Interval Arithmetic Technique for Constrained Reliability Optimization Problems

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This talk will consider:

- the basic ideas in the interval global optimization algorithm, and
- its application finding the global solution of reliability allocation problems

Problem Formulation

Particularly, we deal with the two reliability allocation problems:

$$
\begin{aligned}
\text{Minimize } C(x)&, \\
\text{Subject to } \; R(x) &\geq R_0, \\
x &= (x_1, x_2, \ldots, x_n), \\
x_i &\geq 2, \\
x_i \text{ is integer for } i &= 1, \ldots, n,
\end{aligned}
\tag{1}
$$

or

$$
\begin{aligned}
\text{Maximize } R(x)&, \\
\text{Subject to } \; C(x) &\leq C_0, \\
x &= (x_1, x_2, \ldots, x_n), \\
x_i &\geq 2, \\
x_i \text{ is integer for } i &= 1, \ldots, n
\end{aligned}
\tag{2}
$$

where $C(x)$ and $R(x)$ are differentiable functions, and $0 \leq R_0 \leq 1$.

Introduction

- Many algorithms have been proposed to solve nonlinear programming problems using optimization techniques, but only a few have been demonstrated to be effective when applied to large scale nonlinear programming problems for system reliability with redundancy (Tillman, Ramakumar, Harunuzzaman, Bulfin).
- Another drawback is that the solutions are no integers and hence the true optimal solution which must be integer is not guaranteed.
- The Lagrange multiplier method and the branch-and-bound technique are very commonly used for both the redundancy allocation problem and the mixed integer-type reliability-redundancy allocation problem.

- In recent years, interval techniques have proved to be effective solving nonlinear global optimization problems (Baker, Hansen, Ratschek).

Original System

An original system is assumed to consist of n units in series, with costs $c_1, c_2, \ldots, c_n$ and reliabilities $p_1, p_2, \ldots, p_n$.

$$\boxed{1} \quad \boxed{2} \qquad \cdots \quad \boxed{\text{n}}$$

Figure 1: Original System.

$$\text{Total original system cost :} \qquad C_0 = \sum_{i=1}^{n} c_i$$
$$\text{Total original system reliability:} \quad R_0 = \prod_{i=1}^{n} p_i$$

Unit Redundancies

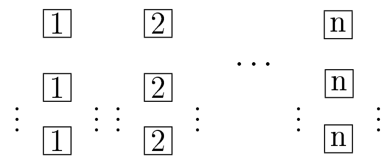

Figure 2: System with *Unit Redundancy Configuration*

If the individual units of the system are replicated, then we have *unit redundancy*, as shown in Figure 2.
Given the basic system with n different units as shown in Figure 1 , *the goal is to improve the overall system reliability to R, by using the unit redundancy with a minimum cost.*

Unit Redundancy

The two main questions to be addressed are:

a) What should be the minimum number of redundancies for the i-th unit, for $i = 1, 2, \ldots, n$, so that the system reliability is maximum?

b) What should be the number of redundancies for the i-th unit, for $i = 1, 2, \ldots, n$, so that the system cost is minimum?

Minimizing the System Cost

The constrained optimization problem is defined by

$$\begin{aligned} \text{minimize } C &= \sum_{i=1}^{n} c_i x_i, \\ \text{subject to } R &= \prod_{i=1}^{n} [1 - (1 - p_i)^{x_i}] \\ &\geq R_0, \end{aligned} \qquad (3)$$

Maximizing the System Reliability

The constrained optimization problem is defined by

$$\begin{aligned} \text{maximize } R &= \prod_{i=1}^{n} [1 - (1 - p_i)^{x_i}] \\ \text{subject to } C &= \sum_{i=1}^{n} c_i x_i \le C_0, \end{aligned} \tag{4}$$

where

C : total system cost
C_0 : the maximum required system cost
R : system reliability
R_0 : the minimum required system reliability
c_i : cost of unit i.
x_i : the number of units in parallel replacing the original unit i.
p_i : reliability of unit i.

Interval Arithmetic (IA)

- IA is an arithmetic on closed intervals of real numbers (Hansen, Kearfott). IA has been used in many optimization applications (Moore, Ratschek, Neumaier).
- $\boldsymbol{x} = (\boldsymbol{x}_{1,} \cdots, \boldsymbol{x}_n)$ is a real interval vector, where $\boldsymbol{x}_i = [\underline{x}_i, \overline{x}_i]$ for $i = 1, \ldots, n$.
- GlobSol is a global optimization package, developed on IA techniques. GlobSol find the global minimum of smooth and nonsmooth objective functions in a given feasible region $\boldsymbol{x}^{(0)}$.
- Assumption: the global optimum will occur at an interior stationary minimum of the objective function on $\boldsymbol{x}^{(0)}$.

Solution Methods

- The Lagrange Multipliers Method, the Kuhn-Tucker Conditions and Newton's Method.
- Branch-and-Bound Technique in Redundancy Allocation.

The branch-and-bound technique of integer programming in reliability optimization is developed by Nakagawa as follows:

1. Solve the problem as if all variables were real numbers. This solution is the upper bound for the maximization problem (or the lower bound for the minimization problem).
2. Choose one variable at a time that has a noninteger value, says x_j, and branch that variable to the next higher integer value for one problem and to the next lower integer value for the other. This results in two

constraints $x_j \geq [x_j] + 1$ and $x_j \leq [x_j]$ that are added in the two branched problems. Solve both problems by the Lagrange multiplier method.

3. Now variable j is an integer in either branch. Fix the integers of x_j for the following steps of branch and bound. Select the branch that results in higher system reliability. Then repeat step2 on another variable $x_k \neq x_j$ for each of the new problems until all variables become integers.

4. Stop branching the problem if the solution is worse than the current best integer solution. Stop the iteration when all the desired integer variables are obtained.

Multivariate Interval Newton Method

The nonlinear simultaneous equations $F(x) = (f_1(x), \ldots, f_n(x))^T = (0, \ldots, 0)^T$ can also be solved by using multivariate interval Newton methods, which are developed for both smooth and non smooth cases. Convergence and existence or uniqueness verification with interval Newton methods have been studied in the past Baker, Moore, and Neumaier.
The solution algorithm presented in this paper is applied to a sequence of intervals, beginning with some initial interval vector $\boldsymbol{x}^{(0)}$ given by the user. The initial interval can be chosen to be sufficiently large to enclose all physically feasible points. It is assumed that the global optimum will occur at an interior stationary minimum of the objective function and not at the boundaries of $\boldsymbol{x}^{(0)}$.

Solution Algorithm

Let $\boldsymbol{x}^{(0)}$ be an initial interval vector. For an interval vector $\boldsymbol{x}^{(k)}$ in the sequence of interval vectors $\boldsymbol{x}^{(0)} \supset \boldsymbol{x}^{(1)} \supset \cdots \supset \boldsymbol{x}^{(k)}$, follow the steps:

1. Compute interval evaluations for the gradient of the objective function, $\nabla C(\boldsymbol{x}^{(k)})$, and the constraint function, $R(\boldsymbol{x}^{(k)})$.
2. *Gradient range test* If the zero vector is not in the gradient of C, $0 \notin \nabla C(\boldsymbol{x}^{(k)})$, then $\boldsymbol{x}^{(k)}$ is discarded, thus no solution of $\nabla C(\boldsymbol{x}^{(k)}) = 0$ exists in this interval vector. Otherwise, the testing of $\boldsymbol{x}^{(k)}$ continues.

3. *Objective range test* Compute an interval evaluation of the objective function, $C(\boldsymbol{x}^{(k)})$. If the lower bound of $C(\boldsymbol{x}^{(k)})$ is greater than a known upper bound on the global minimum of $C(x)$, then $\boldsymbol{x}^{(k)}$ cannot contain the global minimum, and it is discarded. Otherwise, testing of $\boldsymbol{x}^{(k)}$ continues.

4. *Interval Newton test* Solve the linear interval equation system for a new interval N_k

$$C''(\boldsymbol{x}^{(k)})(N_k - \widehat{\boldsymbol{x}^{(k)}}) = -\nabla C(\widehat{\boldsymbol{x}^{(k)}}),$$

   where $C''(\boldsymbol{x}^{(k)})$ is an interval evaluation of the Hessian matrix of $C(x)$, over the current interval $\boldsymbol{x}^{(k)}$, where $\widehat{\boldsymbol{x}^{(k)}}$ is the midpoint of $\boldsymbol{x}^{(k)}$. It can be shown that if $\boldsymbol{x}^*$ is a root of $\nabla C(\boldsymbol{x}^{(k)}) = 0$, then it is also contained in N_k.

a. If $N_k \cap \boldsymbol{x}^{(k)} = \emptyset$, then $\nabla C(\boldsymbol{x}^{(k)}) = 0$ does not have a root in $\boldsymbol{x}^{(k)}$ and $\boldsymbol{x}^{(k)}$ is discarded.

b. Evaluate $C(\widehat{\boldsymbol{x}^{(k)}})$ and find an upper bound for use in Step 3.

c. If $N_k \cap \boldsymbol{x}^{(k)} = N_k$, then there is exactly one root of $\nabla C(\boldsymbol{x}^{(k)}) = 0$ in $\boldsymbol{x}^{(k)}$, which may correspond to the global minimum.

d. If neither of the above is true, then no further conclusion can be drawn.

Numerical Results

The following examples compare GlobSol results with those of the Lagrange multiplier method for the unit redundancy optimization problem (3). G_i represents GlobSol results, LM represents results from the Lagrange multiplier method, and $LMBB$ results from the Lagrange multiplier method with branch-and-bound technique.

Example 1 *A basis series consists of 4 units with costs $c_1 = 1$, $c_2 = 2$, $c_3 = 4$, and $c_4 = 8$ units of money and reliability values $p_1 = 0,2$, $p_2 = 0.4$, $p_3 = 0.6$, $p_4 = 0.8$. Design a new system configuration incorporating the unit redundancy concept to achieve an overall system reliability of $R = 0.995$ at minimum cost. Solve the dual problem maximizing the system reliability with the cost constraint.*
Considering $x = (x_1, x_2, x_3, x_4)$, an integer programming problem corresponding to this example is defined as follows:

$$\text{Minimize } \phi(x) = x_1 + 2x_2 + 4x_3 + 8x_4$$

subject to the constraints

$$(1-0.8^{x_1})(1-0.6^{x_2})(1-0.4^{x_3})(1-0.2^{x_4}) \geq 0.995$$

$$x_i \text{ positive integer for } i = 1, 2, 3, 4.$$

Table 1: Results of Example 1

Method	Solution	Cost	Reliability
G_1	$(28, 14, 8, 4)$	120	.995035
G_2	$(30, 15, 7, 4)$	120	.995062
G_3	$(32, 14, 7, 4)$	120	.995194
G_4	$(30, 13, 8, 4)$	120	.995209
LM	$(30, 14, 8, 4)$	122	.99573

The original system cost is, $C_0 = 15$*, the minimum cost for the unit redundancy concept by using Globsol is* $C = 120 = 8C_0$*, and by using classical LM is* $C = 122 = 8.13C_0$.
Maximizing the overall system reliability with a maximum cost of C=120, the optimum solution was $(x_1, x_2, x_3, x_4) = (30, 13, 8, 4)$*, in Globsol and LM and a maximum system reliability of* $R = .9952$*. When the cost constraint is changed to* $C < 123$*, GlobSol and LM obtain the same* $(x_1, x_2, x_3, x_4) = (30, 14, 8, 4)$

Example 2 *Similar to Example 1, with unit costs* $c_1 = 2$, $c_2 = 3$, $c_3 = 4$, *and* $c_4 = 5$, *and reliability values* $p_1 = p_2 = p_3 = p_4 = 0.5$. *Design a new system configuration incorporating the unit redundancy concept to achieve an overall system reliability of* $R = 0.98$ *at minimum cost.*

Table 2: Results of Example 2

Method	Solution	Cost	Reliability
G_1	$(8, 8, 8, 7)$	107	.980606
LM	$(9, 8, 8, 7)$	109	.982528

The original system cost is $C_0 = 14$, *the minimum cost for the unit redundancy concept by using GlobSol is* $C = 107 = 7.64C_0$ *and by using LM is* $C = 109 = 7.79C_0$ *(see Table 2). The dual optimization problem, to maximize the overall system reliability with a maximum cost of 107, was also solved with GlobSol obtaining similar results. In both problems the optimum solution was*

$(x_1, x_2, x_3, x_4) = (8, 8, 8, 4)$, *with a total cost of* $C = 107$ *and a maximum system reliability of* $R = .980606$.

Example 3 *A 4-stage series system with two linear constraints is formulated as a pure integer programming problem. The decision variables,* $x = (x_1, x_2, x_3, x_4)$, *are the number of redundancies at each stage. The problem is formulated as follows*

$$\begin{aligned} \textit{maximize } R \;&=\; \prod_{i=1}^{4}\left[1-(1-r_i)^{x_i}\right] \\ \textit{subject to } \sum_{j=1}^{4} c_{ij}x_i \;&\le\; b_i, \quad i = 1, 2 \end{aligned} \tag{5}$$

Table 3: Data for Example 3

Stage, j	1	2	3	4
r_j	0.80	0.70	0.75	0.85
c_{1j}	1.2	2.3	3.4	4.5
c_{2j}	5	4	8	7
$b_1 = 56$				
$b_2 = 120$				

With the data given in Table 3, the real solution obtained by the LM and the Kuhn-tucker conditions is, $x = (5.11672, 6.30536, 5.23536, 3.90151)$, *using interval and LMBB techniques give the same integer solution* $x = (5, 6, 5, 4)$. *Even both methods provide the same conclusions about the decision variables, interval techniques provide a more rigorous reasoning by guaranteeing the optimality for this problem.*

Example 4 *A 5-stage series system with three nonlinear constraints is formulated as a mixed integer programming problem. Both the number of redundancies, x_j, and the component reliability, r_j, are to be determined. The problem from Tillman's book is*

$$\text{maximize } R_s(x,r) = \prod_{i=1}^{4}[1-(1-r_i)^{x_i}]$$

$$\text{subject to } g_1(x) = \sum_{j=1}^{5} p_j x_j^2 - P \leq 0$$

$$g_2(x,r) = \sum_{j=1}^{5} \alpha_j\left(\frac{-t}{\ln r_j}\right)^{\beta_j}(x_j + \exp(x_j/4)) - C \leq 0$$

$$g_3(x) = \sum_{j=1}^{5} \omega_j x_j \exp(x_j/4)) - W \leq 0$$

$$x_j \geq 1 \text{ are integers and } \quad 0 < r_j < 1 \text{ for all } j.$$

With the data given in Table 4, the problem was solved with the methods: the LMBB, and with a combination of the sequential method, Hooke and Jeeve Pattern Search, and the heuristic redundancy allocation method HJHRA [21].

The results are shown in Table 5, for the LMBB method with the solution $(R_s, r, x) = (.9298, .7796, .8007, .9023, .7104, .8595, 3, 3, 2, 3, 2)$ *is superior to the HJHRA method with the solution* $(R_s, r, x) = (.9149, .7582, .8000, .9000, .8000, .7500, 3, 3, 2, 2, 3)$ *given in [21]. This mixed integer programming problem has many local optima. The HJHRA method has the drawback of being trapped by a local optimum, and the LMBB method overcomes this drawback and it is quite effective. Interval techniques provide the optimal solution for the redundancy allocation problem related to this problem. We could not verify the solution provided by the LMBB method to the mixed integer programming problem by using interval techniques in our computer systems.*

Table 4: Data for Example 4

j	α_j	p_j	ω_j	P	C	W
1	2.33×10^{-5}	1	7			
2	1.45×10^{-5}	2	8			
3	5.41×10^{-5}	3	8	110	175	200
4	8.05×10^{-5}	4	6			
5	1.95×10^{-5}	2	9			
	$\beta_j = 1.5, \quad j = 1, \ldots, 5$		t=1000			

Table 5: Comparison of Methods

	LMBB	HJHRA
Number of redundancies	$x = (3, 3, 2, 3, 2)$	$x = (3, 3, 2, 2, 3)$
Component reliability	$R = (.77960,$	$R = (.7582,$
	.80065,	.8000,
	.90227,	.9000,
	.71044,	.8000,
	.85947)	.7500)
System reliability	$R_s = .9298$	$R_s = .9149$

Conclusions

Interval arithmetic techniques, proved to be an effective tool to determine optimal design configurations for systems with unit redundancy, and can be used to solve mixed reliability-redundancy allocation problems. Interval arithmetic techniques are competitive alternatives since they provide management with different options and flexibility.

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