_{\tau-p} \oplus}_{\text{fuzzy-AR-component}} \tilde{\mathcal{E}}_\tau \ominus \\ & \underbrace{\underline{B}_1 \odot \tilde{\mathcal{E}}_{\tau-1} \ominus ... \ominus \underline{B}_q \odot \tilde{\mathcal{E}}_{\tau-q}}_{\text{fuzzy-MA-component}} \end{aligned} \tag{29}
$$

The parameters $\underline{A}_1,...,\underline{A}_p$ und $\underline{B}_1,...,\underline{B}_q$ are real-valued $[2n,2n]$ matrices. The factors $\tilde{\mathcal{E}}_\tau$ are elements of a fuzzy-white-noise-process $(\tilde{\mathcal{E}}_\tau)_{\tau\in\mathbf{T}}$ at time point τ and therefore fuzzy random variables. A fuzzy-white-noise-process $(\tilde{\mathcal{E}}_\tau)_{\tau\in\mathbf{T}}$ is characterized by Eqs. (30) to (32).

$$
E[\tilde{\mathcal{E}}_\tau] \;=\; \tilde{m}_{\tilde{\mathcal{E}}_\tau} = \text{constant} \quad \forall\,\tau\in\mathbf{T} \tag{30}
$$

$$
{}_{lr}Var[\tilde{\mathcal{E}}_\tau] \;=\; {}_{lr}\underline{\sigma}^2_{\tilde{\mathcal{E}}_\tau} = \text{constant} \quad \forall\,\tau\in\mathbf{T} \tag{31}
$$

$$
{}_{lr}K_{\tilde{\mathcal{E}}_\tau}(\Delta\tau) \;=\; \begin{cases} {}_{lr}K_{\tilde{\mathcal{E}}_\tau}(0) & \text{for} \quad \Delta\tau = 0 \\ \underline{0} & \text{for} \quad \Delta\tau \neq 0 \end{cases} \tag{32}
$$

3. Parameter estimation

Within the scope of modeling fuzzy time series the parameters $\underline{A}_1, \ldots, \underline{A}_p$ and $\underline{B}_1, \ldots, \underline{B}_q$ of a fuzzy-ARMA$[p, q]$-process have to be determined so that the empirical time series is a representative realization. Fundamental condition is the demand of non-negativity of the $l_\alpha r_\alpha$-increments Δx_j ($j = 1, 2, \ldots n-1, n+2, \ldots 2n$) of all realizations $\tilde{x}_\tau$ of the fuzzy-ARMA-process.

The first method is based on the postulation that the differences between the empirical and model characteristics (first and second order moments) are minimal. This condition results in the optimization problem given by Eq. (33), in which $\underline{P}$ is a shortened form of the process parameters $\underline{A}_1, \ldots, \underline{A}_p$ and $\underline{B}_1, \ldots, \underline{B}_q$

$$\begin{aligned}&\sum_{j=1}^{2n} \left(\Delta\overline{x}_j - \Delta m_j(\underline{P})\right)^2 + \\ &\sum_{\Delta\tau=-\infty}^{\infty} \sum_{k,l=1}^{2n} \left(\hat{k}_{k,l}(\Delta\tau) - k_{k,l}(\Delta\tau, \underline{P})\right)^2 \stackrel{!}{=} \min\end{aligned} \tag{33}$$

The $l_\alpha r_\alpha$-increments $\Delta\overline{x}_i$ of the empirical fuzzy mean value $\tilde{\overline{x}}$ are compared with the $l_\alpha r_\alpha$-increments Δm_i of the fuzzy expected value $\tilde{m}_{\tilde{X}_\tau}$ as well as the elements $\hat{k}_{k,l}(\Delta\tau)$ of the empirical $l_\alpha r_\alpha$-covariance function ${}_{lr}\hat{K}_{\tilde{x}_\tau}(\Delta\tau)$ with the elements $k_{k,l}(\Delta\tau)$ of the theoretical $l_\alpha r_\alpha$-covariance function ${}_{lr}K_{\tilde{X}_\tau}(\Delta\tau)$. The solution of the minimization problem is found with the aid of the modified evolution strategy by (Möller and Beer, 2004). Constraint of the optimization problem is Eq. (22) for all realizations of the process.

The parameter estimation according to Eq. (33) postulates stationary and ergodic fuzzy time series, otherwise it would be obviously futile to estimate the empirical parameters for each point in time. On this account a second approach for parameter estimation of nonstationary fuzzy time series is presented. The aim is to minimize the mean distance $\overline{d}_F$ between optimal one-step forecasts $\mathring{\tilde{x}}_\tau(\underline{P})$ and the known fuzzy values $\tilde{x}_\tau$ of the fuzzy time series for $p < \tau \leq N$ according to Eq. (34). Advantage of this method is the fact, that neither stationary nor ergodic fuzzy time series are presupposed. The approach allows the modeling of nonstationary fuzzy time series with the aid of nonstationary fuzzy random processes without empiric parameters.

$$\overline{d}_F(\underline{P}) = \frac{1}{N-p} \sum_{\tau=p+1}^{N} d_F\left(\tilde{x}_\tau; \mathring{\tilde{x}}_\tau(\underline{P})\right) \stackrel{!}{=} \min \tag{34}$$

Depending on the demanded process parameters $\underline{P}$ (i. e. $\underline{A}_1, \ldots, \underline{A}_p$ and $\underline{B}_1, \ldots, \underline{B}_q$) the optimal one-step-forecasts $\mathring{\tilde{x}}_\tau(\underline{P})$ are computed for each point in time. The distances d_F between $\mathring{\tilde{x}}_\tau(\underline{P})$ and the fuzzy values $\tilde{x}_\tau$ of the fuzzy time series are averaged over time. The minimization of the mean distance $\overline{d}_F$ provides unbiased estimators of the process parameters. The calculation of the optimal one-step forecasts $\mathring{\tilde{x}}_\tau(\underline{P})$ is given in Section 4. The definition of the distance d_F between two fuzzy variables is given as follows.

According to the metrics introduced in (Körner, 1997) the distance $d_F(\tilde{x}; \tilde{y})$ between fuzzy variables $\tilde{x}$ and $\tilde{y}$ is defined as the integral over the Hausdorff distance $d_H(\cdot;\cdot)$ between the α-level sets X_α and Y_α

of $\tilde{x}$ and $\tilde{y}$ given by Eq. (35).

$$d_F\left(\tilde{x};\tilde{y}\right) = \int_0^1 d_H\left(X_\alpha;Y_\alpha\right)d\alpha \tag{35}$$

The Hausdorff distance $d_H\left(X_\alpha;Y_\alpha\right)$ between two non-empty compact sets $X_\alpha;Y_\alpha \subseteq \mathbb{R}$ is defined by Eq. (36).

$$d_H\left(X_\alpha;Y_\alpha\right) = \max\left\{\sup_{x\in X_\alpha}\inf_{y\in Y_\alpha} d_E\left(x;y\right); \sup_{y\in Y_\alpha}\inf_{x\in X_\alpha} d_E\left(x;y\right)\right\} \tag{36}$$

At this $d_E(x;y)$ is the Euclidean distance between two real-valued variables $x, y \in \mathbb{R}$ according to Eq. (37).

$$d_E\left(x;y\right) = |x-y| = \sqrt{(x-y)^2} \tag{37}$$

In the following a third approach for estimation of the parameters $\underline{A}_1$, ..., $\underline{A}_p$ and $\underline{B}_1$, ..., $\underline{B}_q$ of fuzzy-ARMA-processes is presented. This approach also does not presuppose stationary or ergodic fuzzy time series. The concept is to compare the optimal one-step forecasts $\mathring{\tilde{x}}_\tau(\underline{P})$ with the known fuzzy values $\tilde{x}_\tau$ of the fuzzy time series for $p < \tau \leq N$ according to Eq. (38). The error E is defined as the square deviation of the forecasted $l_\alpha r_\alpha$-increments $\Delta\mathring{x}_j(\tau,\underline{P})$ ($j = 1,\, 2,\, ...,\, 2n$) to the known $l_\alpha r_\alpha$-increments $\Delta x_i(\tau)$ of the fuzzy time series. Advantage of this method is that the solution of the minimization problem can be found with the method of gradients.

$$E = \frac{1}{2}\sum_{\tau=1+p}^{N}\sum_{i=1}^{2n}\left(\Delta x_i(\tau) - \Delta\mathring{x}_i(\tau,\underline{P})\right)^2 \stackrel{!}{=} \min \tag{38}$$

After initialization the parameter matrices $\underline{A}_1$, ..., $\underline{A}_p$ and $\underline{B}_1$, ..., $\underline{B}_q$ are improved with the aid of matrices $\Delta\underline{A}_1$, ..., $\Delta\underline{A}_p$ and $\Delta\underline{B}_1$, ..., $\Delta\underline{B}_q$ according to Eqs. (39) and (40).

$$\underline{A}_r(\text{new}) = \underline{A}_r(\text{old}) + \Delta\underline{A}_r \quad \text{with} \quad r = 1, 2, ..., p \tag{39}$$

$$\underline{B}_s(\text{new}) = \underline{B}_s(\text{old}) + \Delta\underline{B}_s \quad \text{with} \quad s = 1, 2, ..., q \tag{40}$$

The matrices $\Delta\underline{A}_1$, ..., $\Delta\underline{A}_p$ and $\Delta\underline{B}_1$, ..., $\Delta\underline{B}_q$ are built proportional to the partial derivativs of the error E with respect to the belonging parameter matrices according to Eqs. (41) and (42). The factor η ($\eta > 0$) defines the increment.

$$\Delta\underline{A}_r = -\eta\frac{\partial E}{\partial\underline{A}_r} = -\eta\begin{bmatrix}\frac{\partial E}{\partial a_{1,1}(r)} & \cdots & \frac{\partial E}{\partial a_{1,2n}(r)} \\ \vdots & \ddots & \vdots \\ \frac{\partial E}{\partial a_{2n,1}(r)} & \cdots & \frac{\partial E}{\partial a_{2n,2n}(r)}\end{bmatrix} \tag{41}$$

$$\Delta\underline{B}_s = -\eta\frac{\partial E}{\partial\underline{B}_s} = -\eta\begin{bmatrix}\frac{\partial E}{\partial b_{1,1}(s)} & \cdots & \frac{\partial E}{\partial b_{1,2n}(s)} \\ \vdots & \ddots & \vdots \\ \frac{\partial E}{\partial b_{2n,1}(s)} & \cdots & \frac{\partial E}{\partial b_{2n,2n}(s)}\end{bmatrix} \tag{42}$$

The partial derivatives $\frac{\partial E}{\partial a_{u,v}(r)}$ and $\frac{\partial E}{\partial b_{u,v}(s)}$ of the error E with respect to the single elements of the parameter matrices are defined by Eqs. (43) and (44) ($u, v = 1, 2, ..., 2n$).

$$\frac{\partial E}{\partial a_{u,v}(r)} = \sum_{\tau=1+p}^{N} \left(\Delta x_u(\tau) - \Delta \mathring{x}_u(\tau, \underline{P})\right) \Delta x_v(\tau - r) \tag{43}$$

$$\frac{\partial E}{\partial b_{u,v}(s)} = \sum_{\tau=1+p}^{N} \left(\Delta x_u(\tau) - \Delta \mathring{x}_u(\tau, \underline{P})\right) \Delta \hat{\varepsilon}_v(\tau - s) \tag{44}$$

The terms $\Delta\hat{\varepsilon}_v(\tau - s)$ are the $l_\alpha r_\alpha$-increments of the estimated realizations $\hat{\tilde{\varepsilon}}_\tau$ of the fuzzy-white-noise-process $(\tilde{\mathcal{E}}_\tau)_{\tau\in\mathbf{T}}$. For each point in time $\tau - s \leq p$ the (not ascertainable) realizations $\Delta\hat{\varepsilon}_v(\tau - s)$ are replaced by the estimated expected value $\hat{E}\left[\Delta\varepsilon_v\right]$.

$$\frac{\partial E}{\partial b_{u,v}(s)} = \sum_{\tau=1+p}^{N} \left(\Delta x_u(\tau) - \Delta \hat{x}_u(\tau, \underline{P})\right) \hat{E}\left[\Delta\varepsilon_v\right] \tag{45}$$

Constraint of the minimization problem is the demand of non-negativity of the estimated $l_\alpha r_\alpha$-increments $\Delta\hat{\varepsilon}_j(\tau)$ and furthermore the $\hat{\tilde{\varepsilon}}_\tau$ have to satisfy the conditions of a fuzzy-white-noise-process.

4. Forecast strategies

Goal of forecast is the determination of future fuzzy data $\tilde{x}_{N+h}$ ($h = 1, 2, ...$) following an observed time series with fuzzy data $\tilde{x}_1, \tilde{x}_2, ..., \tilde{x}_N$. Fundamental precondition for this purpose is the assumption and estimation of an underlying fuzzy random process $(\tilde{\mathrm{X}}_\tau)_{\tau\in\mathbf{T}}$. Thus the validity of a forecast is associated with the validity of the postulated fuzzy random process.

Therefor a time series with fuzzy data $\tilde{x}_1, \tilde{x}_2, ..., \tilde{x}_N$ is interpreted as a realization of a fuzzy random process $(\tilde{\mathrm{X}}_\tau)_{\tau\in\mathbf{T}}$. Consequently forecast is the estimation of fuzzy variables $\tilde{x}_{N+h}$ belonging to the same realization. Analogical the classical time series analysis (Schlittgen and Streitberg, 2001) a forecasted fuzzy data is regarded as a realization $\vec{\tilde{x}}_{N+h}$ of a fuzzy random forecast process $\vec{\tilde{\mathrm{X}}}_{N+h} = \vec{\tilde{\mathrm{X}}}_{N+h}(\tilde{x}_1, \tilde{x}_2, ..., \tilde{x}_N)$, at which $\vec{\tilde{\mathrm{X}}}_{N+h}$ is a random variable depending on the realizations $\tilde{x}_1, \tilde{x}_2, ..., \tilde{x}_N$ of the fuzzy random variables $\tilde{\mathrm{X}}_1, \tilde{\mathrm{X}}_2, ..., \tilde{\mathrm{X}}_N$.

The fuzzy random forecast process $(\vec{\tilde{\mathrm{X}}}_\tau)_{\tau\in\mathbf{T}}$ of an underlying fuzzy-ARMA$[p, q]$-process $(\tilde{\mathrm{X}}_\tau)_{\tau\in\mathbf{T}}$ is defined according to Eq. (46) where $h = 1,\ 2,\ ...$.

$$\begin{aligned}\vec{\tilde{\mathrm{X}}}_{N+h} &= \underline{A}_1 \odot \vec{\tilde{\mathrm{X}}}_{N+h-1} \oplus ... \oplus \underline{A}_p \odot \vec{\tilde{\mathrm{X}}}_{N+h-p} \oplus \tilde{\mathcal{E}}_{N+h} \ominus \\ &\quad \underline{B}_1 \odot \tilde{\mathcal{E}}_{N+h-1} \ominus ... \ominus \underline{B}_q \odot \tilde{\mathcal{E}}_{N+h-q}\end{aligned} \tag{46}$$

$$\text{with} \qquad \vec{\tilde{\mathrm{X}}}_{N+h-u} = \begin{cases} \tilde{x}_{N+h-u} & \text{for } N+h-u \leq N \\ \vec{\tilde{\mathrm{X}}}_{N+h-u} & \text{for } N+h-u > N \end{cases}, \quad u = 1, 2, ..., p$$

$$\text{and} \qquad \tilde{\mathcal{E}}_{N+h-v} = \begin{cases} \tilde{\varepsilon}_{N+h-v} & \text{for } N+h-v \leq N \\ \tilde{\mathcal{E}}_{N+h-v} & \text{for } N+h-v > N \end{cases}, \quad v = 1, 2, ..., q$$

Thereby for each point in time $\tau = N + h - u \leq N$ the observed fuzzy variables $\tilde{x}_{N+h-u}$ are inserted for $\vec{\tilde{\mathrm{X}}}_{N+h-u}$. For each point in time $\tau = N + h - v \leq N$ the $\tilde{\mathcal{E}}_{N+h-v}$ are replaced by the realizations $\tilde{\varepsilon}_{N+h-v}$ of the fuzzy-white-noise-process $(\tilde{\mathcal{E}}_\tau)_{\tau\in\mathbf{T}}$.

4.1. OPTIMAL FORECAST

The optimal forecast $\mathring{\tilde{x}}_{N+h}$ to the time point $\tau = N + h$ is defined as the conditional fuzzy expected value according to Eq. (47).

$$\mathring{\tilde{x}}_{N+h}(\tilde{x}_1, \tilde{x}_2, ..., \tilde{x}_N) = E[\tilde{\mathrm{X}}_{N+h} \,|\, \tilde{x}_1, \tilde{x}_2, ..., \tilde{x}_N] = E[\vec{\tilde{\mathrm{X}}}_{N+h}] \tag{47}$$

In the following the optimal forecast of a fuzzy-ARMA-process, which is the underlying fuzzy random process of an observed sequence of fuzzy data, is introduced.

The optimal one-step forecast of a fuzzy-ARMA$[p, q]$-process $(\tilde{\mathrm{X}}_\tau)_{\tau\in\mathbf{T}}$ according to Eq. (29) is defined by Eq. (48).

$$\begin{aligned}\mathring{\tilde{x}}_{N+1} \;=\; & \underline{A}_1 \odot \tilde{x}_N \oplus ... \oplus \underline{A}_p \odot \tilde{x}_{N+1-p} \oplus E[\tilde{\mathcal{E}}_\tau] \ominus \\ & \underline{B}_1 \odot \tilde{\varepsilon}_N \ominus ... \ominus \underline{B}_q \odot \tilde{\varepsilon}_{N+1-q}\end{aligned} \tag{48}$$

The optimal h-step forecast is obtained by recursive use of the optimal one-step forecast according to Eq. (48). Consequently the forecasted fuzzy data converge with increasing forecast step h on the fuzzy expected value $E[\tilde{\mathrm{X}}_\tau]$. The optimal h-step forecast of a fuzzy-ARMA$[p, q]$-process $(\tilde{\mathrm{X}}_\tau)_{\tau\in\mathbf{T}}$ is defined by Eq. (49).

$$\begin{aligned}\mathring{\tilde{x}}_{N+h} \;=\; & \underline{A}_1 \odot \tilde{x}_{N+h-1} \oplus ... \oplus \underline{A}_p \odot \tilde{x}_{N+h-p} \oplus E[\tilde{\mathcal{E}}_\tau] \ominus \\ & \underline{B}_1 \odot \hat{\tilde{\varepsilon}}_{N+h-1} \ominus ... \ominus \underline{B}_q \odot \hat{\tilde{\varepsilon}}_{N+h-q}\end{aligned} \tag{49}$$

$$\text{with} \qquad \tilde{x}_{N+h-u} = \begin{cases} \tilde{x}_{N+h-u} & \text{für } N+h-u \leq N \\ \mathring{\tilde{x}}_{N+h-u} & \text{für } N+h-u > N \end{cases} , \;\; u = 1, 2, ..., p$$

$$\text{and} \qquad \tilde{\varepsilon}_{N+h-v} = \begin{cases} \tilde{\varepsilon}_{N+h-v} & \text{für } N+h-v \leq N \\ E[\tilde{\mathcal{E}}_\tau] & \text{für } N+h-v > N \end{cases} , \;\; v = 1, 2, ..., q$$

Thereby for each point in time $\tau = N + h - u \leq N$ the optimal forecasts $\mathring{\tilde{x}}_{N+h-u}$ are inserted for $\tilde{x}_{N+h-u}$. For each point in time $\tau = N + h - v > N$ the $\tilde{\varepsilon}_{N+h-v}$ are replaced by the fuzzy expected value $E[\tilde{\mathcal{E}}_\tau]$ of the fuzzy-white-noise-process $(\tilde{\mathcal{E}}_\tau)_{\tau\in\mathbf{T}}$.

4.2. FUZZY FORECAST INTERVALS

A fuzzy interval $\tilde{x}_I$ is refered to as fuzzy forecast interval $\tilde{x}^{\kappa}_{N+h}$, if realizations $\vec{\tilde{x}}_{N+h}$ of the fuzzy random forecast process $(\vec{\tilde{\mathrm{X}}}_\tau)_{\tau\in\mathbf{T}}$ are contained in $\tilde{x}_I$ with the probability κ. Fuzzy forecast intervals $\tilde{x}^{\kappa}_{N+h}$ at time point $\tau = N+h$ of a fuzzy time series $\tilde{x}_1, \tilde{x}_2, ..., \tilde{x}_N$ can be estimated with the aid of monte-carlo-simulation of the fuzzy random forecast process $(\vec{\tilde{\mathrm{X}}}_\tau)_{\tau\in\mathbf{T}}$. The monte-carlo-simulation of the fuzzy random forecast process $(\vec{\tilde{\mathrm{X}}}_\tau)_{\tau\in\mathbf{T}}$ (with an underlying fuzzy-ARMA$[p, q]$-process $(\tilde{\mathrm{X}}_\tau)_{\tau\in\mathbf{T}}$) is obtained by the recursive procedure according to Eq. (50). In the first step a realization $\vec{\tilde{x}}_{N+1}$ at time point $\tau = N + 1$ of the fuzzy random forecast process $(\vec{\tilde{X}}_\tau)_{\tau\in\mathbf{T}}$ is simulated. The realization $\vec{\tilde{x}}_{N+1}$ of the fuzzy random variable $\vec{\tilde{X}}_{N+1}$ depends on the realization $\tilde{\varepsilon}_{N+1}$ of the fuzzy-white-noise-variable $\tilde{\mathcal{E}}_{N+1}$. The fuzzy variables $\tilde{x}_\tau$ and $\tilde{\varepsilon}_\tau$ at time points $\tau \leq N$ are given by the time series.

$$\begin{aligned}\vec{\tilde{X}}_{N+1} \;=\; & \underline{A}_1 \odot \tilde{x}_N \oplus ... \oplus \underline{A}_p \odot \tilde{x}_{N+1-p} \oplus \tilde{\mathcal{E}}_{N+1} \ominus \\ & \underline{B}_1 \odot \tilde{\varepsilon}_N \ominus ... \ominus \underline{B}_q \odot \tilde{\varepsilon}_{N+1-q}\end{aligned} \tag{50}$$

The fuzzy variable $\vec{\tilde{x}}_{N+1}$ is obtained by monte-carlo-simulation of a realization $\tilde{\varepsilon}_{N+1}$. By use of the obtained fuzzy variable $\vec{\tilde{x}}_{N+1}$ and monte-carlo-simulation of a realization $\tilde{\varepsilon}_{N+2}$ the fuzzy variable $\vec{\tilde{x}}_{N+2}$ is obtained in the next step. A successive computing at time points $\tau = N + 1,\ N + 2,\ ...$ results one potential future gradient of the fuzzy time series $(\tilde{x}_\tau)_{\tau\in\mathbf{T}}$. By repetition of this procedure a number of potential future realizations is obtained.

With the aid of s simulated potential future realizations of a fuzzy time series $(\tilde{x}_\tau)_{\tau\in\mathbf{T}}$ fuzzy forecast intervals $\tilde{x}^\kappa_{N+h}$ can be estimated as follows. The interval boundaries $\vec{x}_{\alpha_i l}(N + h)$ and $\vec{x}_{\alpha_i r}(N + h)]$ of the α-level sets $\vec{X}_{\alpha_i}(N + h)$ of the s simulated fuzzy variables $\vec{\tilde{x}}_{N+h}$ are arranged according to size and subscripted according to Eq. (51).

$$\begin{gathered}\vec{x}^1_{\alpha_i l}(N+h) \leq \vec{x}^2_{\alpha_i l}(N+h) \leq ... \leq \vec{x}^s_{\alpha_i l}(N+h) \\[1em] \vec{x}^1_{\alpha_i r}(N+h) \leq \vec{x}^2_{\alpha_i r}(N+h) \leq ... \leq \vec{x}^s_{\alpha_i r}(N+h)\end{gathered} \tag{51}$$

The interval boundaries $x^\kappa_{\alpha_i l}(N + h)$ and $x^\kappa_{\alpha_i r}(N + h)$ of the α-level sets $X^\kappa_{\alpha_i}(N + h)$ of a fuzzy forecast interval $\tilde{x}^\kappa_{N+h}$ at time point $\tau = N + h$ can be estimated according to Eq. (52) for a confidence level κ. Eq. (52) is valid for an even number of s.

$$\begin{aligned} x^\kappa_{\alpha_i l}(N+h) \;&=\; \begin{cases} \leq \vec{x}^1_{\alpha_i l}(N+h) & \text{für} \quad a = 0 \\ \quad \vec{x}^a_{\alpha_i l}(N+h) & \text{für} \quad 0 < a \leq \frac{s}{2} \end{cases} \\ & \text{with} \quad a = \text{int}\left[s \cdot \left(\frac{1}{2} - \frac{\kappa}{2}\right)\right] \\ x^\kappa_{\alpha_i r}(N+h) \;&=\; \begin{cases} \quad \vec{x}^{b+1}_{\alpha_i r}(N+h) & \text{für} \quad \frac{s}{2} \leq b < s \\ \geq \vec{x}^s_{\alpha_i r}(N+h) & \text{für} \quad b = s \end{cases} \\ & \text{with} \quad b = \frac{s}{2} + \text{int}\left[s \cdot \left(\frac{\kappa}{2}\right)\right] \end{aligned} \tag{52}$$

The interval boundaries $x^\kappa_{\alpha_i l}(N + h)$ and $x^\kappa_{\alpha_i r}(N + h)$ of the α-level sets $X^\kappa_{\alpha_i}(N + h)$ according to Eq. (52) correspond with the lower and upper quantile of the empiric distribution of the interval boundaries. Therewith future realizations $\vec{\tilde{x}}_{N+h}$ of a fuzzy time series $(\tilde{x}_\tau)_{\tau\in\mathbf{T}}$ are contained in the fuzzy forecast interval $\tilde{x}^\kappa_{N+h}$ with a probability κ.

4.3. FUZZY RANDOM FORECAST

The forecast strategies presented in sections 4.1 and 4.2 provide concrete fuzzy variables and fuzzy intervals. In the following a fuzzy random forecast is presented, which provides estimators for future fuzzy random variables $\vec{\tilde{\mathrm{X}}}_\tau$ of the fuzzy random forecast process $(\vec{\tilde{\mathrm{X}}}_\tau)_{\tau\in\mathbf{T}}$ at time points $\tau = N+h$. Therewith statements about the probability of future fuzzy variables are feasible.

By monte-carlo-simulation of s potential future realizations $(\vec{\tilde{x}}_{N+h} \,|\, \tilde{x}_1, \tilde{x}_2, ..., \tilde{x}_N)$ of the fuzzy time series $(\tilde{x}_\tau)_{\tau\in\mathbf{T}}$ the fuzzy random variable $\vec{\tilde{\mathrm{X}}}_{N+h}$ can be estimated. For characterization of $\vec{\tilde{\mathrm{X}}}_{N+h}$ the first and second order moments of the fuzzy random variable $\vec{\tilde{\mathrm{X}}}_{N+h}$ are used. With the aid of the simulated fuzzy variables $\vec{\tilde{x}}^c_{N+h}$ $(c = 1,\ 2,\ ...,\ s)$ the estimator of the fuzzy expected value $E[\vec{\tilde{\mathrm{X}}}_{N+h}]$ is obtained as the fuzzy mean value $\bar{\tilde{x}}_{N+h}$ at time point $\tau = N+h$ according to Eq. (53).

$$\hat{E}[\vec{\tilde{\mathrm{X}}}_{N+h}] = \bar{\tilde{x}}_{N+h} = \frac{1}{s}\bigoplus_{c=1}^{s} \vec{\tilde{x}}^c_{N+h} \tag{53}$$

The fuzzy expected value $E[\vec{\tilde{\mathrm{X}}}_{N+h}]$ is identical with the optimal forecast $\mathring{\tilde{x}}_{N+h}$. The $l_\alpha r_\alpha$-subtraction $\hat{E}[\vec{\tilde{\mathrm{X}}}_{N+h}] \ominus E[\vec{\tilde{\mathrm{X}}}_{N+h}]$ (and $\bar{\tilde{x}}_{N+h} \ominus \mathring{\tilde{x}}_{N+h}$ respectively) is a measure for the performance of the simulation. With increasing number of s the norm of the empiric $l_\alpha r_\alpha$-variance ${}_{lr}Var$ of the $l_\alpha r_\alpha$-subtraction $\bar{\tilde{x}}_{N+h}(c) \ominus \mathring{\tilde{x}}_{N+h}$ $(c = 1,\ 2,\ ...,\ s)$ according to Eq. (54) converges on zero. Consequently, with increasing number s of realizations $\vec{\tilde{x}}^c_{N+h}$ $(c = 1,\ 2,\ ...,\ s)$ the simulation represents the characteristics of the fuzzy random forecast process $(\vec{\tilde{\mathrm{X}}}_\tau)_{\tau\in\mathbf{T}}$ superiorly.

$$\lim_{s\to\infty} \left| {}_{lr}Var\left[\bar{\tilde{x}}_{N+h}(c) \ominus \mathring{\tilde{x}}_{N+h} \;|\; c = 1,\ 2,\ ...,\ s\right]\right| = 0 \tag{54}$$

By defining a maximal value η for the norm of the empiric $l_\alpha r_\alpha$-variance ${}_{lr}Var$ according to Eq. (55) a minimum number s_m of realizations can be obtained. In other words, for a wanted performance η of the simulation a number of s_m realizations is needed.

$$\left| {}_{lr}Var\left[\bar{\tilde{x}}_{N+h}(c) \ominus \mathring{\tilde{x}}_{N+h} \;|\; c = 1,\ 2,\ ...,\ s_m\right]\right| \leq \eta \tag{55}$$

The elements of the $l_\alpha r_\alpha$-covariance function ${}_{lr}K_{\vec{\tilde{\mathrm{X}}}_\tau}(\tau_a, \tau_b)$ of the fuzzy random forecast process $(\vec{\tilde{\mathrm{X}}}_\tau)_{\tau\in\mathbf{T}}$ are defined by Eq. 56 where $i, j = 1, 2, ..., n$.

$$\hat{k}^{\alpha_i l}_{\alpha_j r}(\tau_a, \tau_b) = \frac{1}{s}\sum_{c=1}^{s}\Big[(\Delta\vec{x}^c_{\alpha_i l}(\tau_a) - \Delta\mathring{x}_{\alpha_i l}(\tau_a)) \tag{56}$$

$$(\Delta\vec{x}^c_{\alpha_j r}(\tau_b) - \Delta\mathring{x}_{\alpha_j r}(\tau_b))\Big] \tag{57}$$

Thereby the terms $\Delta\vec{x}^c_{\alpha_i l^*}(\tau)$ are the $l_\alpha r_\alpha$-increments of the simulated fuzzy variables $\vec{\tilde{x}}^c_\tau$ at time point $\tau > N$ and the terms $\Delta\mathring{x}_{\alpha_i l^*}(\tau)$ are the $l_\alpha r_\alpha$-increments of the optimal forecast $\mathring{\tilde{x}}_\tau$. The estimator for the $l_\alpha r_\alpha$-variance ${}_{lr}Var[\vec{\tilde{\mathrm{X}}}_\tau] = {}_{lr}\underline{\sigma}^2_{\vec{\tilde{\mathrm{X}}}_\tau}$ corresponds the diagonal elements of the estimated $l_\alpha r_\alpha$-covariance function ${}_{lr}\hat{K}_{\vec{\tilde{\mathrm{X}}}_\tau}(\tau_a, \tau_b)$ with $\tau_a = \tau_b = \tau$.

5. Examples

5.1. EXAMPLE 1

Analysis of time series with fuzzy data is demonstrated by way of heavy goods vehicle traffic over the brigde Blaues Wunder in Dresden. Since October 1999 a weight-in-motion measuring point records the entire traffic over the brigde. The data are kindly provided by the highway board department of Dresden. For the projected analysis the measured data for heavy goods vehicle are revised of weekend and holiday data and thereafter fuzzified based on the histogramms of each weekday. The time series thus obtained is assumed to be stationary. June 2002 to April 2003 is considered as time period analyzed. An section of the time series is shown in Fig. 4.

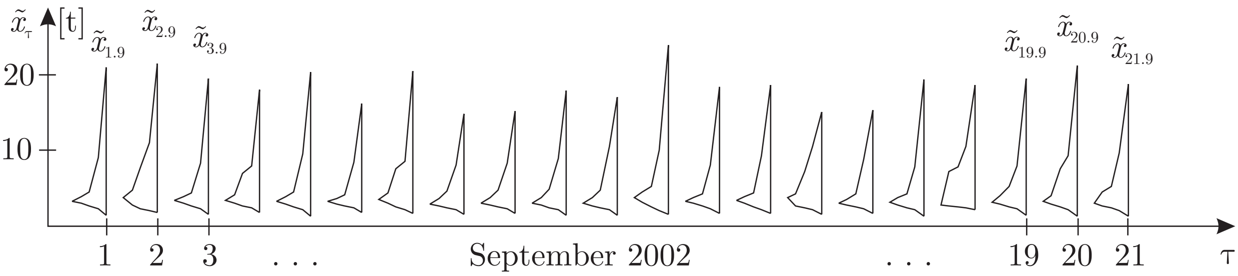

Figure 4. Time series with fuzzy data of heavy goods vehicle traffic over the bridge *Blaues Wunder* in Dresden (section)

The $l_\alpha r_\alpha$-discretization is applied to α-levels $\alpha_1 = 0.0$, $\alpha_2 = 0.25$, $\alpha_3 = 0.5$, $\alpha_4 = 0.75$ and $\alpha_5 = 1.0$. Fig. 5 shows exemplarily the plot of $l_\alpha r_\alpha$-increments $\Delta x_{\alpha_i l}$ and $\Delta x_{\alpha_i r}$.

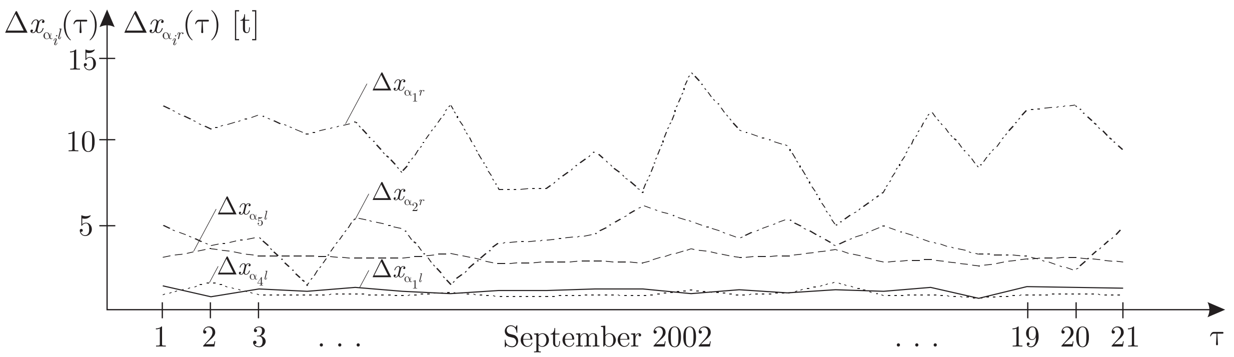

Figure 5. Plot of the $l_\alpha r_\alpha$-increments

Modeling of this time series with fuzzy data bases on a fuzzy-ARMA [10,0]-process. For estimation of the parameters $\underline{A}_1$, $\underline{A}_2, \ldots, \underline{A}_{10}$ the minimization problem according to Eq. (33) is solved. On this account the empirical fuzzy mean value $\tilde{\bar{x}}$ (see Fig. 6), the empirical $l_\alpha r_\alpha$-covariance function, and thus the empirical $l_\alpha r_\alpha$-variance are estimated from the time series under assumption of ergodicity. Consequently, it is demanded that the differences between the empirical and model characteristics (first and second order moments) are minimal.

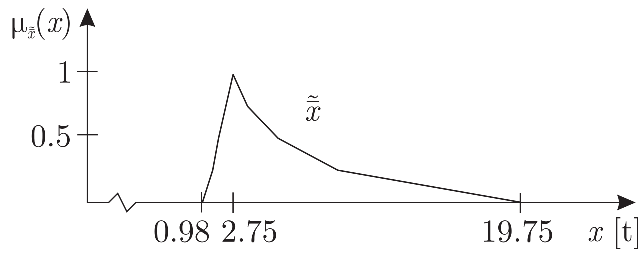

Figure 6. Fuzzy mean value

The solution of the optimization problem yields the following estimators: the process parameters $\underline{A}_1$, $\underline{A}_2, ..., \underline{A}_{10}$ as well as the fuzzy expected value $E[\tilde{\mathcal{E}}_\tau]$, the $l_\alpha r_\alpha$-variance ${}_{lr}Var[\tilde{\mathcal{E}}_\tau]$ and the $l_\alpha r_\alpha$-covariance function ${}_{lr}K_{\tilde{\mathcal{E}}_\tau}(\Delta\tau)$ as parameters of the fuzzy white noise process $(\tilde{\mathcal{E}}_\tau)_{\tau\in\mathbf{T}}$. With the aid of the estimated underlying fuzzy-ARMA[10,0]-process forecast of the following fuzzy data in May 2003 is feasible. The optimal 1-step-forecast of the fuzzy-ARMA[10,0]-process is given by Eq. (58).

$$\mathring{\tilde{x}}_{N+1} \;=\; \underline{A}_1 \odot \tilde{x}_N \oplus ... \oplus \underline{A}_{10} \odot \tilde{x}_{N-9} \oplus E[\tilde{\mathcal{E}}_\tau] \tag{58}$$

A repeated application of Eq. (58) results in the h-step forecast. The forecasted fuzzy data converge on the fuzzy expected value. The resulted fuzzy data in comparison to the real measured data are shown in Fig. 7. The forecast refers to the data for heavy goods vehicle on 12 weekdays in May 2003. The optimal forecasts differ somewhat from the real measurd data. Reason for it is, that the analysed fuzzy time series is characterized by a comparatively minor random influence

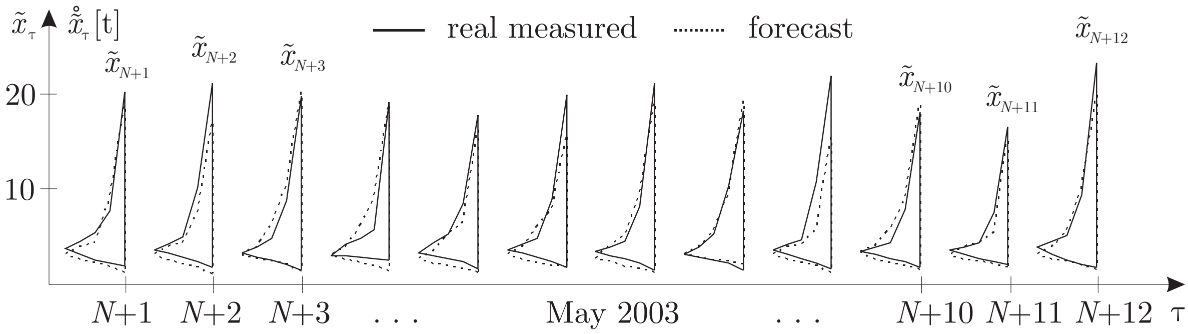

Figure 7. Optimal forecasts in comparison with the measured fuzzy time series

5.2. EXAMPLE 2

Analysis and forecast of nonstationary fuzzy time series is demonstrated by example of extensometer measurements. The given series was measured from 1999 to 2002 and is kindly provided by the EIBS GmbH, Dresden. Table I shows a short section of the measured time series over five days. Three different measuring data exist at each time point. Instead of computing the mean value the measuring difference is considered as uncertainty and modeled as fuzzy variable. The $l_\alpha r_\alpha$-discretization is realized for α-levels $\alpha_1 = 0$ and $\alpha_2 = 1$. Fig. 8 shows the plot of the fuzzy time series.

Table I. Section of extensometer measurements

date	**1st meas. [mm]**	**2nd meas. [mm]**	**3rd meas. [mm]**	**mean value [mm]**
⋮	⋮	⋮	⋮	⋮
30.05.2000	22.51	22.50	22.52	22.510
27.06.2000	22.50	22.52	22.53	22.517
27.07.2000	22.40	22.40	22.41	22.403
30.08.2000	22.35	22.36	22.35	22,353
27.09.2000	21.72	21.80	21.77	21.763
⋮	⋮	⋮	⋮	⋮

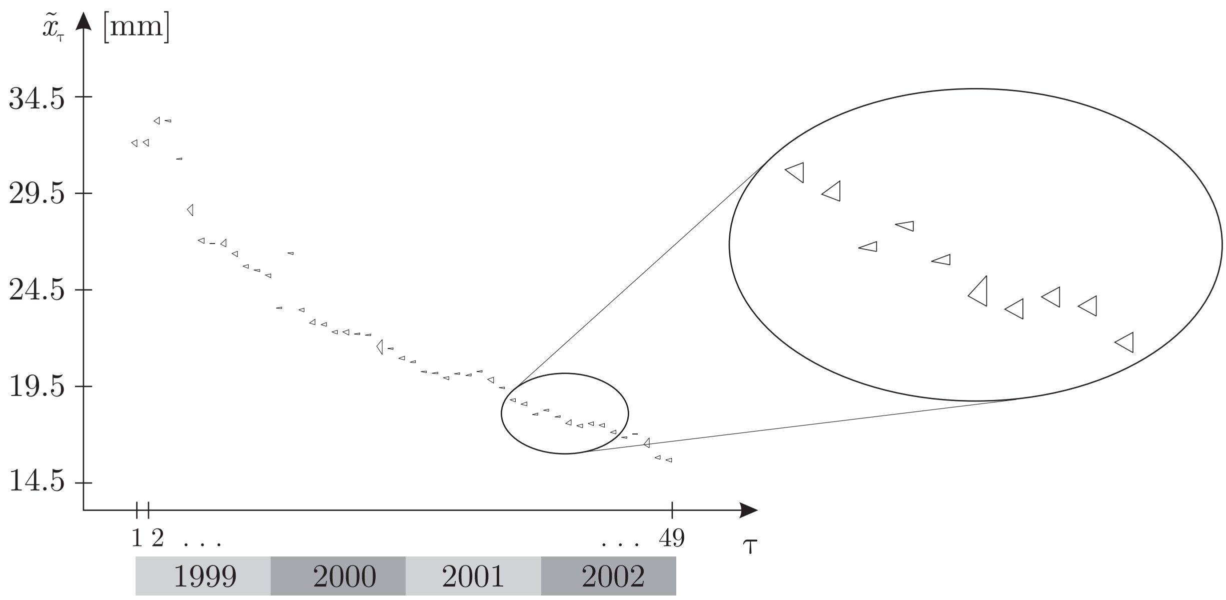

Figure 8. Time series with fuzzified extensometer measurements

The modeling of this fuzzy time series obviously requires a nonstationary fuzzy stochastic process model. The fuzzy time series is specified as nonstationary fuzzy-ARMA-process of the order $p = 10$ and $q = 3$. The estimation of the parameters $\underline{A}_1$, $\underline{A}_2$, ..., $\underline{A}_{10}$ and $\underline{B}_1$, $\underline{B}_2$, $\underline{B}_3$ is done with the aid of the optimization problem given by Eq. (34). This procedure yields optimal 1-step-forecasts with a minimized distance to the empirical fuzzy variables in the considered space of time. The result is shown in Fig. 9.

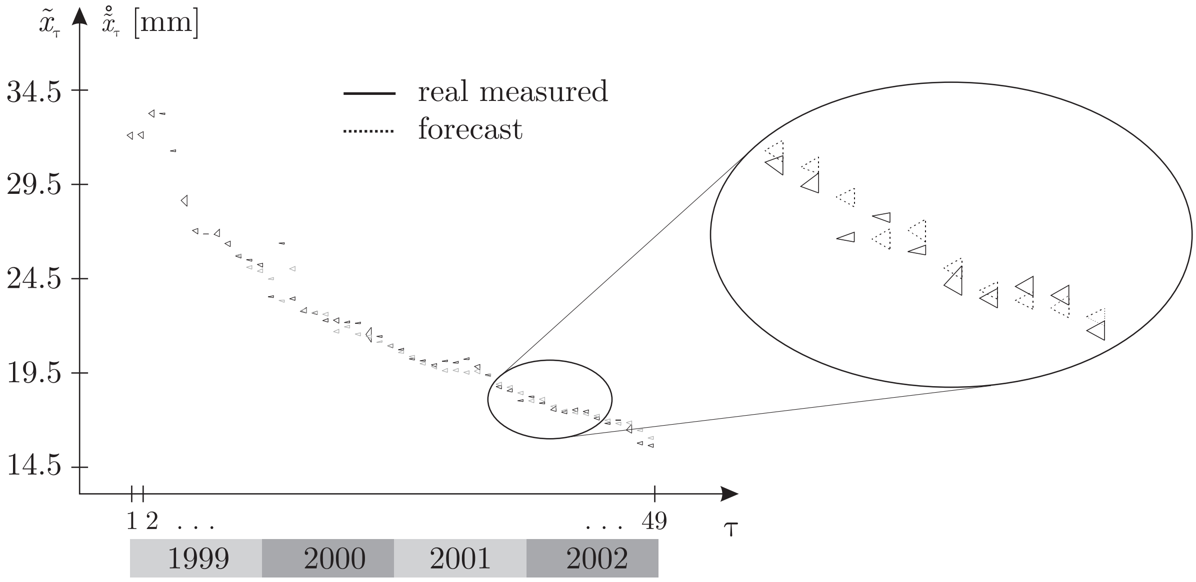

Figure 9. Optimal 1-step-forecasts of the fuzzy time series

For parameter estimation of the underlying fuzzy-ARMA[10,3]-process was based on the empirical fuzzy time series in the space of time from December 1998 until November 2002. The estimated fuzzy stochastic process enables the forecast of future settlements. The optimal long running forecast for the following 37 month is shown in Fig. 10. This is equivalent to a forecasting horizon of 3 years.

With the aid of the fuzzy-ARMA[10,3]-process the estimation of fuzzy forecast intervals is feasible. The fuzzy forecast intervals specify domains in which future realizations are contained with a confidence level κ. Exemplarily the fuzzy forecast intervals with the confidence level 0.95 are shown in Fig. 11.

6. Conclusions

In this paper a new approach for description and modeling of time series with uncertain data is presented. Uncertain data at equally spaced discrete time points are modeled as time series with fuzzy data. In this context a new method for representation of fuzzy data is presented. The $l_\alpha r_\alpha$-discretization enables a new statistical evaluation of fuzzy samples. At this the new fuzzy-ARMA-process is introduced. This process enables analysis and forecast of suitable time series with fuzzy data. The fuzzy-ARMA-process is successfully applied to a time series of heavy goods vehicle traffic data and a time series with uncertain extensometer measurements.

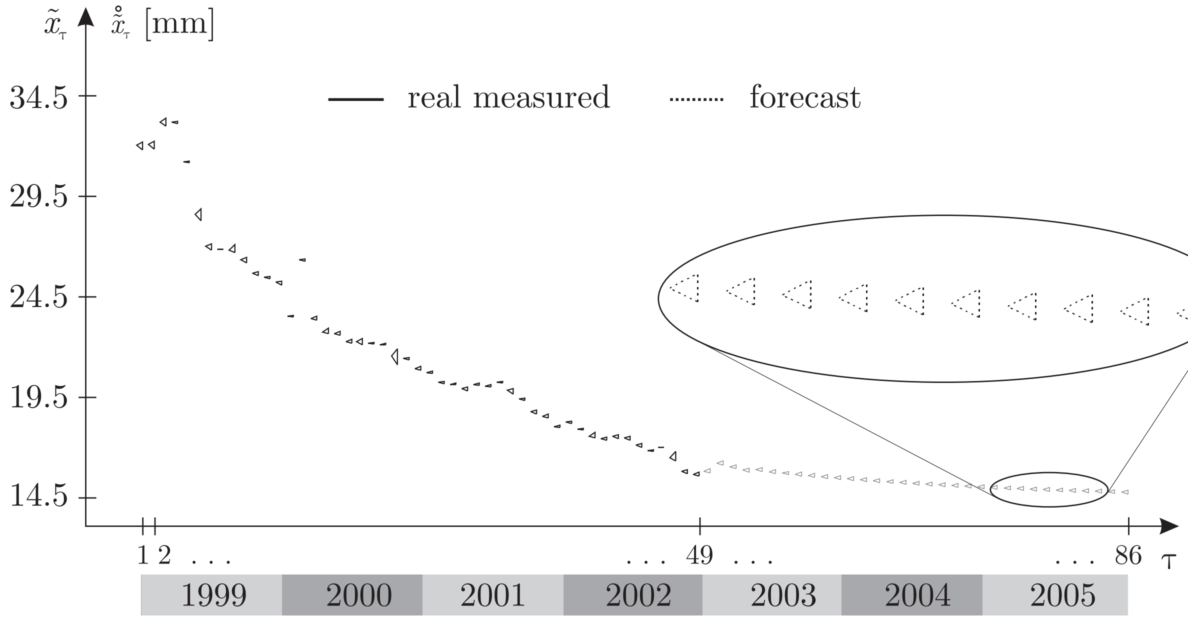

Figure 10. Optimal long running forecast of the fuzzy time series

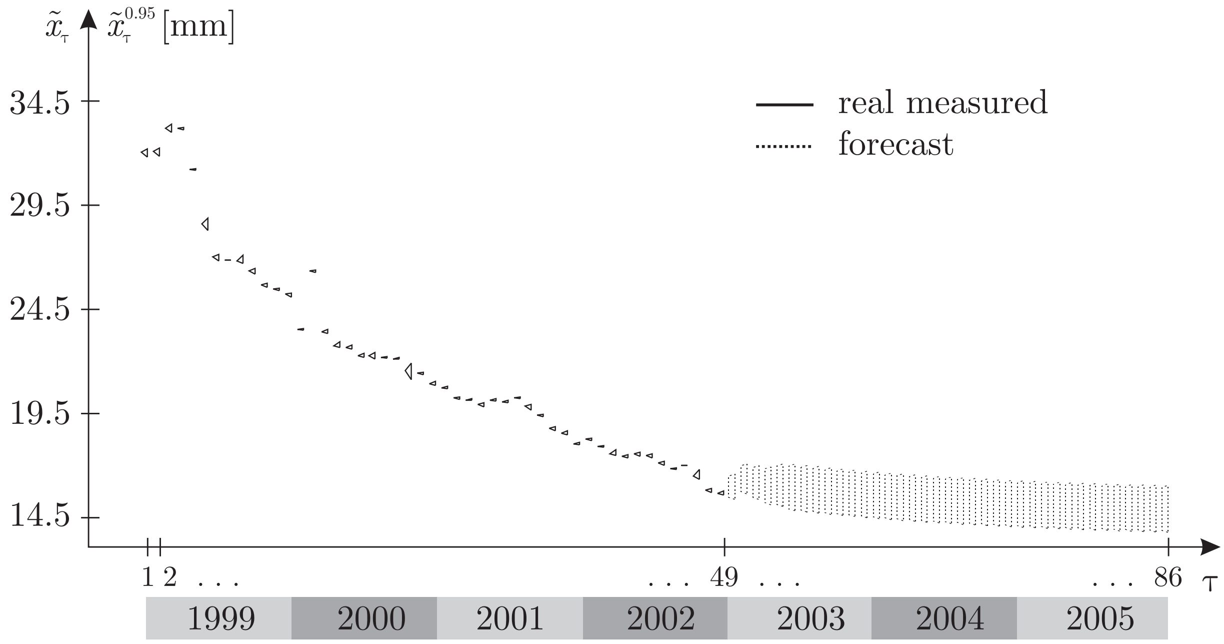

Figure 11. Fuzzy forecast intervals for a confidence level 0.95

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