Comparison of Interval and Convex Analyses

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Abstract: This study shows that the type of the analytical treatment that should be adopted for non-probabilistic analysis of uncertainty depends upon the available experimental data. The main idea is based on the consideration that the maximum structural response predicted by the preferred theory ought be minimal, and the minimum structural response predicted by the preferred theory ought be maximal, to constitute a lower overestimation. Prior to the analysis the existing data ought be enclosed by the minimum volume hyper-rectangle V_1 that contains all experimental data. The experimental data also have to be enclosed by the minimum volume ellipsoid V_2. If V_1 is smaller than V_2 and the response calculated based on it $R(V_1)$ is smaller than $R(V_2)$, then one has to prefer interval analysis. However, if V_1 is in excess of V_2 and $R(V_1)$ is greater than $R(V_2)$, then the analyst ought to utilize convex modeling. If V_1 equals V_2 or these two quantities are in close vicinity, then two approaches can be utilized with nearly equal validity. Some numerical examples are given to illustrate the efficacy of the proposed methodology.

Keywords: uncertainty description, convex modeling, interval analysis, ellipsoid, hyper-rectangle

1. Introduction

Probabilistic approaches are used by numerous analysts for the safety assessment of structures whose parameters or loadings on them are modeled as uncertain variables or functions. In recent decades, some alternatives of it have been suggested. Fuzzy-sets based approaches gain much popularity. There are many discussions on philosophical implications of each of these approaches. Whereas the probabilistic methodology requires the knowledge of probability densities, the fuzzy-sets based approaches demand the knowledge of membership functions. More recently, yet another alternative is embraced by the investigators, that is not based upon any specified measure, either probabilistic or fuzzy, of uncertain variables. It presupposes the knowledge only of bounds of uncertain quantities. These are then called as unknown-but-bounded or uncertain-but-bounded variables. This analysis is both old and new. It is old chronologically but new by its revived use. Apparently the first work on response of a single-degree-of-freedom system under uncertain-but-bounded excitation was written by Bulgakov in 1946. He specially mentioned that the task is to calculate the upper bounds of structural response "under unfavorable circumstances", when the "disturbing action $y_p(t)(p=1,2,\ldots,r)$ satisfy the condition |

$y_p(t)|\leq l_p(l_p$constant) but are otherwise arbitrary one-valued continuous functions of the time t possessing as many derivatives as necessary ". This problem was dubbed by Bulgakov (1946) as the "problem of accumulation of disturbances" (see also his other paper, in 1940, which considers a special case).

There is a considerable literature in the Russian language on the Bulgakov's problem. Independently, in late sixties, Schweppe (1968) developed an analogous thinking based on ellipsoidal modeling, representing the uncertain variables as belonging to an ellipsoid.

Recently, some researchers in uncertain mechanics are developing interval analysis whereas others follow convex modeling (Ben-Haim and Elishakoff, 1990; Rao and Berke, 1997; Lombardi, 1998; Pantelides and Ganzerli, 1998, 1999; Mullen and Muhanna, 1999; Manson, 2005; McWilliam, 2001; Moens and Vandepitte, 2007). The question arises if these analyses are interrelated specifically, should one perform both analyses, or one of them in preferable? This work tries to elucidate the possible reply to this question. Some researchers performed a comparison of results derived by both methods. Elishakoff, Li and Starnes (2001) derived a minimum volume ellipsoid that encloses the minimum volume parallelepiped for buckling analysis. Elishakoff, Cai and Starnes (1994) studied the buckling of elastic column on non-linear elastic foundation by interval analysis whereas Qiu, Ma and Wang (2006) dealt with the same problem via convex modeling. Qiu and Wang (2003) specially distinguished between these two non-probabilistic set theoretical models.

Although convex modeling and interval analysis have been used extensively, in practice, which of the non-probabilistic uncertain descriptions, convex modeling or interval analysis should be preferred? In this study, this problem will be answered. The experimental data are shown to be of the cardinal influence on which of these methods ought be given a preference.

Consider the case that due to high cost of the measurements the experimental points are too scant to determine their statistical information on uncertain parameters: if we choose non-probabilistic set-theoretical convex methods, convex modeling or interval analysis, for uncertain modeling, then the precondition is to seek or determine the suitable set containing the limited experimental points. In fact, there is more than one set to be able to enclose the limited experimental points. However, too big set will produce over-conservative bounds on the structural responses. Of course, it is impossible for us to know the real bounds on uncertain parameters based on the limited experimental points. The enclosing set with minimal volume property may be a better selection, which will produce lower overestimation on the bounds of the structural responses. We can only act on what we know.

2. Description of the Method by Zhu, Elishakoff and Starnes

In this section, the description of the method by Zhu, Elishakoff and Starnes (1996), in which the smallest hyper-rectangle and the smallest ellipsoid containing the given experimental data are determined, is stated in brief.

Suppose that there are m uncertain parameters $a_i (i = 1, 2, \cdots, m)$ describing either the structural properties or the excitation. These parameters constitute an m-dimensional parameter space, namely, $a = (a_1, a_2, \cdots, a_m)$. Suppose that we have limited information on these parameters, represented by M experimental points, $a^{(r)} (r = 1, 2, \cdots, M)$ in this m-dimensional space. Convex modeling assumes that all these experimental points belong to an ellipsoid

$$(a - a_0)^T W (a - a_0) \leq 1 \tag{1}$$

where a_0 is the state vector of the central point of the ellipsoid, and W is the weight matrix. Interval analysis assumes that all experimental points belong to a hyper-rectangle.

By using transformation matrix $T_m(\theta_1, \theta_2, \cdots, \theta_{m-1})$ given in Ref. Zhu et al.(1996), the above M points in the rotated coordinate system will have their new coordinates denoted by $b^{(r)} (r = 1, 2, \cdots, M)$. To obtain the smallest ellipsoid, let us first examine an m-dimensional box of the form

$$\left| b - b_0 \right| \leq d \tag{2}$$

which contains all M points. The vector of semi-axes $d = (d_1, d_2, \cdots, d_m)^T$ and the vector of central points $b_0 = (b_{10}, b_{20}, \cdots, b_{m0})^T$ of the "box" in the rotated coordinate system are given by

$$\begin{aligned} d_k &= \frac{1}{2}\left(\max_r (b_k^{(r)}) - \min_r (b_k^{(r)}) \right), \\ b_{k0} &= \frac{1}{2}\left(\max_r (b_k^{(r)}) + \min_r (b_k^{(r)}) \right), \end{aligned} \qquad (r = 1, 2, \cdots, M;\, k = 1, 2, \cdots, m) \tag{3}$$

We now enclose this box by an ellipsoid

$$\sum_{k=1}^{m} \frac{(b_k - b_{k0})^2}{g_k^2} \leq 1 \tag{4}$$

where g_k are the semi-axes of the ellipsoid. There are infinite number of ellipsoids which contain the box given in Eq.(2). Clearly, the best choice is the one with minimum volume. The volume of an m-dimensional ellipsoid is given by

$$V_e = C_m \prod_{k=1}^{m} g_k \tag{5}$$

where C_m is a constant.

From the monograph by Elishakoff, Li and Starnes (2001) and paper by Qiu (2003), corresponding to the smallest ellipsoid, the semi-axes of the smallest ellipsoid should be

$$g_i = \sqrt{m} d_i, \quad (i = 1, 2, \cdots, m) \tag{6}$$

Thus, once the size of the box Eq.(2) is known, the semi-axes of the minimum-volume ellipsoid enclosing the box of the experimental data are readily determined by utilizing Eq.(6). If there are no experimental points at the corner of the box, the size of such an ellipsoid may further be reduced until one of the experimental points reaches the surface of the ellipsoid. The semi-axes of the ellipsoid in this case may be replaced by ηg_k, where the factor is determined from the condition

$$\eta = \sqrt{\max_r \sum_{k=1}^{m} \frac{\left(b_k^{(r)} - b_{k0}\right)^2}{g_k^2}} \le 1, \quad (r = 1, 2, \cdots, M) \tag{7}$$

If there are some experimental points in the corner of the multidimensional box, the factor η equals unity. The ellipsoid (4) can be written in the form

$$(b - b_0)^T D(b - b_0) \le 1 \tag{8}$$

in which b_0 is the vector of central points whose components are given by Eq.(3), and D is a diagonal matrix

$$D = diag\left((\eta g_1)^{-2}, (\eta g_2)^{-2}, \cdots, (\eta g_m)^{-2}\right) \tag{9}$$

The volume of the ellipsoid now reads

$$V_e = C_m \eta^m \prod_{k=1}^{m} g_k \tag{10}$$

which is a function of a set of parameters $\theta_k (k = 1, 2, \cdots, m-1)$. Therefore, the best ellipsoid among these ellipsoids is the one which contains all given points and possesses the minimum volume, i.e.,

$$V_e = \min_{\theta_1, \theta_2, \cdots, \theta_{m-1}} \left\{V_e(\theta_1, \theta_2, \cdots, \theta_{m-1})\right\} \tag{11}$$

A possible approach to determine this ellipsoid is to search among all possible cases by increasing $\theta_k (k = 1, 2, \cdots, m-1)$ from 0 to π/2 in sufficiently small increments $\Delta\theta_k$, and to compare the volumes of so obtained ellipsoids. Once one finds the ellipsoid with minimum volume in one direction, say $\theta_{k0} (k = 1, 2, \cdots, m-1)$, the ellipsoid can be transformed back into the original coordinate system by applying the transformation matrix T_m. Hence, the vector a_0 of central point and the weight matrix W in Eq.(1) become

$$a_0 = T_m^T b_0, \quad W = T_m^T D T_m \tag{12}$$

where $T_m = T_m(\theta_{10}, \theta_{20}, \cdots, \theta_{m0})$. So Eq.(12) constitutes the smallest ellipsoid containing all experimental points. The "box" corresponding to the smallest ellipsoid is the smallest hyper-rectangle.

3. Convex Modeling and Interval Analysis for the Structural Response

For convenience, in this section, convex modeling method and interval analysis method for the static response analysis of structures with uncertain parameters are reformulated (see Ref. Qiu (2003)). In fact, the presented concept in this study also can be applied to other linear elastic structural mechanics problem with uncertainty, such as the natural frequency analysis, the dynamic response analysis etc.

The matrix equation of static equilibrium in the finite element method can be written as

$$K(a)u(a) = f(a) \tag{13}$$

where $K = (k_{ij})$ is the $n \times n$-dimensional stiffness matrix, $u = (u_i)$ is the *n*-dimensional nodal displacement vector and $f = (f_i)$ is the *n*-dimensional external load vector; $a = (a_1, a_2, \cdots, a_m)^T$ is the structural parameters, such as the physical, material and geometric properties in structures.

Consider a realistic situation in which not enough information is available on the structural parameters to justify an assumption on their probabilistic characteristics. It is assumed that by use of Zhu, Elishakoff and Starnes's method (1996), the derived smallest ellipsoid and the derived smallest hyper-rectangle on the structural parameters can be obtained as, respectively,

$$Z(W, \theta) = \{a : a \in R^m, (a - a_0)^T W (a - a_0) \le \theta^2\} \tag{14}$$

and

$$\underline{a} \le a \le \overline{a} \ \text{ or } \ a_0 - \Delta a \le a \le a_0 + \Delta a \tag{15}$$

where $a_0 = (a_{i0}) \in R^m$ is the nominal value vector of the structural parameter vector a , W is a positive definite matrix and is called the weight matrix, θ is a positive constant and is called the radius of the ellipsoid; $\underline{a}$ and $\overline{a}$ are the lower bound and upper bound of the hyper-rectangle, Δa is the radius of the hyper-rectangle.

The structural parameter of a value slightly different from this nominal value can be denoted as

$$a = a_0 + \delta a \ \text{ or } \ a_i = a_{i0} + \delta a_i \quad , \quad i = 1, 2, \cdots, m \tag{16}$$

where $\delta a = (\delta a_i) \in R^m$ is a small quantity.

By Taylor's series expansion, the static displacement of the structure with uncertain parameter vector $a = a_0 + \delta a$, to first order in δa , is

$$u_i(a) = u_i(a_0 + \delta a) = u_i(a_0) + \sum_{j=1}^{m} \frac{\partial u_i(a_0)}{\partial a_j} \delta a_j, \quad i = 1, 2, \cdots, n \tag{17}$$

For convenience of notation, let us define

$$\varphi^T = \left(\frac{\partial u_i(a_0)}{\partial a_1}, \frac{\partial u_i(a_0)}{\partial a_2}, \cdots, \frac{\partial u_i(a_0)}{\partial a_m} \right) = \left(\frac{\partial u_{i0}}{\partial a_1}, \frac{\partial u_{i0}}{\partial a_2}, \cdots, \frac{\partial u_{i0}}{\partial a_m} \right) \tag{18}$$

By combination of Eq.(17) and Eq.(14), the most and least favourable response for convex modeling method can be obtained as (see Ref. Ben-Haim and Elishakoff (1990))

$$\underline{u}_C = u_0 - \theta\sqrt{\varphi^T W^{-1} \varphi} \quad \text{and} \quad \overline{u}_C = u_0 + \theta\sqrt{\varphi^T W^{-1} \varphi} \tag{19}$$

By combination of Eq.(17) and Eq.(15), the most and least favourable responses for interval analysis method can be obtained as (see Ref. Qiu (2003))

$$\underline{u}_{iI} = u_{i0} - \sum_{j=1}^{m} \left| \frac{\partial u_{i0}}{\partial a_j} \right| \Delta a_j \quad \text{and} \quad \overline{u}_{iI} = u_{i0} + \sum_{j=1}^{m} \left| \frac{\partial u_{i0}}{\partial a_j} \right| \Delta a_j \tag{20}$$

Thus, in the case that the smallest intervals or hyper-rectangle containing uncertain parameters are known, interval analysis method can be adopted to obtain the most and least favorable responses. In the case that the smallest ellipsoid containing uncertain parameters are known, convex modeling method can be adopted to obtain the most and least favorable responses. So, a question will arise. Which method is better? In other words, which method will give the tighter bounds on the structural responses? In the following, a 7-bar planar truss structure and a 60-bar space truss structure are used to reply to this quest.

4. Seven-Bar Planar Truss Structure

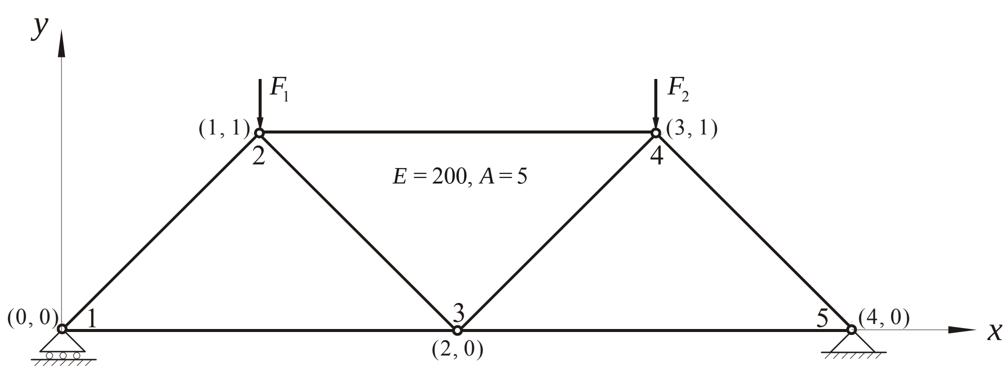

Figure 1. A 7-bar planar truss structure

Let us consider a 7-bar planar truss structure with linear elastic properties depicted in Figure 1. Here, $A = 5$ is the cross-sectional area, $E = 200$ is Young's modulus, F_1 is an external load at node No.2, F_2 is an external load applied at node No.4. The parameters of the truss are given as dimensionless numbers, since the physical values are not relevant to our analysis.

This truss is the same as adopted by Skalna (2003) but here the loads F_1 and F_2 are considered to be uncertain, and the other properties of the truss, such as A and E, are deterministic. Namely, the truss members have deterministic stiffness.

In the following, several sets of hypothesized data for uncertain parameters will be given. By use of the Zhu, Elishakoff and Starnes's method (1996), the smallest ellipse and rectangle can be derived. Based on the derived ellipse and rectangle, the most and least favorable responses of the structure can be calculated by convex modeling method and interval analysis method, respectively.

We will discuss this problem in the following two cases: one is that the principal axes of the derived ellipse and rectangle are parallel to the global coordinate system; the other is that the principal axes of the derived ellipse and rectangle are *not* parallel to the global coordinate system.

4.1. The principal axes of the derived ellipse and rectangle are parallel to the global coordinate system

Case I: Consider a set of hypothesized data for uncertain parameters as shown in Figure 2, and they are listed in Table 1. Here these hypothesized data are randomly generated in order to proceed to the numerical simulations, but in practice the samples for uncertain parameters can be generally obtained by the experiments.

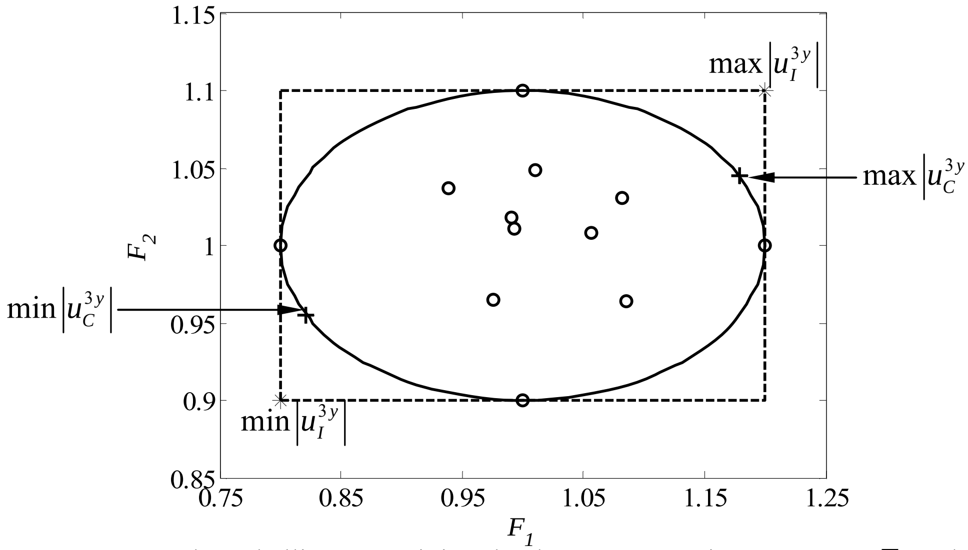

Figure 2. Rectangle and ellipse containing the data on uncertain parameters F_1 and F_2

The smallest rectangle obtained from the set of data by using of Zhu, Elishakoff and Starnes's method (1996) is

$$F_1^I = [0.80, 1.20], \quad F_2^I = [0.90, 1.10] \tag{21}$$

Based on Eq.(21), we conclude that the central values of F_1 and F_2 are, respectively,

$$F_{1c} = (0.80 + 1.20)/2 = 1.0, \quad F_{2c} = (0.90 + 1.10)/2 = 1.0 \tag{22}$$

and the values of radii F_1 and F_2 are, respectively,

$$\Delta F_1 = (1.20 - 0.80)/2 = 0.2, \quad \Delta F_2 = (1.10 - 0.90)/2 = 0.1 \tag{23}$$

Thus, one can analyze the system as subjected to an interval load vector with nominal values $(1.0, 1.0)$ and scatter of $(20\%, 10\%)$.

Table 1. The values of uncertain parameter F_1 and F_2

k	1	2	3	4	5	6	7	8	9	10	11	12
F1	0.991	1.082	1.085	0.938	0.976	0.993	1.011	1.056	0.800	1.200	1.000	1.000
F2	1.018	1.031	0.964	1.037	0.965	1.011	1.048	1.008	1.000	1.000	0.900	1.100

On the other hand, the smallest ellipse can be obtained from the set of data by using of Zhu, Elishakoff and Starnes's method (1996). The optimal rotation angle θ_{10} obtained is 0°, so the transformation matrix T_2 is

$$T_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{24}$$

In the case of $\theta_{10} = 0^\circ$, the vector of semi-axes and the vector of central point of the "box" in the optimal rotated coordinate system are, respectively, $d = (d_1, d_2)^T = (0.2, 0.1)^T$ and $b_0 = (b_{10}, b_{20})^T = (1.0, 1.0)^T$. The semi-axes of the smallest ellipsoid are $g_1 = \sqrt{2}d_1 = 0.2828$ and $g_2 = \sqrt{2}d_2 = 0.1414$. The diagonal matrix D is

$$D = diag\left((\eta g_1)^{-2}, (\eta g_2)^{-2}\right) = diag\left(25, 100\right) \tag{25}$$

where $\eta = \sqrt{2}/2$. Thus, we can get

$$a_0 = T_2^T b_0 = (1.0, 1.0)^T, \quad W = T_2^T D T_2 = \begin{bmatrix} 25 & 0 \\ 0 & 100 \end{bmatrix} \tag{26}$$

It can be seen from Figure 2 that the derived rectangle contains the derived ellipse based on the hypothesized data listed in Table 1.

We can find that the higher-order derivatives of static responses of the 7-bar planar truss structure with respect to uncertain parameters are all zeros. Thus, Eq.(17) based on the first-order Taylor series for this example will be linear and exact, i.e.

$$\begin{aligned} u_i(F_1, F_2) &= u_i(F_{1c} + \delta F_1, F_{2c} + \delta F_2) \\ &= u_i(F_{1c}, F_{2c}) + \frac{\partial u_i(F_c)}{\partial F_1}\delta F_1 + \frac{\partial u_i(F_c)}{\partial F_2}\delta F_2, \quad i = 1, 2, \cdots, n \end{aligned} \tag{27}$$

This is the reason why only the external loads are taken as the uncertain parameters in this study.

Taking the derivative of both sides of Eq.(13) yields

$$\frac{\partial K}{\partial F_j}u + K\frac{\partial u}{\partial F_j} = \frac{\partial f}{\partial F_j}, \quad j = 1, 2 \tag{28}$$

Due to the vanishing of $\dfrac{\partial K}{\partial F_j}$ for this problem, the sensitivity derivative of the structural response with respect to uncertain parameters becomes

$$\frac{\partial u}{\partial F_j} = K^{-1}\frac{\partial f}{\partial F_j}, \; j = 1, 2 \tag{29}$$

Substitution of Eqs.(22), (23) and (29) into Eq.(20) yields the most and least favorable responses in y-direction of node 3 of the 7-bar planar truss structure obtained from interval analysis method as follows

$$\min\left|u_I^{3y}\right| = 0.005803\,,\ \ \max\left|u_I^{3y}\right| = 0.007852 \tag{30}$$

Substitution of Eqs.(26) and (29) into Eq.(19) provides us with the most and least favorable responses in *y*-direction of node 3 of the 7-bar planar truss structure obtained from convex modeling method as follows

$$\min\left|u_C^{3y}\right| = 0.006064\,,\ \ \max\left|u_C^{3y}\right| = 0.007591 \tag{31}$$

The "*" points on the derived rectangle in Figure 2 are the most and least favorable points for interval analysis method. The "+" points on the derived ellipse in Figure 2 are the most and least favorable points for convex modeling method. The two markers "*" and "+" have the same meaning in sequel figures.

Thus, it can be seen from Eqs.(30) and (31) that interval analysis method gives tighter bounds of responses than convex modeling method in the case of data points listed in Table 1.

Case II: Consider another set of hypothesized data for uncertain parameters as shown in Figure 3, and they are listed in Table 2.

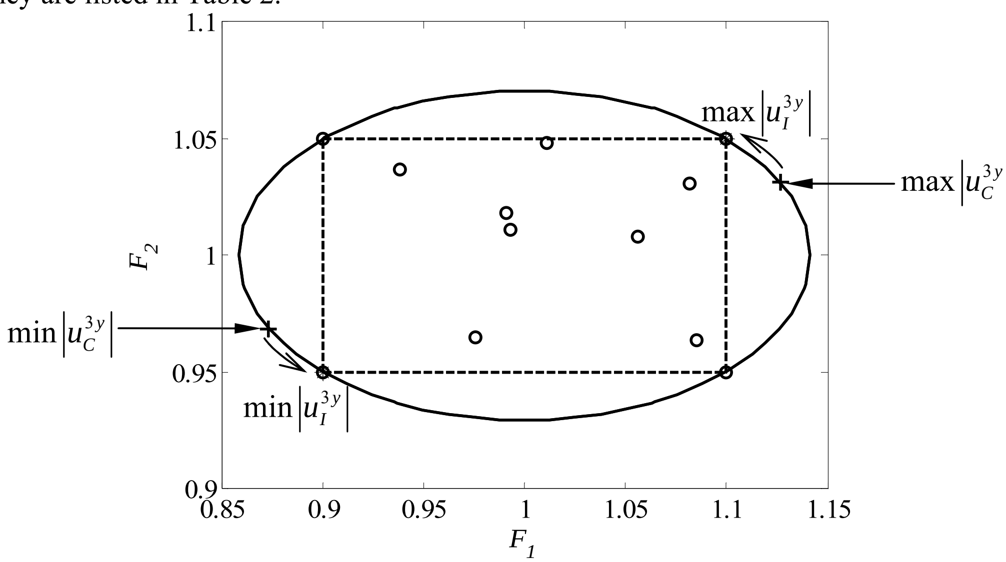

Figure 3. Rectangle and ellipse containing the data on uncertain parameters F_1 and F_2

Table 2. The values of uncertain parameter F_1 and F_2

k	1	2	3	4	5	6	7	8	9	10	11	12

F_1	0.991	1.082	1.085	0.938	0.976	0.993	1.011	1.056	0.900	1.100	1.100	0.900
F_2	1.018	1.031	0.964	1.037	0.965	1.011	1.048	1.008	0.950	0.950	1.050	1.050

The smallest rectangle obtained from the set of data by using of Zhu, Elishakoff and Starnes's method (1996) is

$$F_1^I = [0.90, 1.10], \ F_2^I = [0.95, 1.05] \tag{32}$$

Based on Eq.(32), we conclude that the central values and the values of radii of F_1 and F_2 are, respectively,

$$F_{1c} = 1.0, \ F_{2c} = 1.0 \text{ and } \Delta F_1 = 0.1, \ \Delta F_2 = 0.05 \tag{33}$$

Thus, one can analyze the system as subjected to an interval load vector with nominal values $(1, 1)$ and scatter of $(10\%, 5\%)$.

On the other hand, the smallest ellipse can be obtained from the set of data by using of Zhu, Elishakoff and Starnes's method (1996). The optimal rotation angle θ_{10} obtained is 0°. Similar to Eqs.(24)~(26), the vector a_0 of central point and the weight matrix W can be obtained as

$$a_0 = T_2^T b_0 = (1.0, 1.0)^T, \quad W = T_2^T D T_2 = \begin{bmatrix} 50 & 0 \\ 0 & 200 \end{bmatrix} \tag{34}$$

It can be seen from Figure 3 that the derived ellipse contains the derived rectangle based on the hypothesized data listed in Table 2.

By substituting Eqs.(33) and (29) into Eq.(20) and substituting Eqs.(34) and (29) into Eq.(19) , the most and least favorable responses in y-direction of node 3 of the 7-bar planar truss structure can be, respectively, obtained from interval analysis method and convex modeling method as follows

$$\min\left|u_I^{3y}\right| = 0.006316, \ \max\left|u_I^{3y}\right| = 0.007340 \tag{35}$$

and

$$\min\left|u_C^{3y}\right| = 0.006288, \ \max\left|u_C^{3y}\right| = 0.007367 \tag{36}$$

Thus, it can be seen from Eqs.(35) and (36) that convex modeling method gives tighter bounds of responses than interval analysis method in the case of data points listed in Table 2.

Under this circumstance, an interesting phenomenon can be seen. For convex modeling method, the extreme value points on the ellipse in Figure 3 may be different based on different structural parameters. Namely, the locations of the extreme value points of convex modeling method will change by changing the structural parameters. In certain particular case, the extreme value points of convex modeling method and interval analysis method will coincide.

4.2. THE PRINCIPAL AXES OF THE DERIVED ELLIPSE AND RECTANGLE ARE NOT PARALLEL TO THE GLOBAL COORDINATE SYSTEM.

Case I: Consider a set of hypothesized data for uncertain parameters as shown in Figure 4, and they are listed in Table 3.

The smallest rectangle obtained from the set of data by using of Zhu, Elishakoff and Starnes's method (1996) is shown as Figure 4. The smallest ellipse can be obtained from the set of data by using of Zhu, Elishakoff and Starnes's method (1996). The optimal rotation angle θ_{10} obtained is 30°. Similarly, the vector a_0 of central point and the weight matrix W can be obtained as

$$a_0 = T_2^T b_0 = (0.366, 1.366)^T, \quad W = T_2^T D T_2 = \begin{bmatrix} 43.75 & -32.48 \\ -32.48 & 81.25 \end{bmatrix} \tag{37}$$

As above mentioned, Eq.(17) based on the first-order Taylor series will be exact and linear for this example. Due to the convexity of the derived smallest rectangle, the most and least favorable responses in *y*-direction of node 3 of the 7-bar planar truss structure for interval analysis method will reach on the four vertexes of the smallest rectangle. By calculating and comparing the four responses, the most and least favorable responses or the minimum and maximum values of them are, respectively,

$$\min\left|u_I^{3y}\right| = 0.004855, \quad \max\left|u_I^{3y}\right| = 0.006970 \tag{38}$$

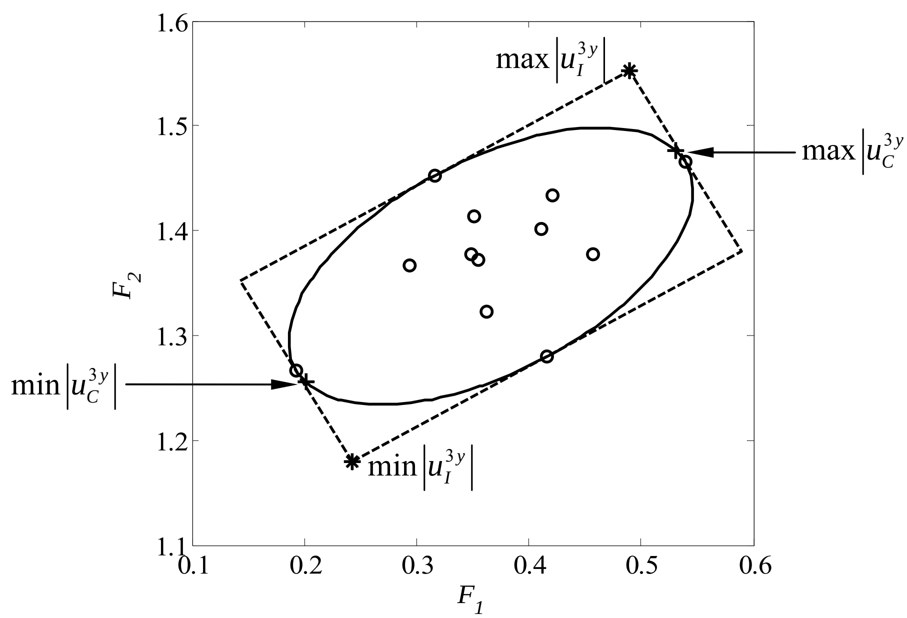

Figure 4. Rectangle and ellipse containing the data on uncertain parameters F_1 and F_2

Table 3. The values of uncertain parameter F_1 and F_2

k	1	2	3	4	5	6	7	8	9	10	11	12
F_1	0.349	0.422	0.458	0.294	0.362	0.355	0.351	0.411	0.193	0.539	0.416	0.316
F_2	1.377	1.434	1.377	1.367	1.323	1.372	1.413	1.401	1.266	1.466	1.279	1.453

By substituting of Eqs.(37) and (29) into Eq.(19), we obtain the most and least favorable responses in *y*-direction of node 3 of the 7-bar planar truss structure obtained from convex modeling method as follows

$$\min\left|u_C^{3y}\right| = 0.004972\,,\quad \max\left|u_C^{3y}\right| = 0.006854 \tag{39}$$

Thus, it can be seen from Eqs.(38) and (39) that interval analysis method gives tighter bounds of responses than convex modeling method in the case of data points listed in Table 3.

Case II: Consider another set of hypothesized data for uncertain parameters as shown in Figure 5, and they are listed in Table 4.

The smallest rectangle obtained from the set of data by using of Zhu, Elishakoff and Starnes's method (1996) is shown as Figure 5. The smallest ellipse can be obtained from the set of data by using of Zhu, Elishakoff and Starnes's method (1996). The optimal rotation angle θ_{10} obtained is 30°. Similarly, the vector a_0 of central point and the weight matrix W can be obtained as

$$a_0 = T_2^T b_0 = (0.366, 1.366)^T\,,\quad W = T_2^T D T_2 = \begin{bmatrix} 87.50 & -64.95 \\ -64.95 & 162.50 \end{bmatrix} \tag{40}$$

In perfect analogy with Eq.(38), the most and least favorable responses in *y*-direction of node 3 of the 7-bar planar truss structure for interval analysis method can be obtained as follows

$$\min\left|u_I^{3y}\right| = 0.005384\,,\quad \max\left|u_I^{3y}\right| = 0.006441 \tag{41}$$

We substitute of Eqs.(40) and (29) into Eq.(19) to get the most and least favorable responses in *y*-direction of node 3 of the 7-bar planar truss structure obtained from convex modeling method as follows

$$\min\left|u_C^{3y}\right| = 0.005247\,,\quad \max\left|u_C^{3y}\right| = 0.006578 \tag{42}$$

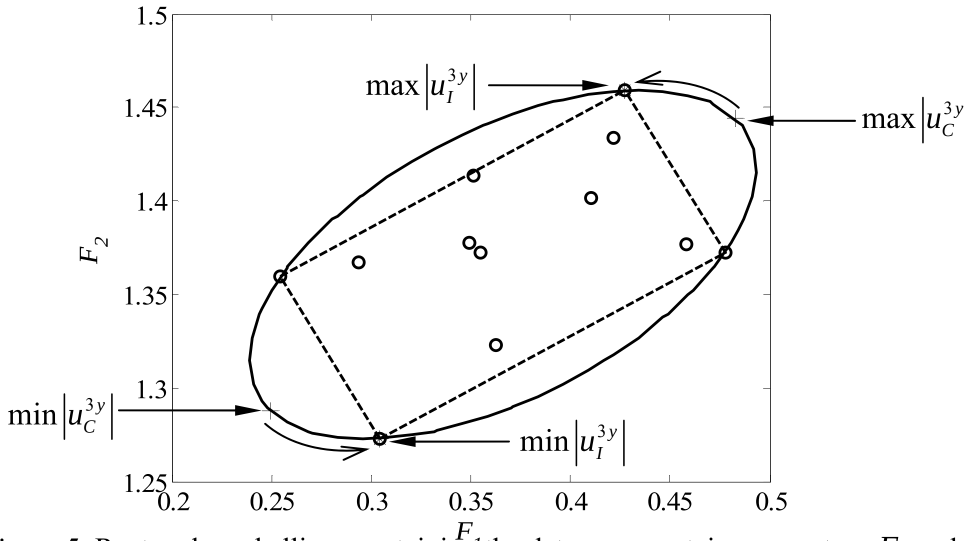

Figure 5. Rectangle and ellipse containing the data on uncertain parameters F_1 and F_2

Table 4. The values of uncertain parameter F_1 and F_2

k	1	2	3	4	5	6	7	8	9	10	11	12
F_1	0.349	0.422	0.458	0.294	0.362	0.355	0.351	0.411	0.304	0.478	0.428	0.254
F_2	1.377	1.434	1.377	1.367	1.323	1.372	1.413	1.401	1.273	1.373	1.459	1.359

Thus, it can be seen from Eqs.(41) and (42) that convex modeling method gives tighter bounds of responses than interval analysis method in the case of data points listed in Table 4. Although only the displacement responses in *y*-direction of node 3 of the 7-bar planar truss structure are compared, the analysis will not change qualitatively if a different aspect of response of the truss structure were used to carry out the comparisons of convex modeling with interval analysis due to the linear elastic properties.

We can find from the above analysis that the choose for two methods, convex modeling or interval analysis, is decided by the distribution of sample data points on uncertain parameters.

5. Sixty-Bar Space Truss Structure

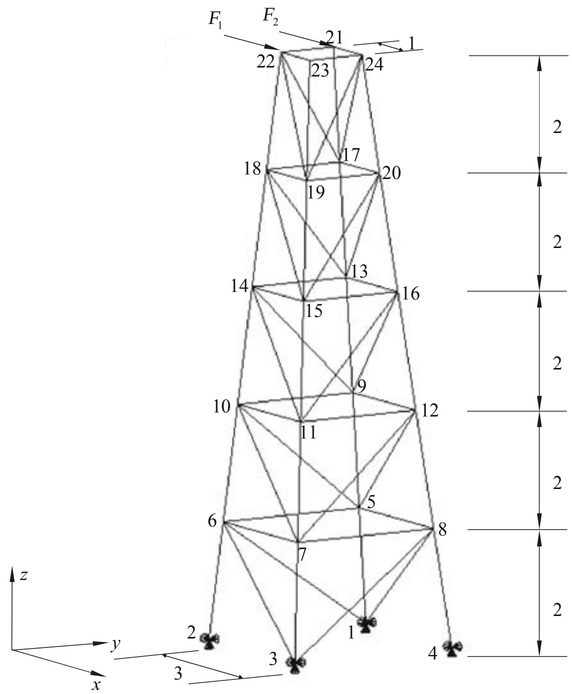

Figure 6. A 60-bar space truss structure

Consider a 60-bar space truss structure with linear elastic properties subject to two *x*-directional loads as shown in Figure 6. The external loads F_1 and F_2, respectively, act on nodes No.21 and No.22. Young's moduli of the bars are $E_i = 2.1\times10^{11}$ $(i = 1, 2\cdots, 60)$. The cross-sectional areas of the bars are $A_i = 1.0\times10^{-3}$ $(i = 1, 2\cdots, 60)$.

Suppose that the external loads F_1 and F_2 are still considered to be uncertain, and the other properties of the truss, such as A and E, are deterministic. Namely, the truss members have deterministic stiffness.

In previous section, the case that there exists the inclusion relation between the derived ellipse and rectangle is studied. In this section, we will consider the non-inclusion relation between them.

Case I: Consider a set of hypothesized data for uncertain parameters as shown in Figure 7, and they are listed in Table 5.

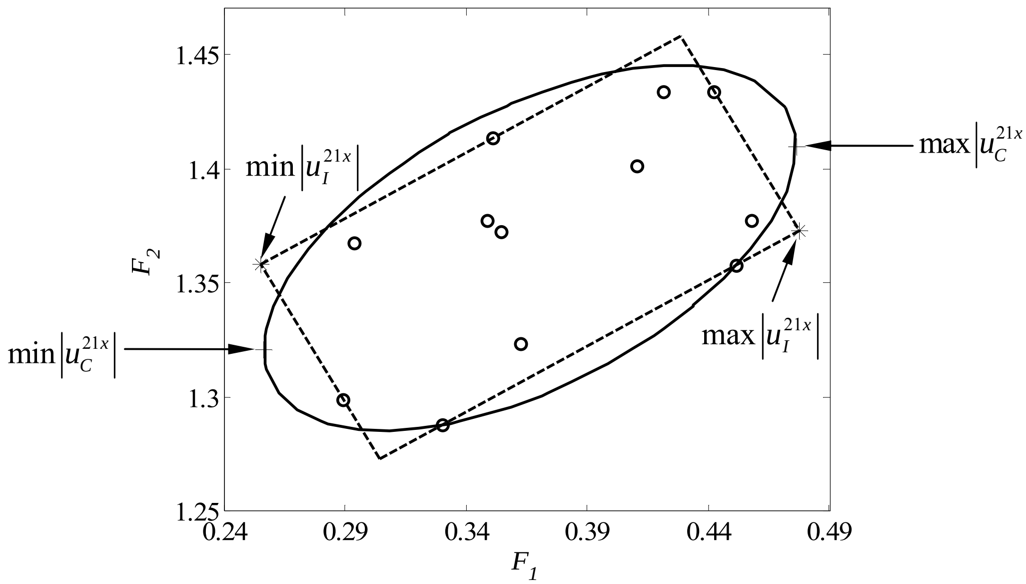

Figure 7. Rectangle and ellipse containing the data on uncertain parameters F_1 and F_2

Table 5. The values of uncertain parameter F_1 and F_2

k	1	2	3	4	5	6	7	8	9	10	11	12
F_1	0.349	0.422	0.458	0.294	0.362	0.355	0.351	0.411	0.330	0.452	0.443	0.289
F_2	1.377	1.434	1.377	1.367	1.323	1.372	1.413	1.401	1.288	1.358	1.433	1.299

The smallest rectangle obtained from the set of data by using of Zhu, Elishakoff and Starnes's method (1996) is shown as Figure 7. The smallest ellipse can be obtained from the set of data by using of Zhu, Elishakoff and Starnes's method (1996). The optimal rotation angle θ_{10} obtained is 30°. Similarly, the vector a_0 of central point and the weight matrix W can be obtained as

$$a_0 = T_2^T b_0 = (0.3664, 1.3653)^T, \quad W = T_2^T D T_2 = \begin{bmatrix} 119.71 & -91.10 \\ -91.10 & 224.91 \end{bmatrix} \tag{43}$$

Similar to Eq.(38) and Eq.(41), the most and least favorable responses in *x*-direction of node 21 of the 60-bar space truss structure for interval analysis method can be obtained as follows

$$\min\left|u_I^{21x}\right| = 1.6491\text{E-7}, \quad \max\left|u_I^{21x}\right| = 3.0862\text{E-7} \tag{44}$$

Substitution of Eqs.(43) and (29) into Eq.(19) yields the most and least favorable responses in *x*-direction of node 21 of the 60-bar space truss structure obtained from convex modeling method as follows

$$\min\left|u_C^{21x}\right| = 1.6575\text{E-7}, \quad \max\left|u_C^{21x}\right| = 3.0777\text{E-7} \tag{45}$$

Thus, it can be seen from Eqs.(44) and (45) that convex modeling method gives tighter bounds of responses than interval analysis method in the case of data points listed in Table 5.

Case II: Consider another set of hypothesized data for uncertain parameters as shown in Figure 8, and they are listed in Table 6.

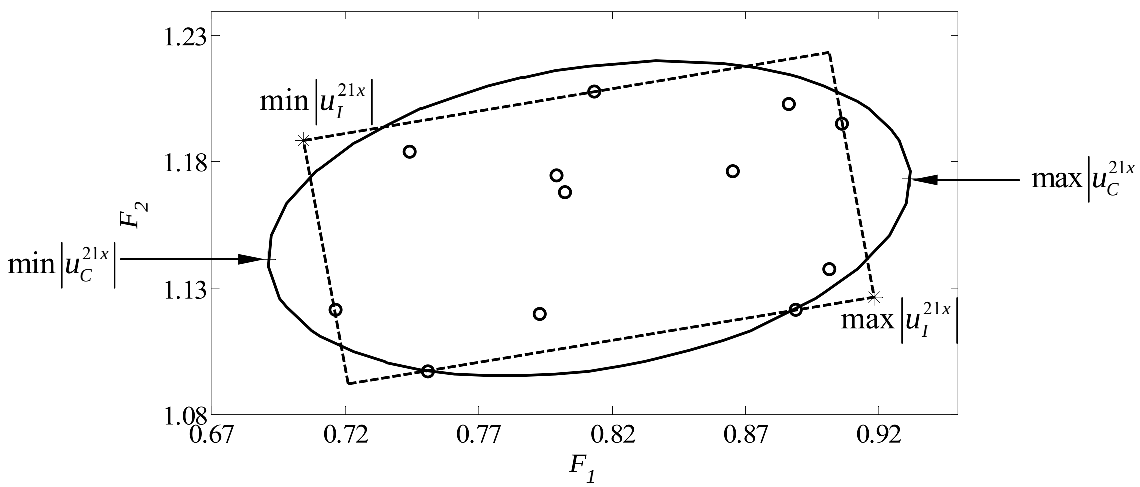

Figure 8. Rectangle and ellipse containing the data on uncertain parameters F_1 and F_2

Table 6. The values of uncertain parameter F_1 and F_2

k	1	2	3	4	5	6	7	8	9	10	11	12
F_1	0.7991	0.887	0.901	0.744	0.793	0.803	0.813	0.865	0.751	0.889	0.906	0.716
F_2	1.175	1.203	1.138	1.184	1.119	1.168	1.208	1.176	1.097	1.121	1.196	1.121

The smallest rectangle obtained from the set of data by using of Zhu, Elishakoff and Starnes's method (1996) is shown as Figure 8. The smallest ellipse is obtained from the set of data by using of Zhu, Elishakoff and Starnes's method (1996). The optimal rotation angle θ_{10} obtained is 10°. Similarly, the vector a_0 of central point and the weight matrix W can be obtained as

$$a_0 = T_2^T b_0 = (0.8113, 1.1576)^T, \quad W = T_2^T D T_2 = \begin{bmatrix} 73.46 & \text{-}35.98 \\ \text{-}35.98 & 271.17 \end{bmatrix} \tag{46}$$

Similar to Eq.(38), the most and least favorable responses in x-direction of node 21 of the 60-bar space truss structure for interval analysis method can be obtained as follows

$$\min\left|u_I^{21x}\right| = 4.5511\text{E-7}, \quad \max\left|u_I^{21x}\right| = 5.9339\text{E-7} \tag{47}$$

Substitution of Eqs.(46) and (29) into Eq.(19) results in the most and least favorable responses in *x*-direction of node 21 of the 60-bar space truss structure obtained from convex modeling method as follows

$$\min\left|u_C^{21x}\right| = 4.4628\text{E-7}, \quad \max\left|u_C^{21x}\right| = 6.0222\text{E-7} \tag{48}$$

Thus, it can be seen from Eqs.(47) and (48) that interval analysis method gives tighter bounds of responses than convex modeling method in the case of data points listed in Table 6.

From the analysis of this section, we still can find that the sample data points decide which of the non-probabilistic uncertainty descriptions, convex modeling or interval analysis, to be prefered.

6. Conclusion

In this study, through numerical examples convex modeling and interval analysis are extensively compared based on the same experimental points. Some explanations are given for the problem that which of the non-probabilistic uncertainty descriptions, convex modeling or interval analysis, ought be utilize. Given the experimental points, the smallest hyper-rectangle and the smallest ellipsoid containing them can be obtained. From these numerical examples it can be concluded that (1) If V_1 is smaller than V_2, then one has to prefer interval analysis; (2) If V_1 is in excess of V_2, then the analyst ought to utilize convex modeling; (3) If V_1 equals V_2 or these two quantities are in close vicinity, then two approaches can be utilized with nearly equal validity. Therefore, the type of the analytical treatment that should be adopted for non-probabilistic analysis of uncertainty depends upon the available experimental data.

Of course, the purpose of the paper is not to replace the probabilistic approach by the non-probabilistic set-theoretic convex methods. The latter is a possible alternative or a supplementary way of the uncertainty analysis when scarce data is available to justify the probabilistic analysis. We conclude that the type of the analysis of uncertainty depends on the type and amount of available information.

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