Using Extended Interval Algebra in Discrete Mechanics

Fulvio Tonon
University of Texas at Austin, USA

Traditional approach to modeling physical systems

1. Model a physical phenomenon using a differential equation (or a system of differential equations) or a variational principle
2. Obtain the algebraic forms of the differential equation(s) or variational principle by forcing them into the mold of discrete time and space
3. In order to commit to algorithms, project real-valued variables onto finite computer words => round-off & truncation

Traditional approach to modeling physical systems

2. **Obtain the algebraic forms of the differential equation(s) or variational principle by forcing them into the mold of discrete time and space**

Peraire and co-workers => Certificates for PDEs (certified bounds on discretization errors).

Elliptic problems, limit theory of plasticity, J-integral

Sauer-Budge, A.M., Bonet, J., Huerta, A., and Peraire, J., Computing bounds for linear functionals of exact weak solutions to Poisson's equation, *SIAM Journal on Numerical Analysis*, 42, 4 1610-1630, (2004).

Xuan, Z.C., Parés, N. and Peraire, J., Computing upper and lower bounds for the J-integral in two-dimensional linear elasticity. *Computer Meth. in Appl. Mech. and Engng.*, 195, 430-443 (2006).

Traditional approach to modeling physical systems

3. In order to commit to algorithms, project real-valued variables onto finite computer words => round-off & truncation

Traditional approach to modeling physical systems

1. Model a physical phenomenon using a differential equation (or a system of differential equations) or a variational principle
2. Obtain the algebraic forms of the differential equation(s) or variational principle by forcing them into the mold of discrete time and space
3. In order to commit to algorithms, project real-valued variables onto finite computer words => round-off & truncation

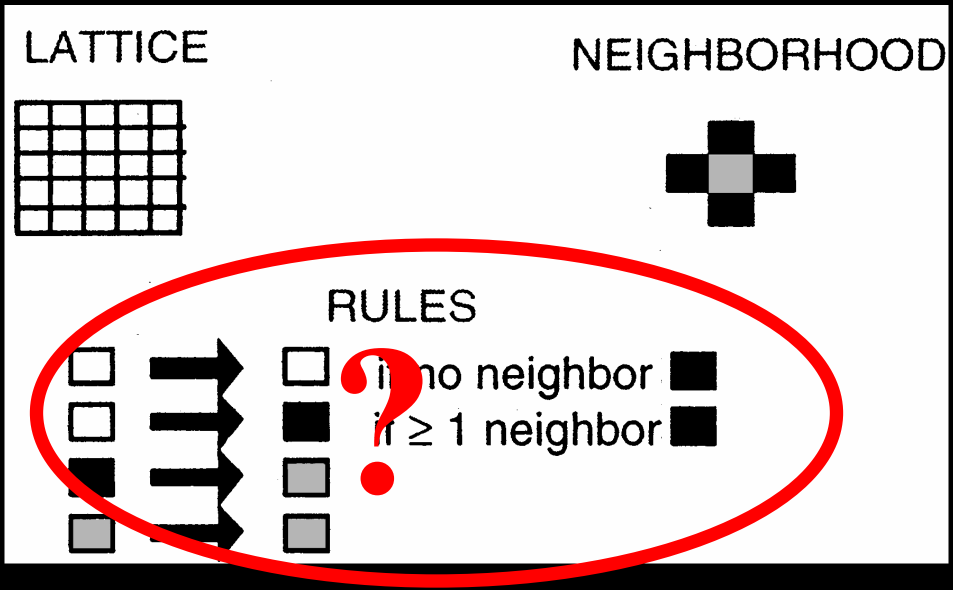
VERIFICATION AND VALIDATION

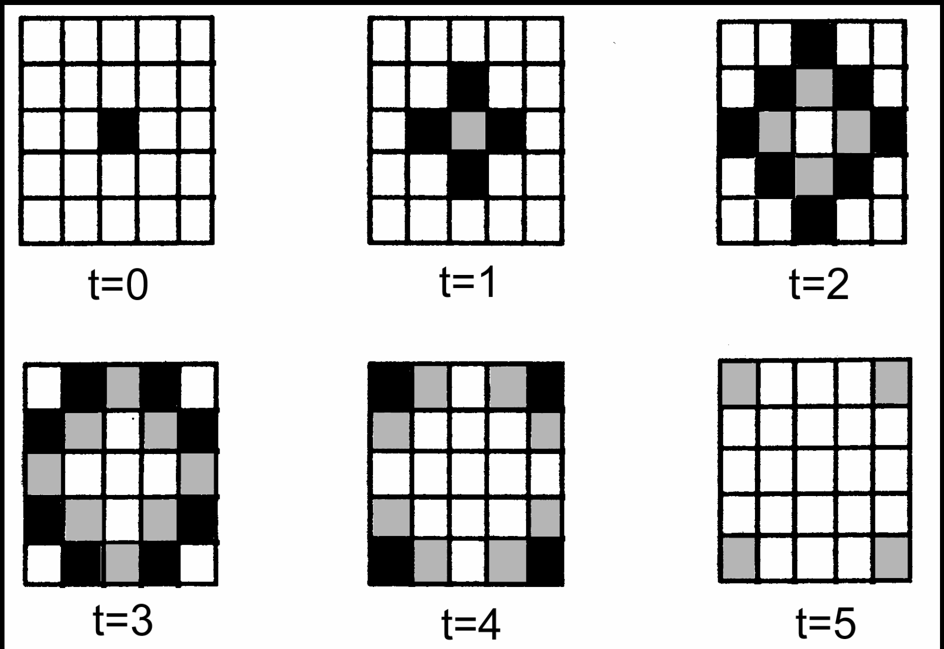
CA approach to modeling physical systems

Physical System

Physical system (computer)

CA approach to modeling physical systems





Exact integration process

Discrete Mechanics (Baez and Gilliam, 1994 & 1996)

- Time evolution proceeds in integer steps
- Configuration space is discrete
- Differential geometry => Algebraic geometry
- Euler-Lagrange Equation
- Noether theorem
- Symplectic techniques

Outline

- Example of mechanical system
- Introduce imprecision using extended intervals

Example of discrete mechanical system

- Configuration space = ring *k* of rational numbers
- Particle of mass m constrained to move on a line: q_0, q_1,..
- Algebra *A* of functions on *k*:

$$A \cong k\left[q\right] = \left\{\lambda_0, \lambda_0 + \lambda_1 q, \lambda_0 + \lambda_2 q^2, \lambda_0 + \lambda_1 q + \lambda_2 q^2 : \lambda_0 + \lambda_3 q^3, \ldots\right\}$$

- Lagrangian

$$A \otimes A \cong k\left[q_1, q_2\right] \quad \mathscr{L}_i = \mathscr{L}\left(q_i, q_{i+1}\right) = \frac{1}{2} m \dot{q}_i^{\,2} - V\left(q_i\right) \quad \dot{q}_i = q_{i+1} - q_i$$

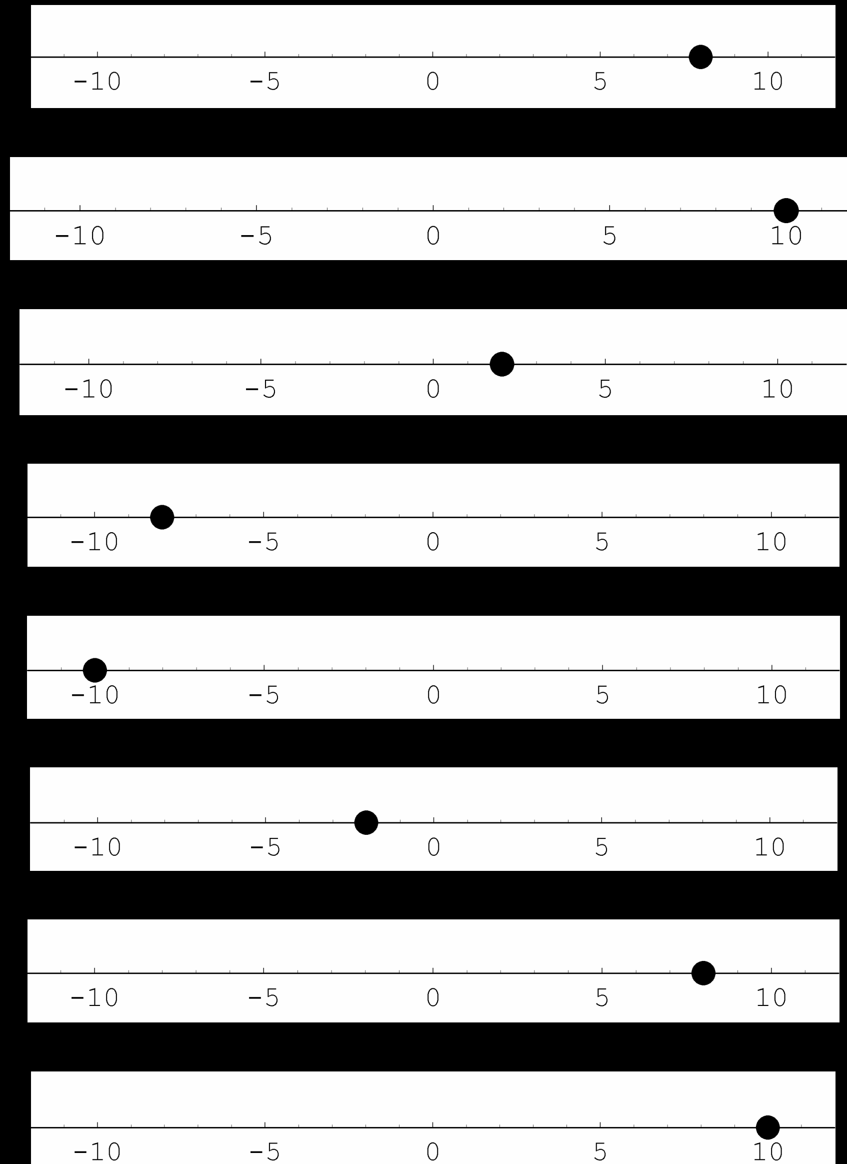
- Space of histories

$$H \cong k\left[q_0, \ldots, q_T\right]$$

- Lagrange equation

$$d_i \mathscr{L}_i + d_i \mathscr{L}_{i-1} = 0 \qquad m\left(\dot{q}_i - \dot{q}_{i-1}\right) = -V'\left(q_i\right)$$

Example of discrete mechanical system



$m = 1$

$q_0 = 8$

$q_1 = 16$

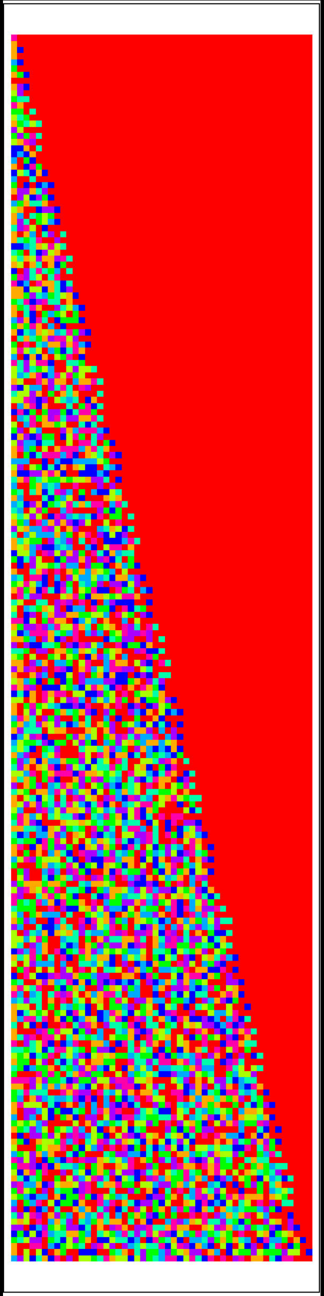
$V = \frac{1}{2} s q^2$

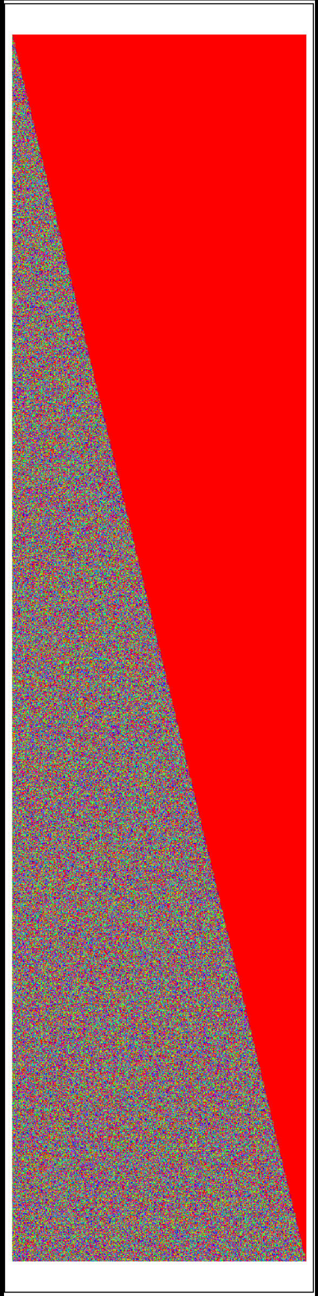
$s = 1$

Example of discrete mechanical system

$m = 1$
$q_0 = 8$
$q_1 = 16$
$V = \frac{1}{2} s q^2$
$s/m = 1/3$

{8,
10,
56/3,
136/9,
176/27,
-344/81,
-3304/243,
-13424/729,
-37384/2187,
-66104/6561,
5936/19683,
624616/59049,
3069656/177147
…}





Example of discrete mechanical system

100 time steps

1,000 time steps

10,000 time steps

Example of discrete mechanical system

- Integer coordinates?
- All coordinates?

- Maximum and minimum coordinates:

100 steps: [-18.68289250959053, 18.682435719981747]

1,000 steps: [-18.68289250959053, 18.683060458305423]

10,000 steps: [- 18.683972612054987, 18.6839719940311]

Discrete mechanical systems on interval algebra

- Configuration space = conditional ring *D* of ext. intervals
- Particle of mass m constrained to move on a line: q_0, q_1,..

- Lagrangian

$$\mathscr{L}_i = \mathscr{L}\left(\mathbf{q}_i, \mathbf{q}_{i+1}\right) = \frac{1}{2}\mathbf{m}\,\dot{\mathbf{q}}_i^{\,2} - \mathbf{V}\left(\mathbf{q}_i\right)_-$$

- Lagrange equation

$$Dual\left(d_i \mathscr{L}_i\right) + d_i \mathscr{L}_{i-1} = 0$$

$$\left(\mathbf{m}\dot{\mathbf{q}}_i\right)\_ - \mathbf{m}\dot{\mathbf{q}}_{i-1} + \mathbf{V}'\left(\mathbf{q}_i\right) = 0$$

Discrete mechanical systems on interval algebra

- **Newton's law:**

$$\left(\mathbf{m}\dot{\mathbf{q}}_i\right)\_ - \mathbf{m}\dot{\mathbf{q}}_{i-1} + \mathbf{V}'\left(\mathbf{q}_i\right) = 0 \Leftrightarrow \left(\mathbf{m}\dot{\mathbf{q}}_i\right)\_ = \left(\mathbf{m}\dot{\mathbf{q}}_{i-1}\right)\_ - \mathbf{V}'\left(\mathbf{q}_i\right)\_$$

$$\Leftrightarrow \dot{\mathbf{q}}_i\,\_ = \dot{\mathbf{q}}_{i-1}\,\_ - \mathbf{m}^{-1}\mathbf{V}'\left(\mathbf{q}_i\right)\_$$

$$\Leftrightarrow \dot{\mathbf{q}}_i\,\_ - \dot{\mathbf{q}}_{i-1} = -\mathbf{m}^{-1}\mathbf{V}'\left(\mathbf{q}_i\right)\_$$

$$\Leftrightarrow \dot{\mathbf{q}}_i - \dot{\mathbf{q}}_{i-1}\,\_ = -\mathbf{m}\,\_^{-1}\,\mathbf{V}'\left(\mathbf{q}_i\right)$$

Discrete mechanical systems on interval algebra

- **Example:**

$$\mathbf{q}_0 = [11{,}000,\ 11{,}100]$$

$$\mathbf{q}_1 = [10{,}000,\ 11{,}000]$$

$$V(\mathbf{q}_1) = \frac{1}{2}\mathbf{s}\mathbf{q}_1^{\,2}$$

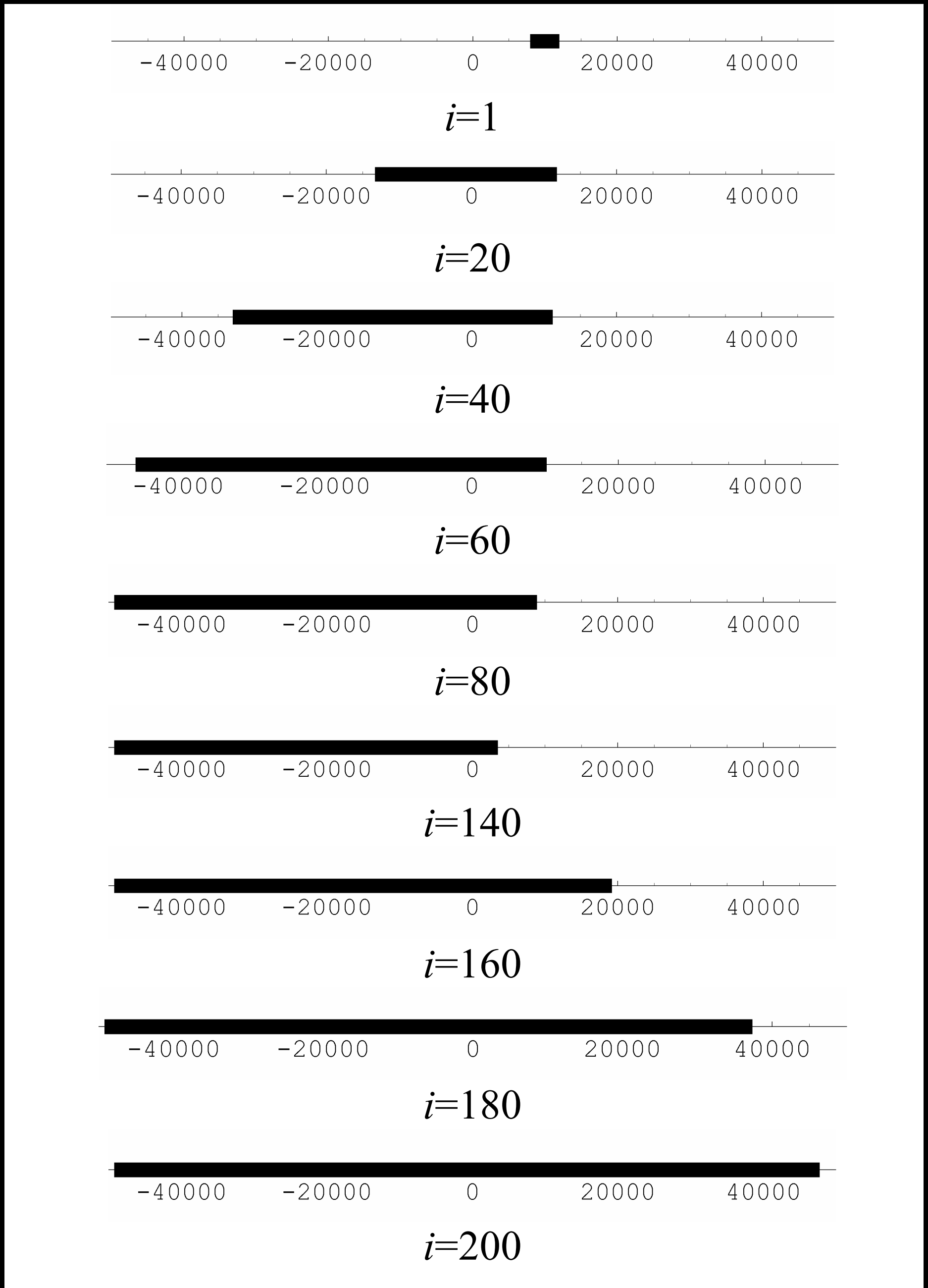
$$\mathbf{s} = [1/5000,\ 3/5500]$$

$$\mathbf{m} = [1,\ 2]$$

Used Popva's algorithm (2005) to eliminate dependency in flow calculation

$$\begin{pmatrix}
\frac{97840}{11} & \frac{109989}{10} \\
\frac{23547324}{3025} & \frac{1099670011}{100000} \\
\frac{27741730507}{4159375} & \frac{10993400549989}{1000000000} \\
\frac{126999172635479}{22876562500} & \frac{109890016499230011}{10000000000000} \\
\frac{557422553635612563}{125821093750000} & \frac{1098350384969200989989}{100000000000000000} \\
\frac{2288250850107591605311}{692016015625000000} & \frac{10976907699076049498790011}{1000000000000000000000} \\
\frac{8301862351155904852855067}{3806088085937500000000} & \frac{109692138576901814927401429989}{10000000000000000000000000} \\
\frac{22075992059906839606189421799}{20933484472656250000000000} & \frac{1096042309491854446854100098350011}{100000000000000000000000000000} \\
-\frac{8361651439670606649600705226397}{1151341645996093750000000000000} & \frac{10950536289837415589895004868001869989}{1000000000000000000000000000000000} \\
-\frac{759751840694239559412888964794437809}{6332379052978515625000000000000000} & \frac{109395544311273029696900192520168297910011}{10000000000000000000000000000000000000} \\
-\frac{8102051036064516583713118612745923335573}{34828084791381835937500000000000000000} & \frac{1092747861697407761918806463410658791002309989}{100000000000000000000000000000000000000000} \\
-\frac{66115762062600742198853274199335175199046281}{191554466352600097656250000000000000000000} & \frac{10914310054959154860924191209732935370254097470011}{1000000000000000000000000000000000000000000000} \\
-\frac{481987991561468735303467618334524740762828702157}{1053549564939300537109375000000000000000000000} & \frac{109000500619387361871742253662383095369480696402749989}{10000000000000000000000000000000000000000000000000}
\end{pmatrix}$$

Discrete mechanical systems on interval algebra



Conclusions

- It may be impossible to determine a priori the evolution of even the simplest discrete mechanical system (sit down and let it evolve)
- When the system takes values on the set of extended intervals:

    1) Introduce "Dual" in Lagrange equation.

    2) Exact bounds can be calculated for i-th time step (monotonicity, Popova 2005)

    3) Taylor models for flow of non-monotonic time evolutions?

Thanks

Evgenija D. Popova (Institute of Mathematics and Informatics, Bulgarian Academy of Sciences)