An interval based technique for FE model updating

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Abstract: Model updating techniques are largely used in civil and mechanical engineering to obtain reliable FE models. The model parameters are iteratively adjusted until the model response matches the measured structural response within a given tolerance. In this work it is assumed to know the response of a structure in terms of uncertain modal quantities. Accordingly, the model response is computed accounting for uncertainty by defining the model parameters as intervals.
The updating problem is formulated in the framework of interval analysis by exploiting the inclusion theorem. The solution is reached when the structural response is completely included by the FE model response and the parameters uncertainty is at a minimum. The presented method offers some advantages that are: each model parameter is included in a physical interval hence the solutions are guaranteed to be physical; the uncertainties of the measured response are naturally embodied into the problem. The method is discussed through a simple numerical example. The interval updating solution is then compared with conventional updating technique by applying it to a real case study.

Keywords: model updating, interval analysis, global optimization, FE model admissibility

1. Introduction

The work is framed in the field of finite element model (FEM) updating procedures (Friswell and Mottershead, 1995), that have the goal of calibrating the model parameters to get the best match between experimental and analytical modal data. In the case of civil engineering, model updating is a useful tool to know the actual state of the structures (diagnosis) and for the construction of predictive structural models (prognosis). The solution of the problem belongs to the field of inverse problems (Sorenson, 1980) and is classically faced using nonlinear programming algorithms. The objective function to be minimized is often chosen as the distance between measured and computed response quantities and the solution strategies can be distinguished by the algorithm used to search for the minimum. In structural dynamics, the updating is classically performed using modal data, that can be expressed either as the modal model or the response model of the structures (Ewins, 1984). In the case of civil engineering it is common practice to refer to modal shapes and related frequencies (Camillacci and Gabriele, 2005).
Two alternative formulations are usually followed: deterministic methods and statistical or Bayesian methods depending on the analyst preference and confidence on the uncertainty related

to the problem. The first formulation makes use of deterministic or crisp parameters and measures, whereas the second includes uncertainty through normal probability density functions or two crisp parameters and measures (average and standard deviation). In both cases the updating problem results to be ill-conditioned because of two fundamental aspects: the dependency of the problem on the ratio between the number of parameters and the number of independent measures (Gola et al., 2001) and the inherent uncertainties that characterize both the FE model (modelling errors) and the experimental data (measurement errors) (Capecchi and Vestroni, 1993).

In general no explicit uncertainty is associated to deterministic methods, for which the optimal parameters gauging is demanded to the search algorithm and to the model sensitivity. On the contrary, Bayesian approaches account for uncertainty by assigning appropriate values to the standard deviation to express confidence on the data. However, the statistical values, at least for the model data, are to be assigned a priori and strictly depend on the skill of the structural analyst (Collins et al., 1974).

The present work proposes an alternative way to treat the uncertainty in updating problems, that is based on the concepts of “interval analysis” (Moore, 1966). This methodology allows to represent uncertain quantities not by means of point values, but by bounding them inside possibility intervals. The interval width define the uncertainty level. In this respect, interval methods offer some advantages as compared to deterministic and stochastic methods in fact: they are capable to account explicitly for the uncertainties of the problem, do not need the introduction of distributions, as in the probabilistic case, let to define interval limits coherent with engineering bounds, do not require initial conditions to start the search algorithm.

Up to date various works have been issued concerning the computation of bounded eigenvalues and eigenvectors of mechanical structures (Shalaby, 2000), while it remains to deepen the possibility of using interval global optimization methods (Ratschek and Rokne, 1988, Hansen and Walster, 2004) to update parameters of FE models. The decision to use an interval approach also implies the use of interval finite element method (IFEM) for the development of numerical model to be updated. A formalized formulation of the IFEM can be found in the works of Muhanna and Mullen (1999) and Muhanna et al. (2006), also with applications in the static case. Previous applications of the static IFEM can be found in the works of Rao and Berke (1997), in Köyluoglu and Elishakoff (1998) where a comparison with probabilistic solutions is also presented. The interval FEM also finds applications in structural optimization procedures of truss structures (Pownuk, 1999). Some deepening in the dynamic case, that is of major interest for the treated arguments, can be found in Moens (2002).

The work is organized in three parts. In section 2 the basic concepts of interval analysis are given and the main properties of interval operations and functions are discussed in view of their subsequent use. In section 3 the interval model updating problem is discussed and applied to simple numerical example. In the last section the method is applied to a real test case, concerning the model updating of a simplified model of a building sub-structure, whose experimental modal data are made available by an independent experimental campaign. The method is discussed according to two possible cases: crisp or certain experimental measures and interval valued or uncertain measures.

2. Interval computations

In interval analysis (Moore 1966, Sunaga 1958) numbers are replaced by intervals in which they are contained, the larger the interval the larger the uncertainty in the evaluation of the number. An interval X could be denoted by infimum and supremum limits (x_{inf}, x_{sup}) or by the central notation, where the interval limits are obtained by respectively adding and subtracting the uncertainty radius Δx to the central value x_c , by way of the unit interval e_Δ = [-1,1] and by applying the interval addition rule.

$$X = \lfloor x_{\text{inf}}, x_{\text{sup}} \rfloor = x_c + \Delta x \cdot e_\Delta \tag{1}$$

The result of a generic interval operation "op" is the interval set of all the possible solutions when any operand varies independently in its own limits. From this definition follow the *inclusion property*, that is any possible result from the crisp operation "x op y" is included in the interval operation "X op Y", providing that $x \in X$ and $y \in Y$.
In the standard interval computations a result is generally overbounded. In this case the word "standard" means that any interval expression is evaluated according to the assumption of independency between operands. From this follows that the sharpest computed interval is evaluated from an expression that contains a minimum number of occurrences of the same operand. An example is given from the so called sub-distributivity property in equation (2).

$$X \cdot (Y + Z) \subseteq X \cdot Y + X \cdot Z \tag{2}$$

Let be $f(\mathbf{x})$ a real valued function that depends on the crisp parameters $\mathbf{x} = (x_1, \dots, x_i, \dots, x_n)$, there exist some ways to define its interval extension $F(\mathbf{X})$ (Moore, 1966). The interval functions considered in the paper are called natural extensions and are obtained by replacing every single occurrence x_i in the expression of f with the correspondent interval X_i in F. $F(\mathbf{X})$ maps $\mathbf{X} = (X_1,\dots,X_n)$ into the real interval space and converges to f, i.e. $F(\mathbf{x}) = f(\mathbf{x})$, whenever $\mathbf{X}$ shrinks to crisp $\mathbf{x}$.
The inclusion property is settled for natural extensions by the inclusion theorem (Hansen and Walster, 2004). This theorem ensures, for various kind of interval extensions, that the inclusion range of $F(\mathbf{X})$ bounds all minima and maxima of $f(\mathbf{x})$ over $\mathbf{X}$. This theorem was firstly demonstrated for interval natural extensions that are also inclusion monotonic, i.e. $F(\cdot)$ is inclusion monotonic if taken $\{\mathbf{X} \subset \mathbf{Y} \mid X_i \subset Y_i, \forall \text{ i}\}$, it follows that

$$F(\mathbf{X}) \subset F(\mathbf{Y}) \tag{3}$$

In this work it is of interest to discuss the interval analysis aspects related to inverse engineering problems. One of this is the convergence to crisp values of interval functions, and two different type of interval functions are considered. The first type, also called *thin* interval function, possesses interval variables $\mathbf{X}$ and crisp parameters p, $F(\mathbf{X},\mathbf{p})$. The thin attribute is referred to the kind of convergence of the function as the radius $\Delta\mathbf{x}$ decrease and tends to zero. This is shown in the graphical example of Figure 1a, where the continuous line represents the crisp evaluated function $f(\mathbf{x},\mathbf{p})$, whereas the progressive decreasing monotonic boxes are the interval representation of its natural extension. From the figure is seen that as $\Delta x_i \rightarrow 0$, $\forall$ i, then $X_i \rightarrow x_{ci}$ and $F(\mathbf{X},\mathbf{p}) \rightarrow f(\mathbf{x}_c,\mathbf{p})$. The second type, also called *thick* interval function, possesses both interval variables $\mathbf{X}$ and parameters $\mathbf{P}$, $F(\mathbf{X},\mathbf{P})$. For thick functions only a relaxed type of convergence can be defined. In fact, if $F(\cdot,\mathbf{P})$ is evaluated on crisp $\mathbf{x}_c$ of the Figure 1b, then the best that can be obtained is that $F(\mathbf{x}_c,\mathbf{P}) \supset f(\mathbf{x}_c,\mathbf{p})$, $\forall$ $\mathbf{p}\in\mathbf{P}$, but not equals it at $\mathbf{x}_c$. In this case the thick attribute refers to the impossibility of converging to crisp values of this type of functions, as a consequence of the presence of interval parameters $\mathbf{P}$ inside the function expression. From a geometrical point of view, as $\Delta x_i \rightarrow 0$, $\forall$ i, $F(\mathbf{X},\mathbf{P})$ converge to a segment, and covers a bundle of crisp functions.

This distinction between thin and thick function and their different kind of convergence are the main concepts embodied into the presented method together with the inclusion property. In fact, solutions to mechanical problems are considered physically plausible only if the method guarantees the inclusion of the experimental outcomes.

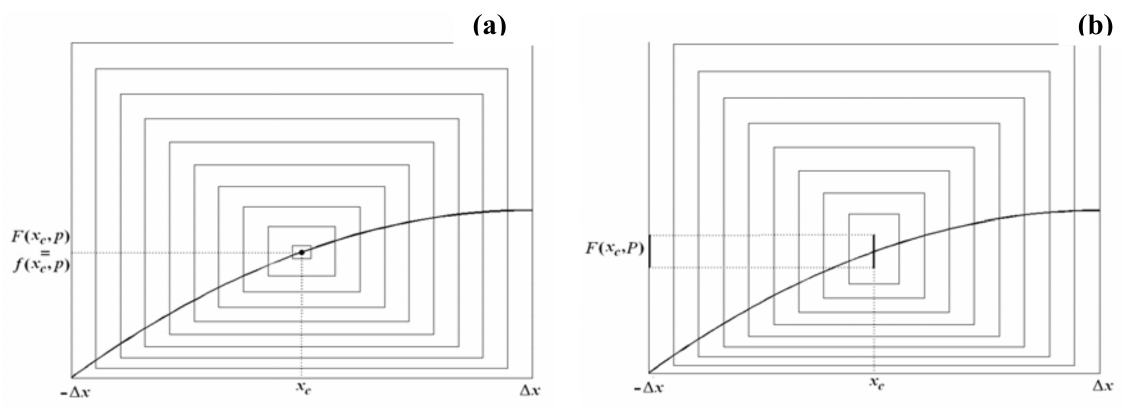


Figure 1 – Monotonic convergence of (a) thin function, (b) thick function

2.1 MODEL FUNCTION AND INTERVAL SOLUTION

The previously presented concepts about interval functions are now applied to define which kind of model are used in the present work, which kind of quantities are affected by uncertainty and which kind of interval solutions need to be computed.
One remembers that the purpose of the paper is to present an interval model updating procedure and the models to be updated are finite element (FE) representations of mechanical structures.
The formulation of the interval FE method can be found in Muhanna (1999) and Moens (2002), where uncertainties can appear in the mass and stiffness coefficients as well as in the geometry of the system. However, in the present context, only the constitutive parameters of the model are considered affected by uncertainty, therefore the stiffness matrix **K** is an interval matrix.
In view of model updating applications eigenvalues and eigenvectors of the system need to be computed and then compared with the experimental counterparts. In general these measured quantities are uncertain, due to experimental errors, and could be bounded in confidence intervals, that one wants to reproduce with the updated FE model.
For this purpose the model functions are defined as $\{\Lambda(\mathbf{K}), U(\mathbf{K})\}$, respectively the interval eigenvalues and eigenvectors, that depend on the interval stiffness variables that are present in the stiffness matrix **K**. $\Lambda(\mathbf{K})$ and $U(\mathbf{K})$ can also be considered as the interval extensions of the crisp eigenfunctions $\lambda(\mathbf{k})$ and $u(\mathbf{k})$.
According to the interval FE method, interval enclosure $\mathbf{\Lambda}=\Lambda(\mathbf{K})$ and $\mathbf{U}=U(\mathbf{K})$ can be computed as the solution of the generalized algebraic problem:

$$\mathbf{KU} = \mathbf{\Lambda m U} \tag{4}$$

To solve this problem one should find the inclusion set for the eigenvalues, defined as

$$\mathbf{\Gamma} = \left\{ \boldsymbol{\lambda} \in \mathrm{R}^{\mathrm{n}}, \mathbf{u} \in \mathrm{R}^{\mathrm{n\times n}} \mid \mathbf{ku} = \boldsymbol{\lambda}\mathbf{mu}, \mathbf{u} \neq 0, \mathbf{k} \in \mathbf{K} \right\} \tag{5}$$

Unfortunately the methods proposed in the literature are capable to find the true solution $\mathbf{\Lambda} = \mathbf{\Gamma}$ only in limited cases (Shalaby, 2000) being the wider solution $\mathbf{\Lambda} \supset \mathbf{\Gamma}$ the only available solution for the general cases. Presently, a solution strategy similar to that developed by Qiu and Chen (1995) is followed. The choice comes from the observation that the interval computations can be replaced by crisp operations on the interval limits yet preserving the monotonic inclusion of the solution (Chiao, 1999). According to Qiu and Chen the problem (4) specialised for the j-th eigenvalue is written as:

$$\left(\mathbf{k}_{\mathrm{c}} + \Delta\mathbf{k} \cdot e_{\Delta}\right)\mathbf{U}_j = \Lambda_j \mathbf{m} \mathbf{U}_j \tag{6}$$

and the solution bounds are then computed by solving two crisp sub-problems obtained according to the following general interval property:

$$
\begin{aligned}
&\mathbf{K}\mathbf{u}_j = \mathbf{k}_c\mathbf{u}_j + \Delta\mathbf{k}\left|\mathbf{u}_j\right| \cdot e_\Delta = \mathbf{k}_c\mathbf{u}_j + \Delta\mathbf{k}\left(\mathbf{s}_j\mathbf{u}_j\right) \cdot e_\Delta ;\\
&\text{with } \left|\mathbf{u}_j\right| = \mathbf{s}_j\mathbf{u}_j \text{ and } \mathbf{s}_j = diag\left(sign\left(\mathbf{u}_j\right)\right)
\end{aligned}
\tag{7}
$$

that guarantees the monotonic inclusion in the working range. The infimum and supremum limits are obtained by the following expressions:

$$
\begin{cases}
\left(\mathbf{k}_c - \mathbf{s}_j^T \Delta\mathbf{k}\ \ \mathbf{s}_j\right)\mathbf{u}_{j,\inf} = \lambda_{j,\inf}\mathbf{m}\,\mathbf{u}_{j,\inf}\\
\left(\mathbf{k}_c + \mathbf{s}_j^T \Delta\mathbf{k}\ \ \mathbf{s}_j\right)\mathbf{u}_{j,\sup} = \lambda_{j,\sup}\mathbf{m}\,\mathbf{u}_{j,\sup}
\end{cases}
\tag{8}
$$

Equations (8) gives $\boldsymbol{\Lambda} = [\lambda_{\inf}, \lambda_{\sup}] \supset \boldsymbol{\Gamma}$ where the over-bounding depends on the uncertainty radius $\Delta\mathbf{k}$. It has been shown in Moens (2002) that the above formulation ensures to include all the true solutions when the eigenvalues of the system are properly spaced.
The solution of the equations (8) is found under the hypothesis of sign invariance of the j-th eigenvector (Deif and Rhon, 1994). This restriction on the allowable eigenvectors is again necessary to guarantee the preservation of the inclusion property. In view of the updating problem it is also important to guarantee that the interval method used to calculate the function values $\{\boldsymbol{\Lambda}=\Lambda(\mathbf{K}), \mathbf{U}=U(\mathbf{K})\}$ is inclusion monotonic, in this case it is demonstrated (Hansen and Walster, 2004) that, for the defined extensions, the inclusion theorem holds. The authors are aware that exist many methods to compute interval eigenvalues and that the selected method is characterized by a great overbounding of the interval estimation. But the inclusion theorem validity it is, at author's judgment, more important for optimization problems applied to physical systems than the overbounding, at this work stage. This will be better explained in the following sections.

3. Interval model updating

The inverse model updating problem is classically formulated as the search for the minimum of a predefined objective function $l(\mathbf{x})$ that depends on the vector of the updating unknowns $\mathbf{x}$:

$$
\min_{\substack{\mathbf{x}\in\mathbf{D}\\ \mathbf{D}\in R^n}} l(\mathbf{x}) \tag{9}
$$

In those cases in which a matching between two sets of quantities is sought for, $l(\mathbf{x})$ is conveniently expressed as a measure of the distance between experimental and numerical quantities and a least squares formulation is followed (Camillacci and Gabriele, 2005). Therefore, in the present case, the objective function is specialized as the 2 norm distance:

$$l(\mathbf{k}) = \left\|\boldsymbol{\lambda}_s - \lambda(\mathbf{k})\right\|_2^2 \tag{10}$$

where the unknowns are stiffness variables, that are present in the matrix **k**, and where only the contribution of the eigenvalues is considered. A more general form than (10) would comprise the contribution of the eigenvectors as well (Gola et al. 2001), but this further sophistication is not within the purposes of the paper.
On the contrary, the interval model updating problem is here discussed.

3.1 INTERVAL GLOBAL MINIMIZATION

First of all the minimization of the error norm defined by equation (10), is a nonlinear programming problem and is possible to solve it in an interval space by interval global optimization algorithms (Hansen and Walster, 2004; Ratschek and Rockne, 1988). Such algorithms are generally comprised in the so called branch and bound methods (B&B), where an initial search domain is iteratively subdivided in smaller sub-domains and, for each created sub-domain, a first criterion (bounding step) is applied to verify if the sought solution could be contained in it or not. In the first case a sub-domain survives and a second criterion is applied to subdivide it again. In the second case the evaluated sub-domain is discarded from the search. The found solution is finally given by the surviving sub-domains. Interval B&B methods can be developed thanks to the existence of the interval inclusion theorem. In fact a non verified inclusion into generated sub-domains can be used as discarding criterion. Inclusions need to be verified by defining a proper interval extension of the crisp function to be minimized and a proper extension is that for which the inclusion theorem can be demonstrated, for example the class of inclusion monotonic extensions.
All the concepts briefly explained are now applied to the original updating problem defined by equation (10). The model updating problem is not only a mathematical programming problem, it is also a physical problem defined with some uncertainties, and is here important to underline the differences that arise with respect to a conventional setting. In fact, different cases should be accounted for, depending on which quantities are affected by the uncertainty.

1. The uncertainty source is only in the FE model stiffness parameters (**K**); in this case a natural extension for (10) is written as (11)

$$L(\mathbf{K}) = \left\|\boldsymbol{\lambda}_s - \Lambda(\mathbf{K})\right\|_2^2 \tag{11}$$

where $\boldsymbol{\lambda}_s$ is the crisp vector of the experimental eigenvalues, $\Lambda(\mathbf{K})$ is the interval extension by which the FE model interval eigenvalues, Λ, are calculated. If $\Lambda(\mathbf{K})$ is evaluated by the equations (8), it is inclusion monotonic and hence $L(\mathbf{K})$ is also inclusion monotonic.
$L(\mathbf{K})$ is a thin interval extension, because as $\Delta k_i \rightarrow 0$, $\forall$ i, then $K_i \rightarrow k_{ci}$ and $L(\mathbf{K}) \rightarrow f(\mathbf{k}_c)$. That means that the crisp updating problem (10) and the interval updating problem (11) have the same crisp solution, in the limit that the uncertainty approach to zero.
The solution in this case can be effectively obtained through standard interval B&B optimization techniques (Hansen and Walster, 2004; Jansson and Knüppel, 1995) that give good results even in the case of ill-conditioned problems and that, in the limit $\Delta\mathbf{k} \rightarrow 0$, converge to crisp solutions.

2. In a second case the uncertainty source is both in the experimental measures and in the model parameters; in this case it is required to match the interval vectors $\boldsymbol{\Lambda}_s$ and $\boldsymbol{\Lambda}=\Lambda(\mathbf{K})$.
The natural extension of equation (10) is in this case given by

$$L(\mathbf{K}) = \left\|\boldsymbol{\Lambda}_s - \Lambda(\mathbf{K})\right\|_2^2 \tag{12}$$

and the above expression for $L(\mathbf{K})$ cannot be used unless a metric between intervals is introduced to replace the standard metric between crisp values (Moore, 1966).
The equation (12) also defines an interval thick extension of (10), due to the fixed uncertainty in the experimental measures vector $\boldsymbol{\Lambda}_s$. In this case the objective of reducing the stiffness uncertainty, with the goal to converge around an optimal solution, is limited by this fact and numerical solutions can only be found with a final fixed uncertainty.

In the case 2., instead of introducing a metric between intervals, an approach coherent with the principles of interval analysis is proposed and named *Interval Intersection Method* (INTIM, Gabriele, 2004), where the basic branch and bound optimization technique present in Hansen (2004) is adopted.
One supposes to define the interval search domain $\mathbf{D}$, with $K_i \in \mathbf{D}$, $\forall$ i. The branching step is left unchanged and its repeated application produces progressively smaller sub-domains, $\mathbf{D}_j \subset \mathbf{D}$, in which the searched stiffness parameters are included. In the bounding step the basic operations between sets are applied to interval solutions $\boldsymbol{\Lambda}$, to verify the inclusion of the experimental vector $\boldsymbol{\Lambda}_s$. If the function $\Lambda(\mathbf{K})$ is inclusion monotonic and the inclusion theorem is verified, then the verified inclusion of $\boldsymbol{\Lambda}_s$ in $\boldsymbol{\Lambda}$ means that the FE model, endowed with the interval parameters $\mathbf{K}$, is capable to represent the experimental solution. This capability is here intended as FE *model admissibility*.
The degree of admissibility of a model, in the parameters domain, with respect to the known measured response is hence simply checked using the intersection operation to verify the inclusion:

$$\left\{ \mathbf{\Lambda}_s \cap \mathit{\Lambda}(\mathbf{K}) = \mathbf{\Lambda}_s, K_{\mathrm{i}} \in \mathbf{D}_{\mathrm{j}}, \forall \mathrm{i} \right\} \tag{13}$$

It can be assumed that if the total inclusion is not verified the model is not admissible to represent the real structure in the considered domain, in fact if the inclusion theorem holds no other eigensolution can be found outside the calculate interval (Λ).
By inverting the previous statement, the equation (13) can be taken as exclusion criterion in the B&B search algorithm, for the sub-domains $\mathbf{D}_j$, in the following pessimistic form:

$$\left\{ \mathbf{\Lambda}_s \cap \mathit{\Lambda}(\mathbf{K}) = \varnothing, K_{\mathrm{i}} \in \mathbf{D}_{\mathrm{j}}, \forall \mathrm{i} \right\} \tag{14}$$

In the interval updating algorithm the solution is iteratively found. Starting from the whole parameters space $\mathbf{D}$, this is consecutively branched in sub-domains $\mathbf{D}_j$ that, in turn, are preserved or discarded according to (13) and (14). The procedure stops when for some $\mathbf{D}_j$ the criterion (13) holds and a pre-fixed radius of minimum uncertainty tolerance is reached, so they cannot be further branched.
It is important to note that the above procedure is indeed general and can be applied to case 1. as well, when λ_s is a crisp measures vector. Both cases and solution procedures are illustrated according to the numerical simulation discussed below.
In Figure 2 is depicted a graphical representation of the branching and the bounding steps, by thinking to apply admissibility criteria (13) and (14) for each generated sub-domain in the initial search box $\mathbf{K}_0$. In the figure the arrows represent the applications of the interval extension $\mathit{\Lambda}(\mathbf{K})$, in order to obtain the intervals Λ to be compared with Λ_s in the measures space.

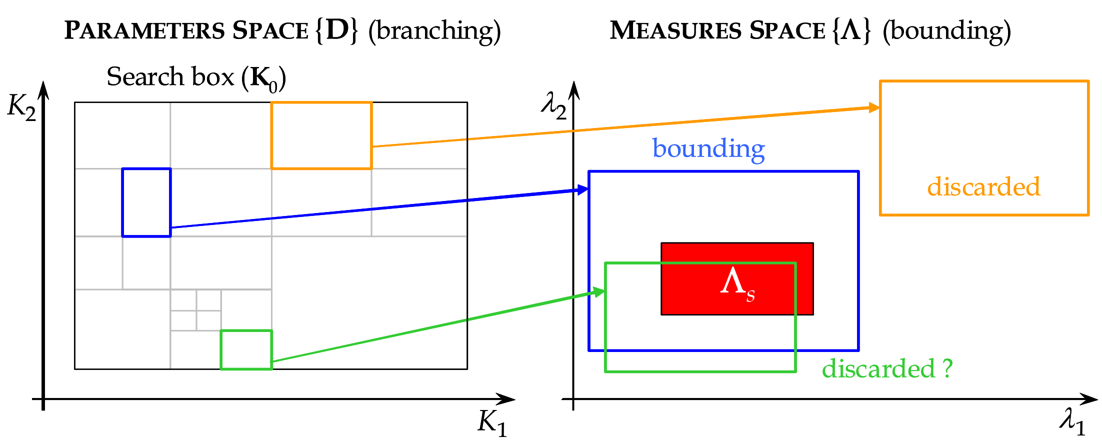
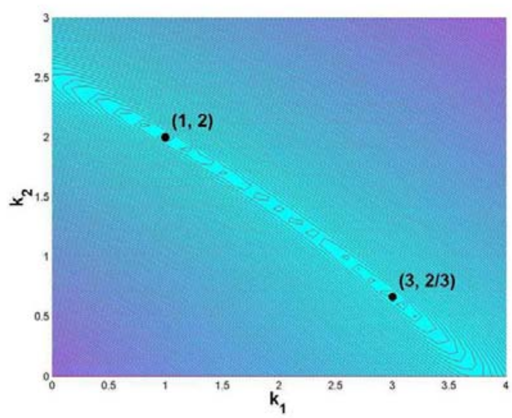


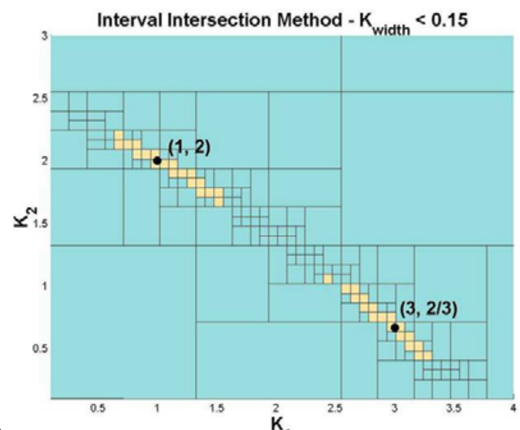
Figure 2 – Interval intersection method – branching and bounding step

3.2 NUMERICAL EXAMPLE

A simple mechanical system composed by a 2dofs mass-spring system is considered. The problem is again to find the parameters vector $\mathbf{k} = [k_1, k_2]$ that produces the best match between given experimental and numerical modal data. The mechanical system is used either to generate pseudo-experimental data or to compute the numerical frequencies according to the formulation given in section 2.1. It is initially assumed that the experimental frequencies are exactly identified (case 1.) and only the parameters are affected by uncertainty. Then, also the experimental frequencies are assumed to be identified within an interval (case 2.).
The uncertainty free pseudo-experimental frequencies are $\mathbf{f}_s = [0.082, 0.307]$ Hz, ($\lambda_s = 2\pi f_s^2$), and it corresponds to $\mathbf{k}_0 = [1, 2]$. In a full deterministic setting the 2-norm objective function (10) applies and conventional minimization schemes can be used. However, even in this simple situations the objective function can have more than one minimum as shown in Figure 3a where it is plotted in the form of a contour plot representation. In particular, the function has two global minima: one for the true vector of parameters $\mathbf{k}_0$ and the other for $\mathbf{k} = [3, 2/3]$. Depending on the search domain, the solution algorithm and the initial value of the parameters the false minimum can be reached by the a crisp updating procedure.



(a)

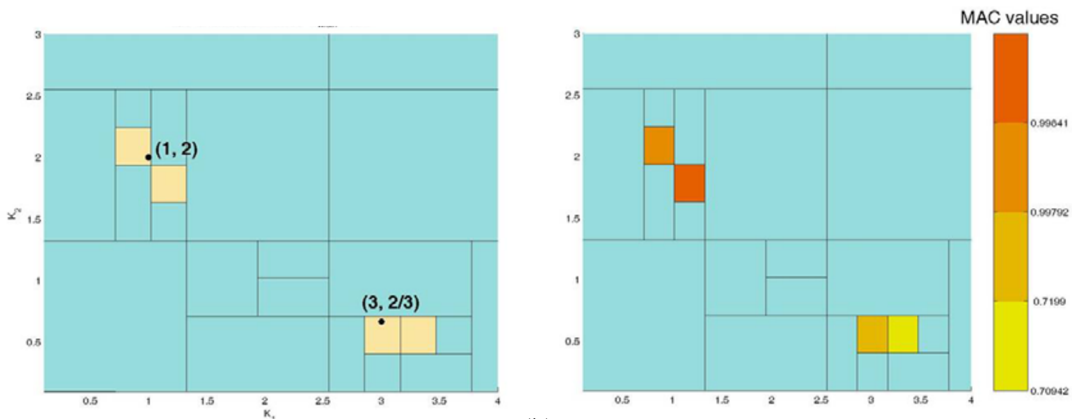


(b)

Figure 3 – (a) Crisp objective function, (b) INTIM solution for case 1.

In the case of crisp experimental (solution case 1.) INTIM technique is applied starting from the initial parameters domain $\mathbf{D} \supseteq \mathbf{K}_0 = [[0,4], [0,3]]$ and the result is shown in Figure 3b, where the progressive partition in finer sub-domains tending to accumulate around the minima. The partitioning stops when for some $\mathbf{D}_j$ the criterion (13) holds and the radius of minimum

uncertainty tolerance k_w, is reached, so that $\mathbf{D}_j$ cannot be further branched. In the figure the solution domain is given by the collection of the lightest boxes that are those for which $k_w < 0.15$. The uncertainty in the obtained solution is measured by the spread of the lightest boxes around the crisp minima.
In the case of interval valued experimental data the considerations done for solution case 2. are valid. Now the pseudo-experimental eigenvalues are collected in the interval vector $\mathbf{\Lambda}_s = [[0.08, 0.45], [3.55, 3.92]]$ and it corresponds to $\mathbf{K}^* = [[0.99, 1.01], [1.98, 2.02]]$.
The solution in the parameter space is given in Figure 4a. In the present case $\mathbf{k}_w = 0.31$ and the solution is slightly more confined with a reduced number of branches, but with larger final boxes. Here again two distinct sub-domains solutions are detected.



(a) (b)

Figure 4 – INTIM solutions for case 2. (a) bounded solution, (b) solution choice

3.2.1. *Solution choice*

As shown by the example above, one of the main advantages of INTIM is the capability to find all the minima in the parameters space, in the form of a collection of boxes. Anyway, no further distinction between minima can be done to locate the box of the true parameters.
To make a choice among all the possible solutions an a posteriori processing of the solution is performed on the base of a choice criterion. If the experimental modal shapes are known the modal assurance criterion MAC can be used:

$$\mathrm{MAC}_{s,n} = \frac{\left(\mathbf{u}_s^T \cdot \mathbf{u}_n\right)^2}{\left(\mathbf{u}_s^T \cdot \mathbf{u}_s\right)\cdot\left(\mathbf{u}_n^T \cdot \mathbf{u}_n\right)} \tag{15}$$

where $\mathbf{u}_n$ and $\mathbf{u}_s$ stand for numerical and experimental central values of the eigenvectors. It is worth recalling that $0 \le \mathrm{MAC} \le 1$ and $\mathrm{MAC} = 1$ whenever $\mathbf{u}_n = \mathbf{u}_s$.

The MAC values have been computed for all the solution boxes in Figure 4a and have been reported in Figure 4b as a color scale superposed to the parameters domain. The darkest box is the parameters interval endowed with the highest MAC that is therefore chosen as the updating solution.

4. Real case study

The case study is taken from the ILVA-IDEM project in which one of the authors is involved (Mazzolani et al., 2004; Cardellicchio, Spina and Valente, 2004; Valente, Spina and Nicoletti, 2006). The experimental results were obtained during a large experimental campaign aimed at evaluating the mechanical and strength characteristics of an existing reinforced concrete building that can be considered representative of many gravity-load designed reinforced concrete buildings located in the South of Italy, Figure 5a.

(a) (b)

Figure 5 – Case study: (a) Original building, (b) Selected structural module

	TR - Transient tests	SS - Steady state tests
1^{st} longitudinal mode	[1.76, 1.87]	[1.74, 1.85]
2^{nd} longitudinal mode	[5.65, 5.72]	[5.20, 5.65]

Table 1 – Identified experimental frequencies (Hz) of the longitudinal main frame.

	First floor	Second floor
Beams	1.75e10	1.75e10
Columns	[1.35, 1.90]e10	[0.89, 1.21]e10

Table 2 – Measured Young modulus (N/m^2) of the structural members.

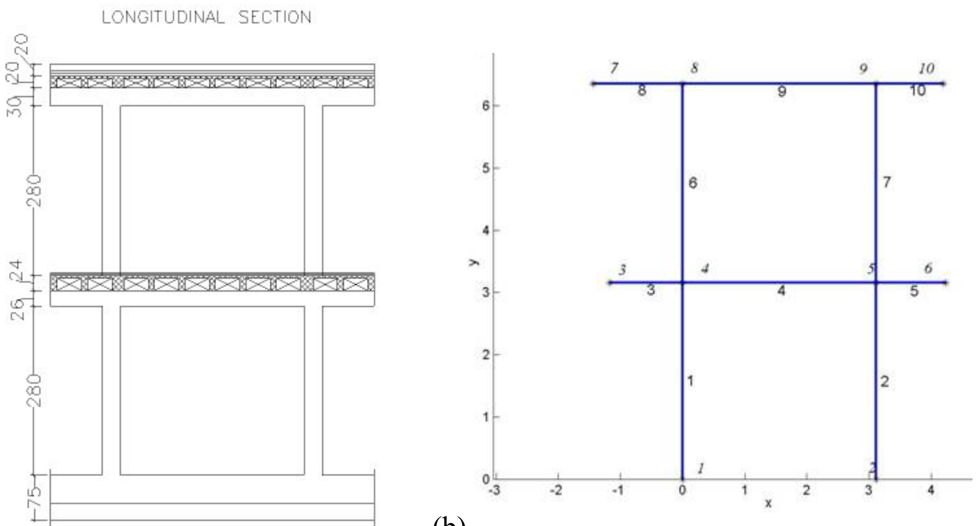
Partition walls and external claddings were removed and, then, the original building was divided into six separate smaller structures (Figure 5b). Four of them, nominally identical each other, were subjected to dynamic testing aimed at identifying the modal model and at evaluating the scatter in the results. Different types of tests were performed by changing the excitation type. Impulse excitations were used to provide transient response (test TR) and harmonic excitations were used to provide steady state response (test SS). The analysis of the dynamic response was performed through well established methods (Ewins, 1984) and frequencies and modal shapes were identified for the first six modes. For the present purposes, only a small set of the whole available data are considered. They are referred to the structural module marked in Figure 5b. Further, for simplicity, only the modal behaviour in the plane of the main frames is considered Figure 6a. The frequencies of the first two longitudinal modes have been used in the updating procedure and their interval variation is shown in Table 1.
The mechanical properties of the concrete were measured in laboratory on core samples extracted from the structure and on site using NDT tests to check for the concrete uniformity. The results are given in Table 2, from which it is apparent that the uncertainty is limited to the columns.

4.1 RESULTS

The INTIM described in section 3 is applied to the 2D model of Figure 6b in order to update the stiffness of the columns. The Young modulus E is the parameter to update since it acts as a scale factor for the columns stiffness. It is assumed that the columns of a floor have all identical stiffness, therefore two interval values E_1 and E_2 are sought for, one per floor. A wide and identical intervals $E_1 = E_2 = [0.1, 3]\times10^{10}$ N/m^2 has been chosen to be the initial search domain **D**. A physical justification can be given to this choice in consideration of the large uncertainties related to the NTD tests, but it is unnecessary since it is the ability of the technique to work with box domains and its numerical efficiency that suggest to widen **D** in order to get a complete picture of the solution.

4.1.1. *Solution case 1.*

It is interesting to observe that if a crisp model updating procedure would be used, together with the initial conditions equal to the average values of the measured elastic moduli of Table 2 ($\mathbf{E}_s$), the following crisp values would be found: $e_1 = 2.27\times10^{10}$ N/m^2, $e_2 = 0.33\times10^{10}$ N/m^2 (Figure 7). They are very far from those listed in Table 2 that can be considered the physical solution range so that one can wonder if the adopted FE model is adequate to the problem or it should be revised.
In this case the interval solution calculated by INTIM and applied by choosing λ_s as the central values of the uncertain experimental measures (solution case 1.), is again far from the measured elastic moduli box ($\mathbf{E}_s$). But INTIM solution puts in evidence the presence of a second solution sub-domain that have a not null intersection with $\mathbf{E}_s$ only along the E_2 axis.



(a) (b)

Figure 6 – Case study: (a) Longitudinal main frames, (b) 2D FE model

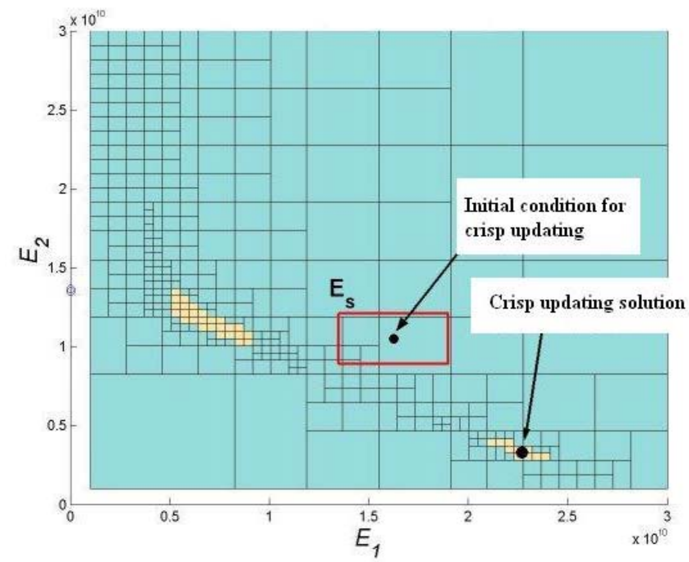


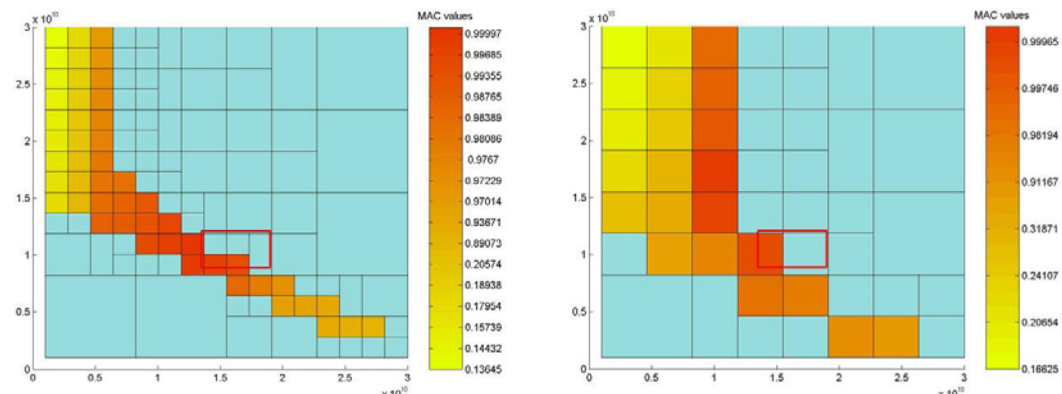
Figure 7 – Comparison between crisp updating solution and INTIM solution case 1.

4.1.2. *Solution case 2.*

The presented case study is really uncertain, as it clear from the values in Tables 1 and 2, and could be wrong to consider crisp values of the experimental frequencies as objective of the updating procedure.

If one considers the full measures uncertainty, firstly the interval updating technique can be used initially to check the admissibility of the FE model and then to find the solution. In the presence of experimental evaluations for the elastic moduli $\mathbf{E}_s$, admissible FE models are those for which the experimental eigensolution (Λ_s) is completely included by the model response (Λ), and parameters solution ($[E_1,E_2] \in \mathbf{D}_j$) has at least one non vanishing intersection with the experimental box $\mathbf{E}_s$.

The solution of the interval updating procedure is shown in the parameter space in Figure 8, where the results obtained from test TR and test SS are both reported. The empty rectangle shown in the figures is the box $\mathbf{E}_s$ of Table 2 and the most feasible solutions are in the color scale of the MAC values.



(a) (b)

Figure 8 – INTIM solution case 2. (a) test TR, (b) test SS

A finer partitioning in Figure 8a than in Figure 8b can be appreciated. In fact, for test TR and SS the stop tolerances are different, 1.83×10^9 N/m^2 and 3.67×10^9 N/m^2 respectively, because of the different level of uncertainties in the identified frequencies, see Table 1. Anyway, as expected, the qualitative behaviour of the solution is similar in the two cases.

It is worth to recall that all the boxes that are possible solutions have a full intersection in the frequency space (13). The MAC values are then used to discriminate among the possible solutions. From the figures, it can be observed that the darkest boxes are also those closest to the rectangle of the measured elastic moduli. Finally it deserves to underline that one could want to shrink the boxes radius to a point in order to have a crisp solution. However the convergence to a point value is not guaranteed in the parameters space, since in the frequency space it can happen that further radius reductions pushes the results of $\Lambda(\mathbf{K})$ to interval boxes for which the inclusion rule (13) does not apply, that is to say that in this case the objective function is an interval thick extension.

5. Conclusions

In this work the application of an interval updating technique is discussed and applied in both numerical and experimental cases. The technique is applied to uncertain interval FE models by taking distinct the case of certain (crisp) measures (solution case 1.) from the case of uncertain (interval) measures (solution case 2.). For this second case the updating approach, consistent with the principles of interval analysis and set theoretic comparisons, is presented and here called interval intersection method, INTIM. In the solution case 1. standard updating techniques, based on crisp objective function, and the new one are compared, by applying them to a simple 2dofs mechanical system. For the interval solution case 1. it is pointed out that all the admissible solutions can be found.

In the solution case 2. only the INTIM technique has been applied. In this case further developments are given for the choice of physical solutions in the FE model parameters space, based on MAC comparison of modal shapes. The interval intersection method is first numerically validated by applying it to the previous 2dofs system, is then applied for updating the column stiffness of a 2D model of an r/c experimented structure, by tacking its longitudinal modal behaviour. The obtained interval best results, in the parameters space, are found to be intersected with the interval of the equivalent measured mechanical properties.

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