General Interval FEM Program Based on Sensitivity Analysis Method

Andrzej Pownuk
Department of Mathematical Science
University of Texas at El Paso
http://andrzej.pownuk.com

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Uncertainty in mechanics

Figure: Material properties and geometrical parameters of damaged structures

Uncertainty in mechanics

Figure: Material and geometrical properties of rocks

Uncertainty in mechanics

Figure: Material and geometrical properties of soil

Random variables

- Definition $X : \Omega \ni \omega \rightarrow X(\omega) \in R$
- Probability density function $P\{a \leq X \leq b\} = \int_a^b f(x)dx$

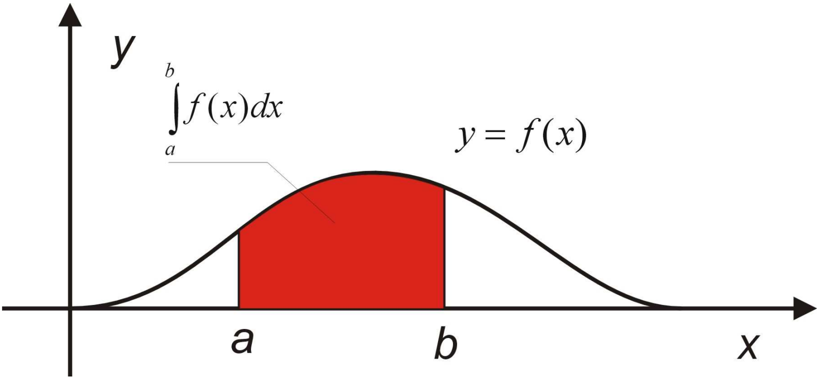

Figure: Probability that $P\{a \leq X \leq b\} = \int_a^b f(x)dx$

Distributed load as a random variable

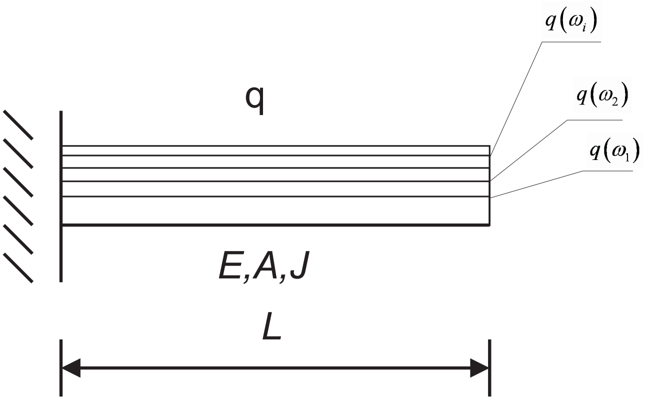

Figure: Beam with random distributed load

Distributed load as a random field

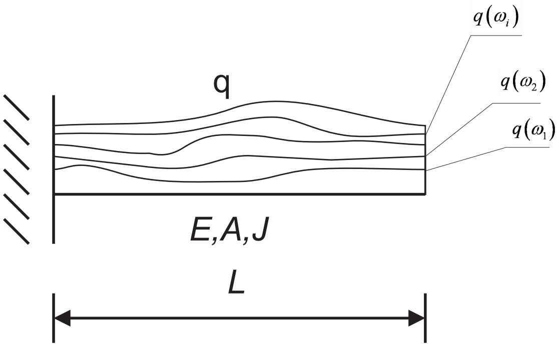

Figure: Beam with random distributed load (random field)

At this moment interval methods are not able to take into account more complicated types of dependency.

Main problem:
How to get probabilistic characteristics (e.g. μ, Σ)?

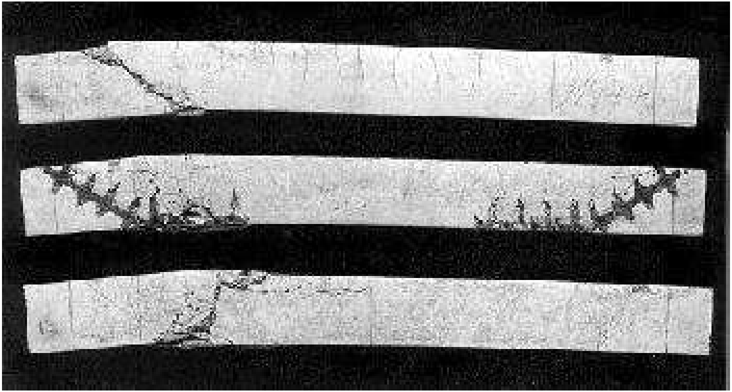

Figure: Concrete beams with cracks

Limitations of probabilistic methods

Elishakoff I., Possible Limitations of Probabilistic Methods in Engineering, ASME. Applied Mechanics Reviews, Vol.53, pp 19-36, 2000

- Lack of probabilistic data (because there is no time and money for collecting these data).
- Controversy related to likelihood interpretation of reliability and safety.
- **Some researchers claim that probability doesn't exist (pure random object doesn't exist).**
- In many cases the problems are unique (particularly civil engineering applications) and it is hard to get reliable probabilistic data.
- In some cases data are unavailable because it is very hard to get the information about the values of particular parameter (e.g. material parameters of soil 2000 m under ground level).
- etc.

Safety factors

- Semi-probabilistic methods. Reliability index

$$\beta = -\Phi^{-1}(P_f) \tag{1}$$

Calibration of partial safety factors

$$\min_{\gamma} W(\gamma, \beta) \tag{2}$$

where W is some penalty function.

- Non-probabilistic definition

$$\gamma = \frac{x^{max}}{x^{design}} \tag{3}$$

where x^{design} is a design value, x^{max} is characteristic value.

Simplest case of worst case analysis: interval parameters

- $\mathbf{p} = [p^-, p^+]$ or $\mathbf{p} = [p_1^-, p_1^+] \times [p_2^-, p_2^+] \times \ldots \times [p_m^-, p_m^+]$.
- Solution set of equations with interval parameters

$$u(\mathbf{p}) = \{u : F(u, p) = Q(p), p \in \mathbf{p}\} \tag{4}$$

  or

$$\Box u(\mathbf{p}) = \Box\{u : F(u, p) = Q(p), p \in \mathbf{p}\} \tag{5}$$

  where $\Box u(\mathbf{p})$ is the smallest set which contain the set $u(\mathbf{p})$. Above definition us valid also in the case of differential and integral equation.
- In particular case we have system of linear equation with interval parameters.

$$\Box u(\mathbf{p}) = \Box\{u : K(p)u = Q(p), p \in \mathbf{p}\} \tag{6}$$

Convex model of uncertainty

- Ben-Haim, Y., and Elishakoff, I. (1990). Convex models of uncertainty in applied mechanics, Elsevier, New York.
- Ellipsoidal uncertainty

$$\tilde{p} = \left\{ (p_1, p_2) : \frac{p_1^2}{a^2} + \frac{p_2^2}{b^2} \leqslant 1 \right\} \tag{7}$$

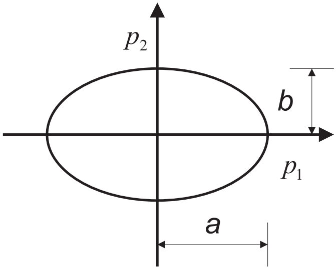

Figure: Ellipsoidal uncertainty

Solution of the interval equations using endpoint combination method

- Let us consider interval equation

$$f(u, p) = 0, \quad \textit{or equivalently} \quad u = u(p) \tag{8}$$

- Additionally lets assume that $u = u(p)$ is monotone, then

$$\underline{u} = min\{u(\underline{p}), u(\overline{p})\}, \quad \overline{u} = max\{u(\underline{p}), u(\overline{p})\} \tag{9}$$

In multidimensional case in order to find the solution we have to solve 2^m (where m is a number of uncertain parameters).

$$\underline{u} = \min\{u(p_1^{\pm}, p_2^{\pm}, ..., p_m^{\pm})\} \tag{10}$$

$$\overline{u} = \max\{u(p_1^{\pm}, p_2^{\pm}, ..., p_m^{\pm})\} \tag{11}$$

Taylor expansion method (first order)

- Let us consider interval equation

$$f(u,p) = 0, \quad \textit{or equivalently} \quad u = u(p) \tag{12}$$

- Function $u = u(p)$ can be approximated using Taylor expansion

$$u(p) = u(p_0) + \sum_i \frac{\partial u(p_0)}{\partial p_i}(p_i - p_{i0}) \tag{13}$$

$$\underline{u} = u(p_0) + \sum_i \left|\frac{\partial u(p_0)}{\partial p_i}\right|(\underline{p}_i - p_{i0}) \tag{14}$$

$$\overline{u} = u(p_0) + \sum_i \left|\frac{\partial u(p_0)}{\partial p_i}\right|(\overline{p}_i - p_{i0}) \tag{15}$$

Extreme values of monotone function

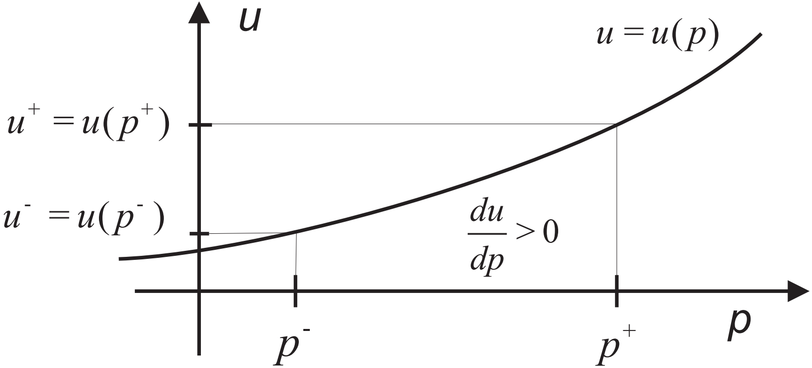

Figure: Extreme values of a monotone function $u = u(p)$ can be calculated by using upper and lower bounds of the parameters i.e. $p^-, p^+ \in R$.

Sensitivity analysis

$$u = u(p), \quad p \in [\underline{p}, \overline{p}] \tag{16}$$

$$\frac{du(p)}{dp} \geqslant 0, \quad for \quad p \in [\underline{p}, \overline{p}] \tag{17}$$

$$\underline{u} = u(\underline{p}), \quad \overline{u} = u(\overline{p}) \tag{18}$$

$$\frac{du(p)}{dp} < 0, \quad for \quad p \in [\underline{p}, \overline{p}] \tag{19}$$

$$\underline{u} = u(\overline{p}), \quad \overline{u} = u(\underline{p}) \tag{20}$$

Interval functional parameters

- Uncertain Young modulus

$$E(x) \in \mathbf{E}(x) = [E^{-}(x), E^{+}(x)] \tag{21}$$

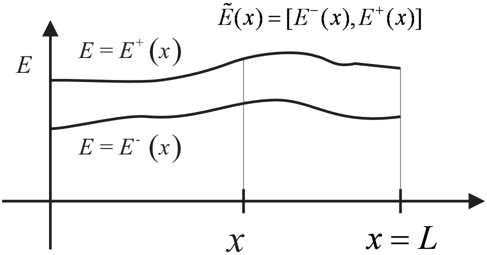

Figure: Set-valued Young modulus

Solution of equation with functional parameters

$$u(\mathbf{p}) = \{u : F(u, p) = 0, p(x) \in \mathbf{p}(x)\}, \tag{22}$$

$$\mathbf{u}(\mathbf{p}) = \square u(\mathbf{p}) = \square\{u : F(u, p) = 0, p(x) \in \mathbf{p}(x)\}. \tag{23}$$

General concept of monotonicity

A map $T : X \to Y$ is monotone if $(X, \geqslant)$ is a partially ordered set and $x, y \in X$, $x \geqslant y \Rightarrow T(x) \geqslant T(y)$.

A partial order is a binary relation "$\leqslant$" over a set P which is reflexive, antisymmetric, and transitive, i.e., for all a, b, and c in P, we have that:
$a \leqslant a$ (reflexivity);
if $a \leqslant b$ and $b \leqslant a$ then $a = b$ (antisymmetry);
if $a \leqslant b$ and $b \leqslant c$ then $a \leqslant c$ (transitivity).

Sensitivity analysis: functional parameters case

$$\boxed{u(p_0 + \delta p) - u(p_0) \approx \delta u(p_0, \delta p)} \tag{24}$$

Algorithm 1 General sensitivity analysis with functional parameters

1. **if** $\delta u(p, \delta p) \geqslant 0$ **then** $p^{min} = \underline{p}$, $p^{max} = \overline{p}$.
2. **if** $\delta u(p, \delta p) < 0$ **then** $p^{min} = \overline{p}$, $p^{max} = \underline{p}$.
3. $\underline{u} = u(p^{min})$, $\overline{u} = u(p^{max})$.

Sensitivity analysis: functional parameters case

$$u(p) = \int_{\Omega} L(x, p(x))dx \tag{25}$$

$$\delta u(p_0, \delta p) = \int_{\Omega} \frac{\delta u(p)}{\delta p(x)} \delta p(x) dx = \int_{\Omega} \frac{\partial L(x, p(x))}{\partial p(x)} \delta p(x) dx \tag{26}$$

Theorem If $\frac{\delta u(p)}{\delta p(x)} \geqslant 0$ for $p \in [\underline{p}, \overline{p}] \subset X$, then the function $u = u(p)$ is monotone in the interval $\mathbf{p}$.

Algorithm 2 Sensitivity analysis based on functional derivative

1. **if** $\frac{\delta u}{\delta p(x)} \geqslant 0$ **then** $p^{min} = \underline{p}$, $p^{max} = \overline{p}$.
2. **if** $\frac{\delta u}{\delta p(x)} < 0$ **then** $p^{min} = \overline{p}$, $p^{max} = \underline{p}$.
3. $\underline{u} = u(p^{min})$, $\overline{u} = u(p^{max})$.

Non-monotone case, continuous gradient method

$$u(p_0 + \delta p) - u(p_0) \approx \delta u(p_0, \delta p) = \int\limits_{\Omega} \frac{\delta u(p)}{\delta p(x)} \delta p(x) dx \geqslant 0 \tag{27}$$

$$\delta p^u(x) = \lambda(x) \frac{\delta u(p_0)}{\delta p(x)} \tag{28}$$

$$\delta p^l(x) = -\lambda(x) \frac{\delta u(p_0)}{\delta p(x)} \tag{29}$$

$$u(p_0 + \delta p^u) \geqslant u(p_0) \tag{30}$$

$$u(p_0 + \delta p^l) \leqslant u(p_0) \tag{31}$$

Non-monotone case, continuous gradient method

Algorithm 3 Calculation of upper bound $\overline{u}$

1. $p(x) = p_0(x)$
2. choose the function $\lambda(x)$
3. $\delta p^u(x) = \lambda(x)\frac{\delta u(p)}{\delta p(x)}$
4. $p_{old}(x) = p(x)$
5. $p(x) := p(x) + \delta p^u(x)$
6. **if** $p(x) > \overline{p}(x)$ **then** $p(x) = \overline{p}(x)$
7. **if** $p(x) < \underline{p}(x)$ **then** $p(x = \underline{p}(x)$
8. **if** $\|p_{old} - p\| > \varepsilon$ **then** goto step 2
9. $\overline{u} = u(p)$
10. **stop**

Sensitivity with respect of changes of the region of integration

$$u(\Omega) = \int\limits_{\Omega} L(x)dx \tag{32}$$

$$u(\Omega+\Delta\Omega) - u(\Omega) = \int\limits_{\Omega+\Delta\Omega} L(x)dx - \int\limits_{\Omega} L(x)dx = \int\limits_{\Delta\Omega} L(x)dx \tag{33}$$

$$\int\limits_{\Delta\Omega} L(x)dx = |\Delta\Omega| L(x^*) \tag{34}$$

$$\frac{u(\Omega+\Delta\Omega) - u(\Omega)}{|\Delta\Omega|} = L(x^*) \tag{35}$$

$$\frac{\delta u}{\delta\Omega(x)} = \lim_{|\Delta\Omega(x)|\to 0} \frac{u(\Omega+\Delta\Omega(x)) - u(\Omega)}{|\Delta\Omega(x)|} = L(x). \tag{36}$$

Sensitivity with respect of changes of the region of integration

The inclusion $\subset$ can be treat as partial order relation $\geqslant$. Because of that it is possible to take into account "set intervals"

$$[\underline{\Omega}, \overline{\Omega}] = \{\Omega : \Omega \subset \Omega \subset \overline{\Omega}\}. \tag{37}$$

Algorithm 4 Sensitivity analysis based on functional derivative

1. **if** $\frac{\delta u}{\delta \Omega(x)} \geqslant 0$ for $x \in \overline{\Omega} - \underline{\Omega}$ **then** $\Omega^{max} = \overline{\Omega}$, $\Omega^{min} = \underline{\Omega}$.
2. **if** $\frac{\delta u}{\delta \Omega(x)} < 0$ for $x \in \overline{\Omega} - \underline{\Omega}$ **then** $\Omega^{max} = \underline{\Omega}$, $\Omega^{min} = \overline{\Omega}$.
3. $\underline{u} = u(\Omega^{min})$, $\overline{u} = u(\Omega^{max})$.

Conclusions

- Using functional derivative it is possible to find solution of equation with uncertain functional parameters.
- The method can be applied to solution of large class of engineering problems with uncertain filed.
- The method can be applied to solution of linear and nonlinear problems of computational mechanics with uncertain filed.
- The algorithm of sensitivity analysis method method can be parallel.