Computing on Sets – **Bounds on the Local Error in Numerical Solutions of Partial Differential Equations**

Robert L. Mullen
Frank H. Neff Professor of Civil Engineering
Case Western Reserve University
Cleveland, Ohio

Center for Reliable Engineering Computing (REC)

We handle computations with care

CASE SCHOOL OF ENGINEERING

REC

Georgia Institute of Technology

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Imprimers

- Ivo Babushka
- Vladik Kreinovich
- Ray Moore
- A. Neumier

Why do engineers calculate?

- To gain information useful to design

- Information –

$$I = \log\left[\frac{\text{Probability of event after message received}}{\text{Probability of event before message received}}\right]$$

Definition from Stanford Goldman “Information Theory” Prentice-Hall 1953.

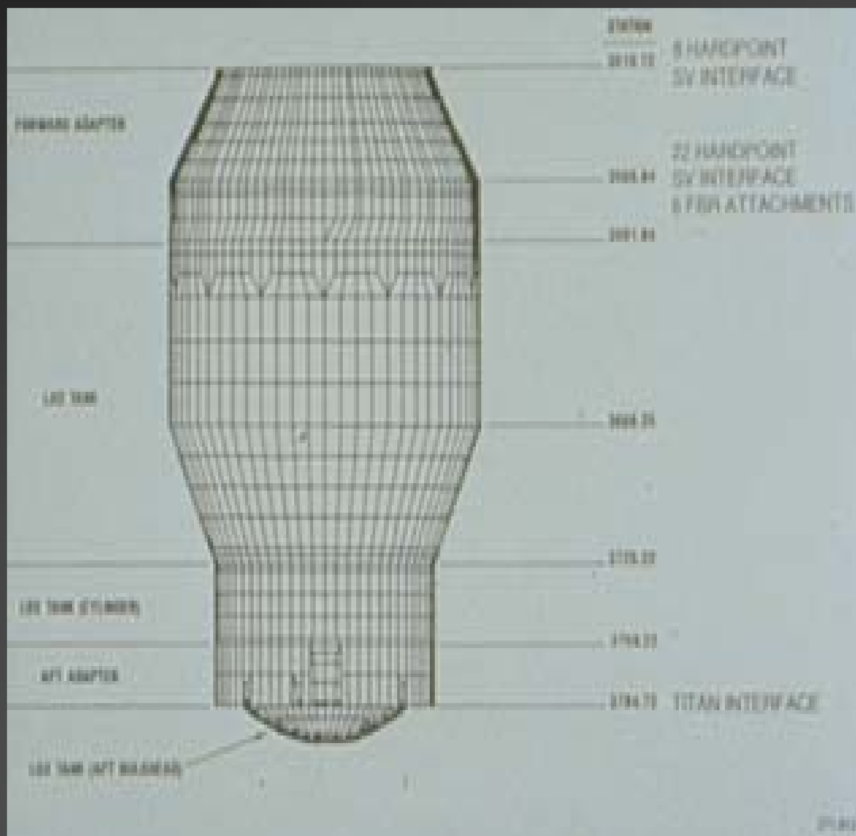
Why do engineers calculate?

- To gain information useful to design

- Information –

$$I = \log\left[\frac{\text{Uncertainty in predicting system behavior after calulation}}{\text{Uncertainty in predicting system behavior before calculation}}\right]$$

What information is provided by a finite element analysis?



What is the probability that the n^{th} mode of the Centar is .324571447 Hz?

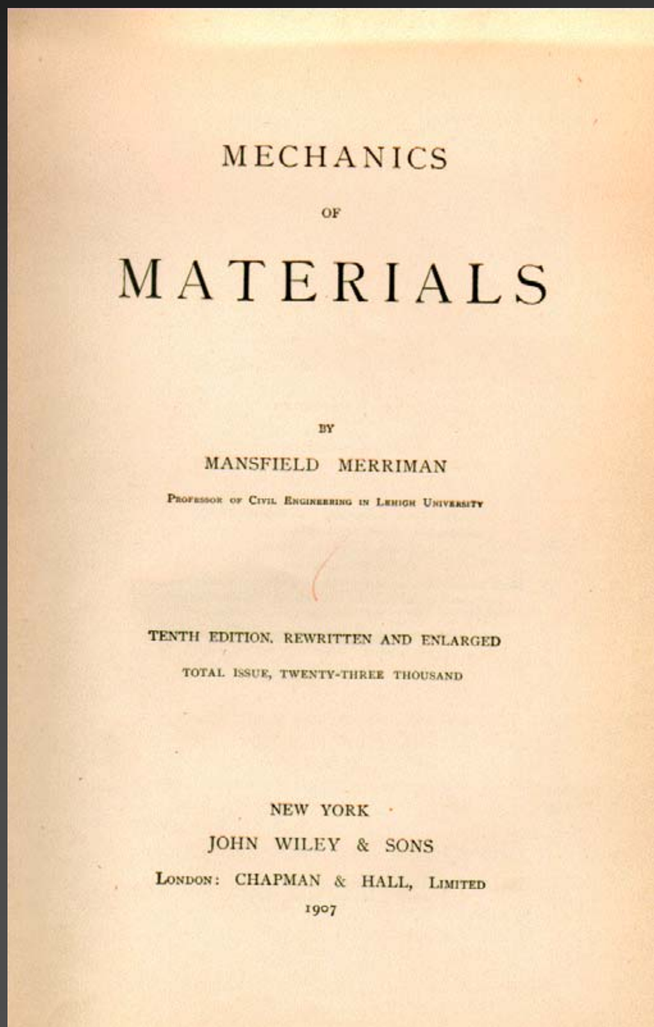
Information provided by FEM

$$I = \log\left[\frac{\text{Probability of event after calulation}}{\text{Probability of event before calculation}}\right]$$

$$I = \log\left[\frac{p}{p}\right]$$

I = ? or I = 0 or I = - ∞

Historical Considerations


MECHANICS

OF

MATERIALS

BY

MANSFIELD MERRIMAN

Professor of Civil Engineering in Lehigh University

TENTH EDITION, REWRITTEN AND ENLARGED

TOTAL ISSUE, TWENTY-THREE THOUSAND

NEW YORK

JOHN WILEY & SONS

London: CHAPMAN & HALL, Limited

1907

.30 and .3 are not the same answer

On Computations

- Numerical computations required in the mechanics of materials should be performed so the the precision of the results fairly corresponds with the precision of the data.
- The values of the ultimate strength given in the preceding tables are indefinite in the second significant figure.

Significant Figures

- Represent information as well as limits to knowledge.
- Reduced computational cost.
- Better representation of limits to knowledge is the use of probability but how does one pick a probability density function?

Modern Engineering Computations

- Significant figures is 1930s technology.

- What should we be doing to address the reliability of engineering computations in the future?

Sources of unreliability in engineering analysis

- Errors in model representing the physics
- Errors in parameters used in model
- Errors in discretization of a model
- Errors in computations (ie. truncation errors)

Impact of uncertainty in engineering analysis

- A Patriot Missile Killed 28 and Injured >100 (GAO/IMTEC-92-26)

- The failure of the Sleipner A offshore platform, August 23, 1991

Impact of uncertainty in engineering analysis

- The Patriot battery at Dhahran failed to track and intercept the Scud missile because of a software problem in the system's weapons control computer. This problem led to an inaccurate tracking calculation that became worse the longer the system operated. At the time of the incident, the battery had been operating continuously for over 100 hours. By then, the inaccuracy was serious enough to cause the system to look in the wrong place for the incoming Scud.

Impact – Sleipner Failure

- The conclusion of the investigation was that the loss was caused by a failure in a cell wall, resulting in a serious crack and a leakage that the pumps were not able to cope with. The wall failed as a result of a combination of a serious error in the finite element analysis and insufficient anchorage of the reinforcement in a critical zone.

Impact – Sleipner Failure

the failure caused a seismic event registering 3.0 on the Richter scale, and left nothing but a pile of debris at 220m of depth. The failure involved a total economic loss of about $700 million.

More examples

- From Ivo Babuška and R. Tempone Proceedings of REC-2006

The Columbia Space Shuttle accident caused by a piece of foam broken off the fuel tank. After the hit was observed the potential damage was computationally judged non-serious.

Reason: The used model was based on the effect of small meteorites and not on a large piece of foam.

Loss of Mars Climate Orbiter:

Reason: Unintended mixture of English and metric units.

Even more examples

Tacoma Narrows Bridge: The first Suspension Bridge Across Puget-Sound (Washington State) collapsed Nov. 7th1940.

Reason: Incorrect Model –Not respecting aerodynamic forces. (Effect of von Kármánvortices)

Collapsed roof of The Hartford Civic Center Jan. 18th1978.

Reason: Linear model and model of the joints was not adequate.

Collapsed roof in Katowice(Poland) Jan. 28th2006, 65 dead.

Reason: Design was not adjusted for heavy snow and avoiding total collapse.

- Also see the work of Norm Delatte and Paul Bosela - Engineering Forensics (NSF workshop CSU June 2005)

GOAL

- Develop tools to allow uncertainty in an engineering analysis to be computed with:
- 1) Limited Information on defining parameters
- 2) Computationally efficient (not solutions to Kolmorgorov type equations).
- 3) Integrate Verification and Validation errors.

Outline

- **COMPUTING ON SETS: Random Sets, Fuzzy Sets, Convex Sets, Clouds, Probability Boxes**
- Interval Operations
- Examples of progress in Reliable Engineering Analysis
- Validation Errors:

  Errors in representing the physics

  Errors in parameters used in model

  Interval Finite Elements
- Verification Errors

  Errors in discretization of model

  Errors in computations (truncation errors)

  Interval Boundary Elements
- Conclusions

Computation on sets

Probability theory.
(measure on Borel sets)

Fuzzy Sets
(membership function)

Random Sets - Rough sets
Convex Sets (strict Zermelo-Fraenkel base theory)

Computing on sets

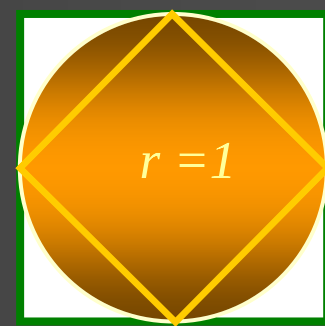
- Closed sets on real numbers - Interval Sets

- Interval number represents a range of possible values within a closed set

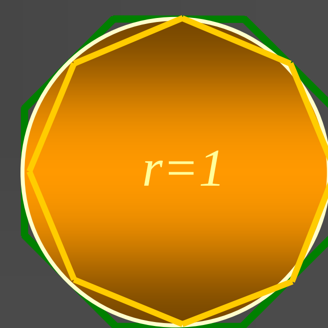
$$x \equiv [x^l, x^u] := \{\tilde{x} \in R \mid x^l \leq \tilde{x} \leq x^u\}$$

Interval arithmetic – Background

- Archimedes (287 – 212 B.C.)
  - A circle of radius one has an area equal to π
  - $2 < \pi < 4$



$$3\frac{10}{71} < \pi < 3\frac{1}{7}$$



$\pi = [3.14085, 3.14286]$

Interval arithmetic – Background

- Modern interval arithmetic
  - Physical constants or measurements

    $g \in [9.8045, 9.8082]$

  - Representation of numbers

    $1/3 \approx 0.33333$ $\quad \sqrt{2} \approx 1.4142$ $\quad \pi \approx 3.1416$

    $1/3 \in [0.33333, 0.33334]$ $\quad \sqrt{2} \in [1.4142, 1.4143]$ $\quad \pi \in [3.1414, 3.1416]$

  - Rounding errors

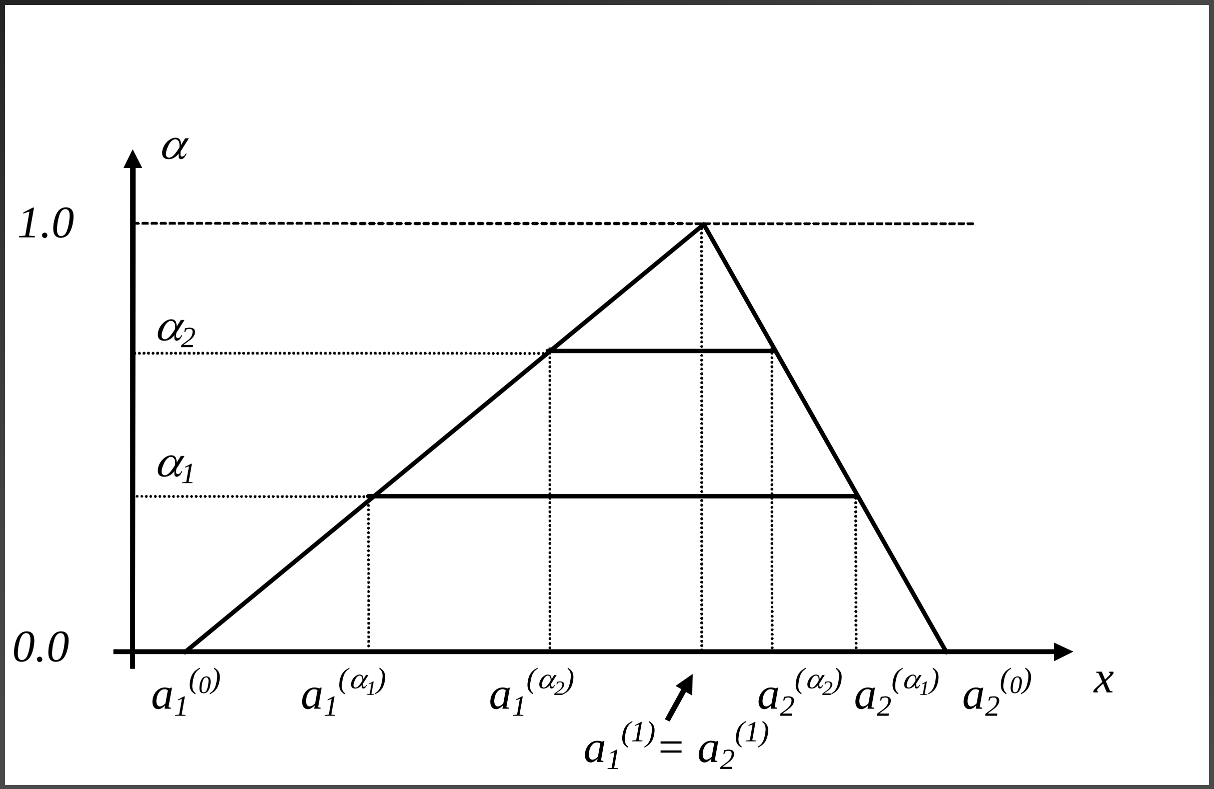
    $1/0.12345 \approx 8.1004$ $\quad 1/0.12345 \in [8.1004, 8.1005]$

R. E. Moore, E. Hansen , A. Neumaier, G. Alefed, J. Herzberger

Intervals as a Measure of Uncertainty

- Interval construction of fuzzy set membership functions
- Convex Sets
- Intervals as a set of all bounded probability density functions
- Intervals as a set of all bounded, symmetric, non- increasing from mid-point Probability density functions (Lagoa and Barmish 2001)

Intervals as α-cut for fuzzy set membership functions



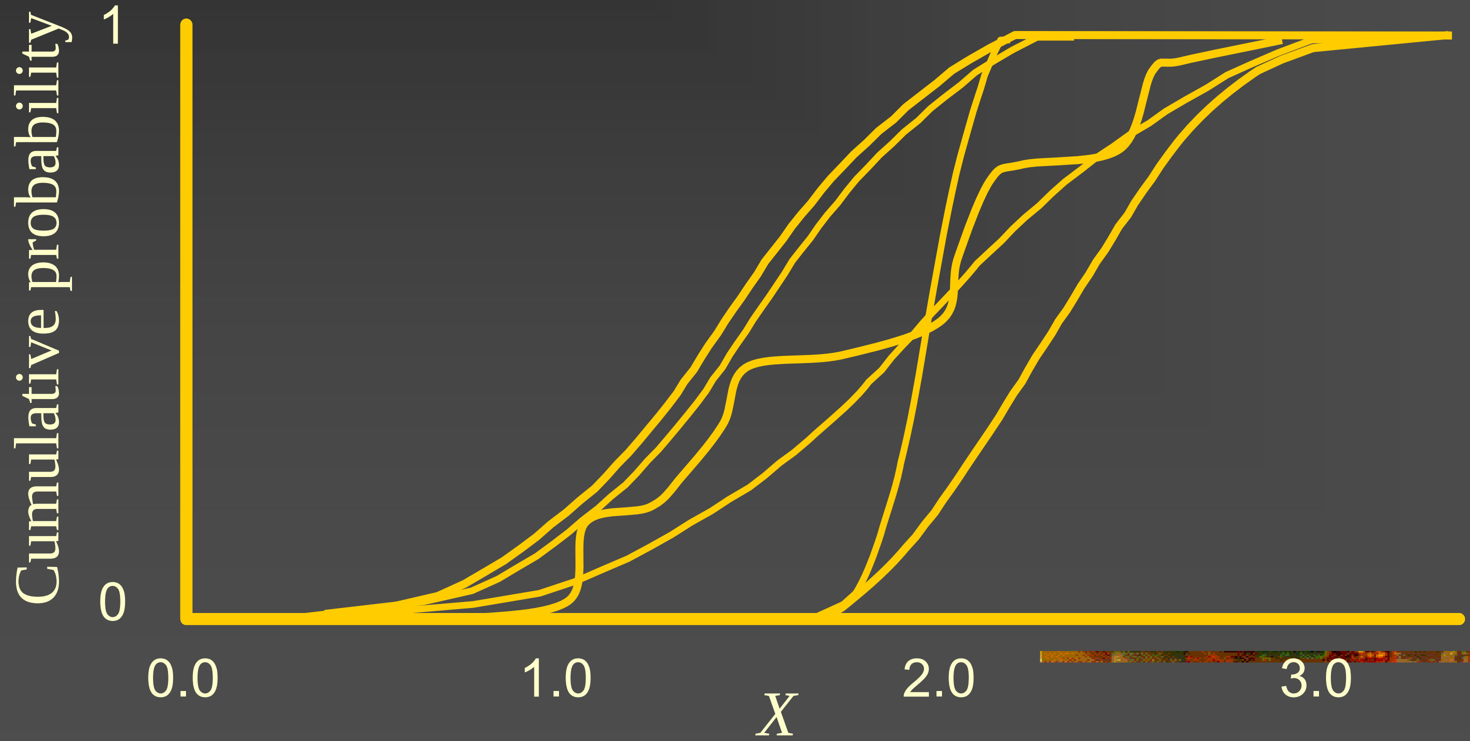
Intervals as a Measure of Uncertainty (2)

- Dempster-Shafer evidence theory
- Random set theory
- Probability Boxes (P-Box) analysis

Computing on Sets

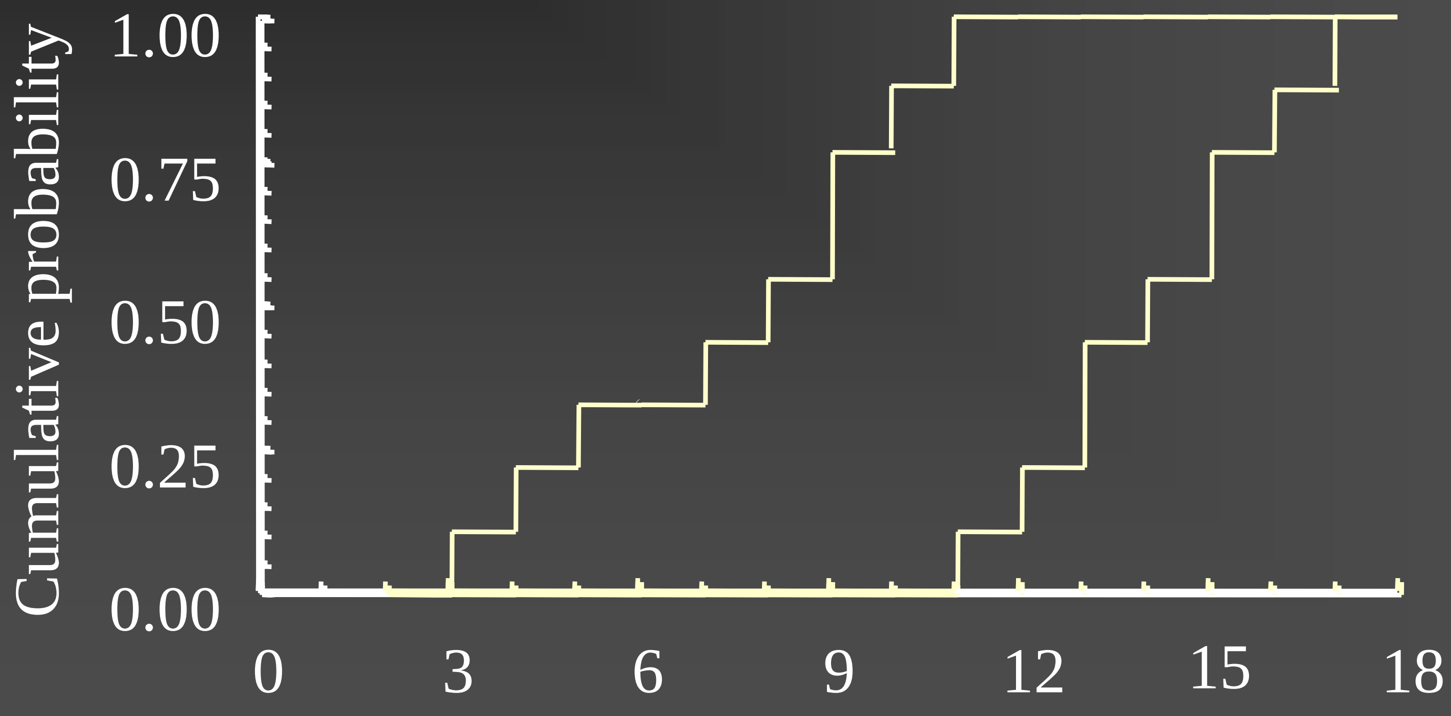
- Imprecise Probabilities

Figure from Scott Ferson and Vladik Kreinovich-REC 2006



Interval Representation of P-boxes

Interval bounds on an cumulative distribution function



Scott Ferson and Vladik Kreinovich-REC 2006

Computational Justification

- Conventional probability requires a function for each uncertain variables. (Problems grow in dimension). Current record for solution of White noise loading on a structure system with limit states is 5 nodes. (see work of Larry Bergman at Illinois on Fokker-Plank equation)

Wojtkiewicz, Johnson, Bergman, Grigoriu Spencer

Outline

- COMPUTING ON SETS: Random Sets, Fuzzy Sets, Sets, Convex Sets, Clouds
- **Interval Operations**
- Examples of progress in Reliable Engineering Analysis
- Validation Errors:

  Errors in representing the physics

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  Interval Finite Elements
- Verification Errors

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  Interval Boundary Elements
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Interval Representation

- Interval number represents a range of possible values within a closed set

$$x \equiv [x^l, x^u] := \{\tilde{x} \in R \mid x^l \leq \tilde{x} \leq x^u\}$$

- Uncertainty can be represented as interval, if the length of a body $l = 10 \pm 0.25$ the interval representation is

$$\boldsymbol{l} = [9.75, 10.25]$$

Interval vectors and matrices

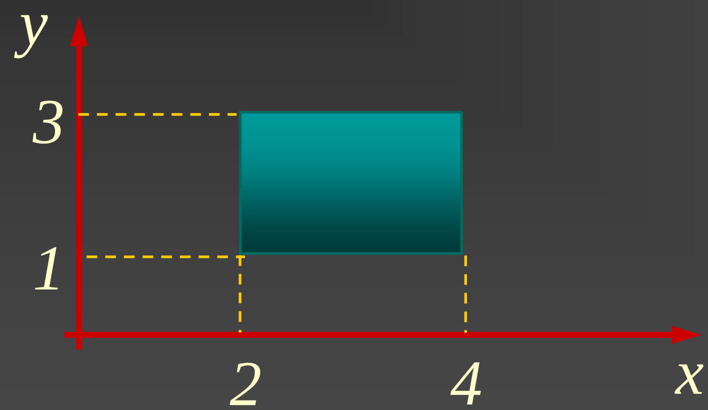
- Interval Vectors and Matrices

➢ An interval matrix is such matrix that contains all real matrices whose elements are obtained from all possible values between the lower and upper bounds of its interval components

$$A = \{\tilde{A} \in R^{m \times n} \mid \tilde{A}_{ij} \in A_{ij} \ \text{for}\ i = 1,\ldots,m;\ \ j = 1,\ldots,n\}$$

Interval Representation - Vectors

$$A(\boldsymbol{x}, \boldsymbol{y}) = A([2,4],[1,3])$$



Interval

Interval Operations

Let $x = [a, b]$ and $y = [c, d]$ be two interval numbers

1. Addition

$$x + y = [a, b] + [c, d] = [a + c, b + d]$$

2. Subtraction

$$x - y = [a, b] - [c, d] = [a - d, b - c]$$

3. Multiplication

$$xy = [\min(ac, ad, bc, bd), \max(ac, ad, bc, bd)]$$

4. Division

$$1 / x = [1/b, 1/a]$$

Properties of Interval Arithmetic

Let x, y and z be interval numbers

1. Commutative Law

$$x + y = y + x$$

$$xy = yx$$

2. Associative Law

$$x + (y + z) = (x + y) + z$$

$$x(yz) = (xy)z$$

3. ***Distributive Law does not always hold,*** *but*

$$x(y + z) \subseteq xy + xz$$

Properties of Interval Arithmetic

- Interval arithmetic is an elegant tool for practical work with inequalities, approximate numbers, error bounds, and more generally with certain convex and bounded sets.

  - ***BUT it should be applied with care!***

Sharp Results – Overestimation

- If a, b and c are interval numbers, then:

$a\,(b \pm c) \subseteq ab \pm ac$

- If we set

$a = [-2, 2]$; $b = [1, 2]$; $c = [-2, 1]$, we get

$a\,(b + c) = [-2, 2]([1, 2] + [-2, 1]) = [-2, 2]\,[-1, 3] = [-6, 6]$

- However,

$ab + ac = [-2, 2][1, 2] + [-2, 2][-2, 1] = [-4, 4] + [-4, 4] = [-8, 8]$

Sharp Results – Overestimation

- The ***DEPENDENCY*** problem arises when one or several variables occur more than once in an interval expression

$$f(x) = x - x, \qquad x = [1, 2]$$

$$f(x) = [1 - 2, 2 - 1] = [-1, 1] \neq 0$$

$$\cancel{f(x, y) = \{ f(\tilde{x}, \tilde{y}) = \tilde{x} - \tilde{y} \mid \tilde{x} \in x, \tilde{y} \in y \}}$$

$$f(x) = x(1 - 1) \quad \Rightarrow \quad f(x) = 0$$

$$f(x) = \{ f(\tilde{x}) = \tilde{x} - \tilde{x} \mid \tilde{x} \in x \}$$

Sharp Results – Overestimation

- Let ***a, b, c*** and ***d*** be independent variables, each with interval [*1, 3*]

$$\tilde{A} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}, \quad A \times B = \begin{pmatrix} [-2,2] & [-2,2] \\ [-2,2] & [-2,2] \end{pmatrix}$$

$$\tilde{A} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad B_{phys} = \begin{pmatrix} b & -b \\ -b & b \end{pmatrix}, \quad A \times B_{phys} = \begin{pmatrix} [b-b] & [b-b] \\ [b-b] & [b-b] \end{pmatrix}$$

$$\tilde{A} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad B^{*}_{phys} = b \times \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad A \times B^{*}_{phys} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Interval methods

- One of the best and worst events in the history of interval methods was the release of interval libraries and the compiler extension (FORTRAN, C, Java) allowing an interval variable type.
- Naive use of interval methods results in catastrophic over estimation of the width of interval bounds.
- “I tried intervals and they are useless”

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Interval Finite Elements

- Follows conventional FEM
- Loads, nodal geometry and element materials are expressed as interval quantities
- Integration of stiffness and load requires interval integration

Interval Finite Elements

- Interval Linear System of Equations

$A\,x = b$

$$\begin{pmatrix} 2 & [-1,0] \\ [-1,0] & 2 \end{pmatrix} \times \begin{pmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{pmatrix} = \begin{pmatrix} 1.2 \\ -1.2 \end{pmatrix}$$

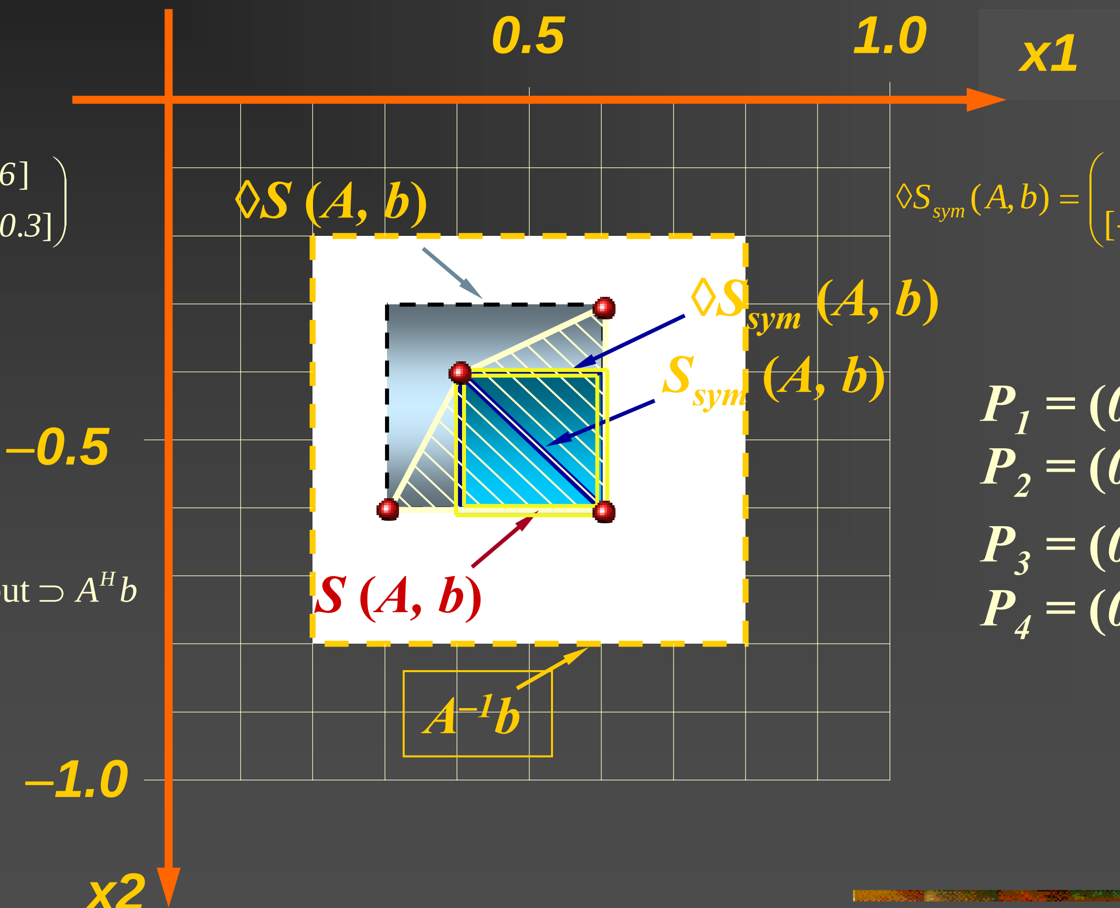
$$Then \quad \tilde{A} \in A \quad iff$$

$$A := \begin{pmatrix} 2 & -\alpha \\ -\beta & 2 \end{pmatrix} \qquad \text{with } \alpha, \beta \in [0,1]$$

Interval Finite Elements

$$A^H b = \lozenge S(A,b) = \begin{pmatrix} [0.3, 0.6] \\ [-0.6, -0.3] \end{pmatrix}$$

$$A^{-1} b = \begin{pmatrix} [0.2, 0.8] \\ [-0.8, -0.2] \end{pmatrix} \neq \text{ but} \supset A^H b$$



$$\lozenge S_{sym}(A,b) = \begin{pmatrix} [0.4, 0.6] \\ [-0.6, -0.4] \end{pmatrix} \neq A^H b$$

$$P_1 = (0.3, -0.6)$$
$$P_2 = (0.6, -0.6)$$
$$P_3 = (0.6, -0.3)$$
$$P_4 = (0.4, -0.4)$$

Computational Complexity

- The general problem of an exact hull of the solution of an interval system of equations is NP hard (Sahni 1974 or Kreinovich et. al. 1997)
- All combinations of upper and lower bounds with a parametric system may not yield worst-case bounds.
- Goal – provide sharp outer bounds to hull in polynomial time.

Finite Element – Sharp Results

1. Load Dependency
2. Stiffness Dependency

Finite Element – Sharp Results

1. Load Dependency

$$P_b = \sum L^T \int_l N^T b(x)\, dx$$

The global load vector P_b can be written as

$$P_b = M\,F$$

where F is the vector of interval coefficients of the load approximating polynomial

Finite Element – Sharp Results

The sharp solution for the interval displacement can be written as:

$$U = (K^{-1}\ M)\ F$$

Thus all non-interval values are multiplied first, the last multiplication involves the interval quantities

- ***If this order is not maintained, the resulting interval solution will not be sharp***

Finite Element – Sharp Results

2. Stiffness Dependency

   - Element level
   - Coupling (assemblage process)

Finite Element – Sharp Results

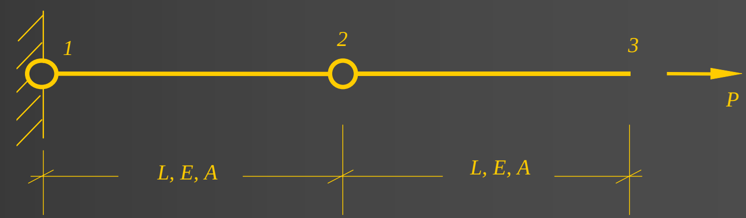
- Element Level

$$k = \begin{pmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{pmatrix}$$

$$k_{phys} = k_1 \times \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

Finite Element – Sharp Results

- Coupling



$$\boldsymbol{k} = \begin{pmatrix} \boldsymbol{k}_1 + \boldsymbol{k}_2 & -\boldsymbol{k}_2 \\ -\boldsymbol{k}_2 & \boldsymbol{k}_2 \end{pmatrix}, \quad \boldsymbol{k}^{-1} = \begin{pmatrix} \dfrac{1}{\boldsymbol{k}_1} & \dfrac{1}{\boldsymbol{k}_1} \\ \dfrac{1}{\boldsymbol{k}_1} & \dfrac{\boldsymbol{k}_1 + \boldsymbol{k}_2}{\boldsymbol{k}_1 \boldsymbol{k}_2} \end{pmatrix}, \qquad \boldsymbol{p} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$u_2 = \frac{1}{k_1}, \quad u_3 = \frac{k_1 + k_2}{k_1 k_2} \qquad (over\ estimation\ in\ u_3, r_3 = 3r_{3-exact})$$

$$u_2 = \frac{1}{k_1}, \quad u_3 = \frac{1}{k_1} + \frac{1}{k_2} \qquad (exact\ solution)$$

Finite Element – Present Formulation

- In steady-state analysis-variational formulation

$$\Pi = \frac{1}{2}\boldsymbol{U}^T\boldsymbol{K}\boldsymbol{U} - \boldsymbol{U}^T\boldsymbol{P}$$

$$\boldsymbol{C}\boldsymbol{U} = \boldsymbol{V}$$

$$\Pi^* = \frac{1}{2}\boldsymbol{U}^T\boldsymbol{K}\boldsymbol{U} - \boldsymbol{U}^T\boldsymbol{P} + \lambda^T(\boldsymbol{C}\boldsymbol{U} - \boldsymbol{V})$$

- Invoking the stationarity of Π^*, that is $\delta\Pi^* = 0$

$$\begin{pmatrix} \boldsymbol{K} & \boldsymbol{C}^T \\ \boldsymbol{C} & 0 \end{pmatrix}\begin{pmatrix} \boldsymbol{U} \\ \lambda \end{pmatrix} = \begin{pmatrix} \boldsymbol{P} \\ \boldsymbol{V} \end{pmatrix}$$

Finite Element – Present Formulation

Element-by-Element

$$\tilde{C}U = 0$$

$$\begin{pmatrix} K & \tilde{C}^T \\ \tilde{C} & 0 \end{pmatrix}\begin{pmatrix} U \\ \lambda \end{pmatrix} = \begin{pmatrix} P \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{aligned} KU + \tilde{C}^T\lambda &= P \\ \tilde{C}U &= 0 \end{aligned}$$

1 ———— 2   3 ———— 4

$$K = \begin{pmatrix} \frac{E_1A_1}{L_1} & -\frac{E_1A_1}{L_1} & 0 & 0 \\ -\frac{E_1A_1}{L_1} & \frac{E_1A_1}{L_1} & 0 & 0 \\ 0 & 0 & \frac{E_2A_2}{L_2} & -\frac{E_2A_2}{L_2} \\ 0 & 0 & -\frac{E_2A_2}{L_2} & \frac{E_2A_2}{L_2} \end{pmatrix} \qquad \tilde{C}^T = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

Finite Element – Present Formulation

- Element-by-Element

$$\begin{aligned} D\tilde{S}U &= P - \tilde{C}^T\lambda \\ D\tilde{C}^T\tilde{C}U &= 0 \end{aligned} \quad \Rightarrow \quad D(\tilde{S}U + \tilde{C}^T\tilde{C}U) = (P - \tilde{C}^T\lambda)$$

$$D = \begin{pmatrix} E_1 & 0 & 0 & 0 \\ 0 & E_1 & 0 & 0 \\ 0 & 0 & E_2 & 0 \\ 0 & 0 & 0 & E_2 \end{pmatrix} \qquad \tilde{S} = \begin{pmatrix} \frac{A_1}{L_1} & -\frac{A_1}{L_1} & 0 & 0 \\ -\frac{A_1}{L_1} & \frac{A_1}{L_1} & 0 & 0 \\ 0 & 0 & \frac{A_2}{L_2} & -\frac{A_2}{L_2} \\ 0 & 0 & -\frac{A_2}{L_2} & \frac{A_2}{L_2} \end{pmatrix}$$

Finite Element – Present Formulation

Element-by-Element

$$D(\tilde{S}+\tilde{Q})U=(P-\tilde{C}^T\lambda) \quad \Longrightarrow \quad D\tilde{R}U=(P-\tilde{C}^T\lambda)$$

$$D\tilde{R}U=(P-\tilde{C}^T\lambda) \quad \Longrightarrow \quad U=\tilde{R}^{-1}D^{-1}(P-\tilde{C}^T\lambda)$$

$$Q=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Finite Element – Present Formulation

Element-by-Element

$$U = \tilde{R}^{-1} D^{-1} (P - \tilde{C}^T \lambda) \quad \Rightarrow \quad U = \tilde{R}^{-1} M \ \delta$$

$$M = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \qquad \delta = \begin{pmatrix} \dfrac{1}{E_1} \\ \dfrac{1}{E_2} \end{pmatrix}$$

Linear interval equation

Finding the solution of a linear system $Ax = b$

- Transform into fixed point equation $g(x) = x$

  $g(x) = x - R(Ax - b) = Rb + (I - RA)x$

  (R is nonsingular)

- From Brouwer's fixed point theorem, for some interval vector $\mathbf{x} \in \mathrm{IR}^n$

  $$Rb + (I - RA)x \in \mathbf{x} \quad \forall x \in \mathbf{x}$$

  implies $\quad \exists x \in \mathbf{x}: Ax = b$

Linear interval equation

Finding the enclosure of a linear interval equation $\mathbf{Ax} = \mathbf{b}$

- *If* $R\,\mathbf{b} - \boldsymbol{R}Ax_0 + (I - R\mathbf{A})\,\mathbf{x}^* \subseteq \mathrm{int}\,(\mathbf{x}^*)$

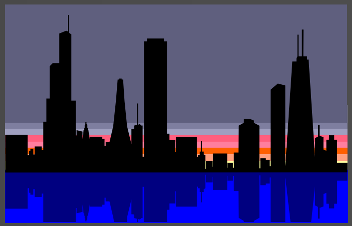
  then $\Sigma(\mathbf{A}, \mathbf{b}) \subseteq x_0 + \mathbf{x}^*$
- Fixed point Iteration
  - $\mathbf{x}^{*n+1} = \mathbf{z} + \mathbf{C}(\varepsilon\mathbf{x}^{*n})$ for $n = 0, 1, 2, \ldots$

    $\mathbf{z} = R\,\mathbf{b} - \boldsymbol{R}Ax_0$, $\mathbf{C} = I - R\mathbf{A}$, ε: inflation parameter
  - Stopping criteria: $\mathbf{x}^{*n+1} \subseteq \mathrm{int}(\mathbf{x}^{*n})$
  - Enclosure: $\Sigma(A, b) \subseteq x_0 + \mathbf{x}^{*n+1}$

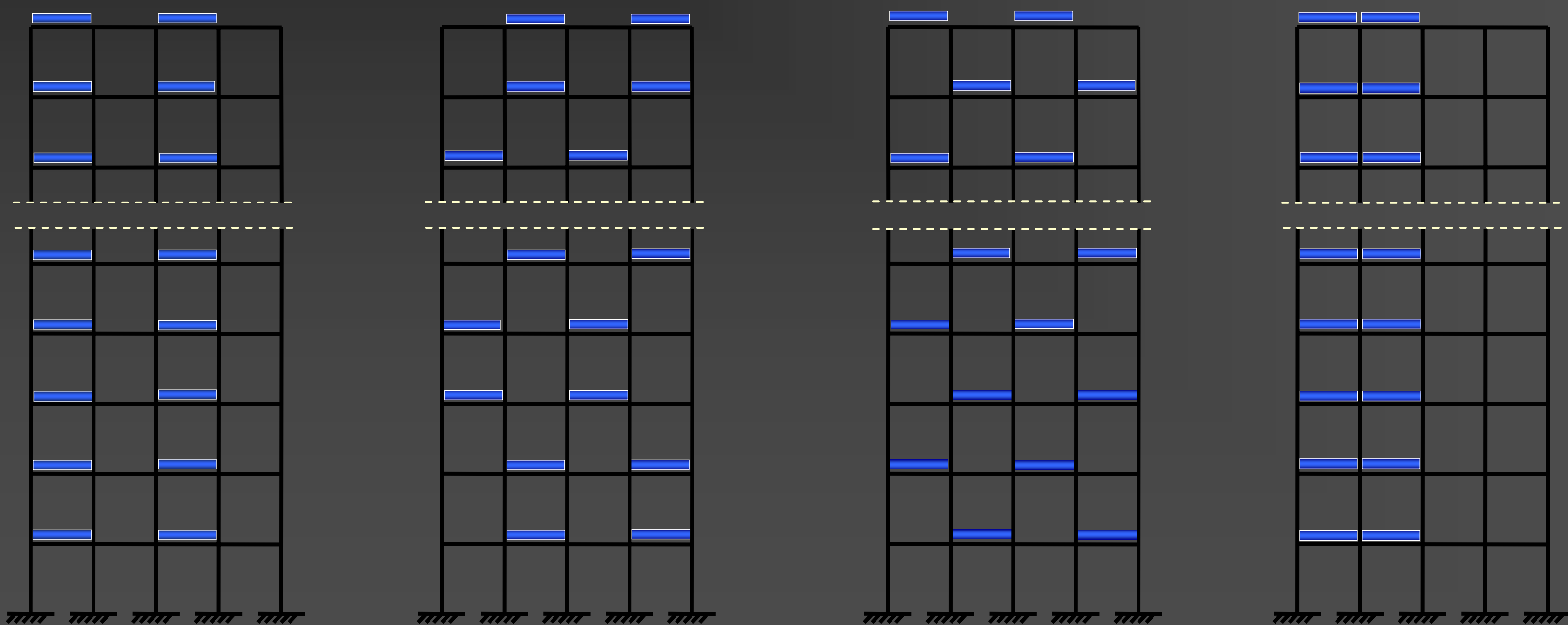
Examples

- Load uncertainty
  - Four-bay forty-story frame
  - Continuum
- Stiffness uncertainty
  - Two-bay truss
  - Four-bay truss

Examples – Load Uncertainty

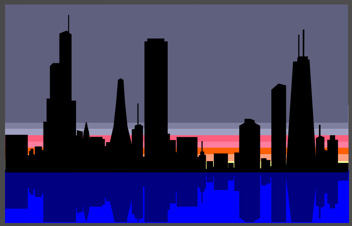


- **Four-bay forty-story frame**



Loading A **Loading B** **Loading C** **Loading D**

Examples – Load Uncertainty



- **Four-bay forty-story frame**

Total number of floor load patterns

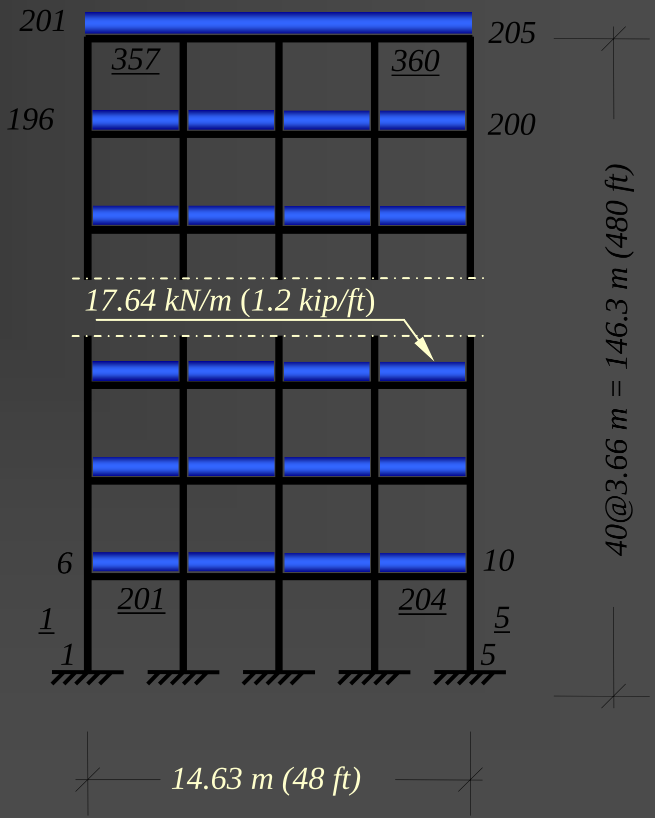
2^{160} = 1.46 x 10^{48}

If one were able to calculate

10,000 patterns / s

there has not been sufficient time since the creation of the universe ***(4-8)*** billion years ? to solve all load patterns for this simple structure using load combinations

Material *A36*, Beams *W24* x *55*, Columns *W14* x *398*



Examples – Load Uncertainty

- **Four-bay forty-story frame**

Four bay forty floor frame - Interval solutions for shear force and bending moment of first floor columns

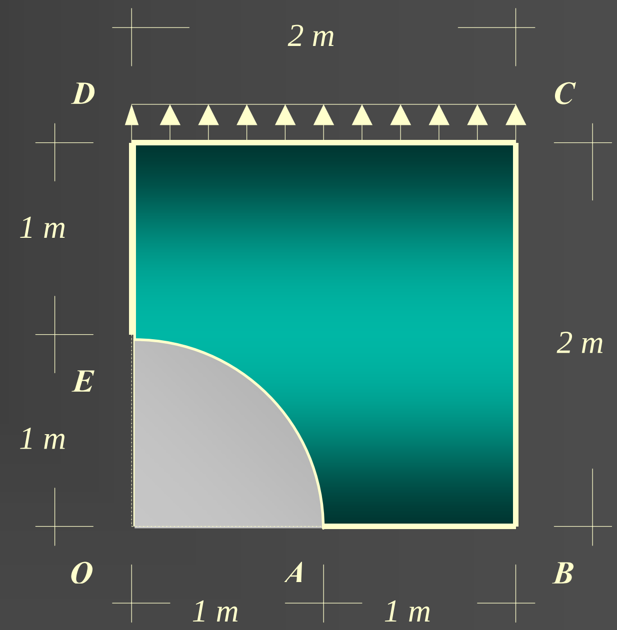
Elements		1		2		3	
Nodes		1	6	2	7	3	8
Combination solution		Total number of required		combinations = $1.461501637 \times 10^{48}$			
Interval	Axial force (kN)	[-2034.5, 185.7]		[-2161.7, 0.0]		[-2226.7, 0.0]	
solution	Shear force (kN)	[-5.1, 0.9]		[-5.8, 5.0]		[-5.0, 5.0]	
	Moment (kN m)	[-10.3, 4.5]	[-15.3, 5.4]	[-10.6, 9.3]	[-17, 15.2]	[-8.9, 8.9]	[-16, 16]

Structural loading

- Interval results always contain Monte Carlo simulations.
- Interval results match results from optimization formulations for an element within machine precision.
- Interval methods are computationally equivalent to influence line methods for structures.

Examples – Continuum Load Uncertainty

- **Square plate with opening**
  - Modulus of elasticity = 200 GPa
  - Poisson's ratio = 0.25
  - Load = [0, 6.9] kPa
  - 1600 linear displacement triangular elements

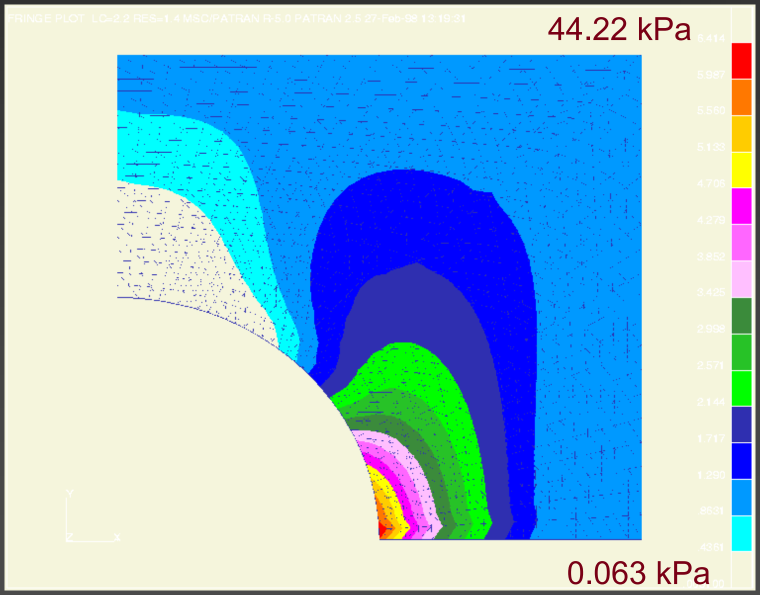


Selected displacements, load = [0, 6.9].kPa

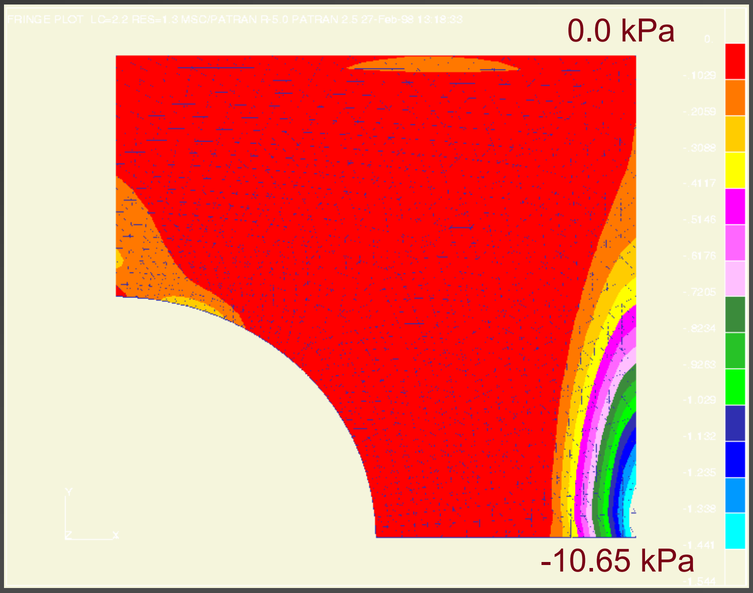
Point	$U_x \times 10^{-7}$ (mm)	$U_y \times 10^{-7}$ (mm)
A	[-42.2, 4.96]	[0.0, 0.0]
B	[-44.02, 3.82]	[0.0, 0.0]
C	[-5.90, 16.77]	[-9.97, 16.39]
D	[0.0, 0.0]	[-2.38, 637]
E	[0.0, 0.0]	[-2.51, 60.05]

Examples – Continuum Load Uncertainty

- **Square plate with opening**

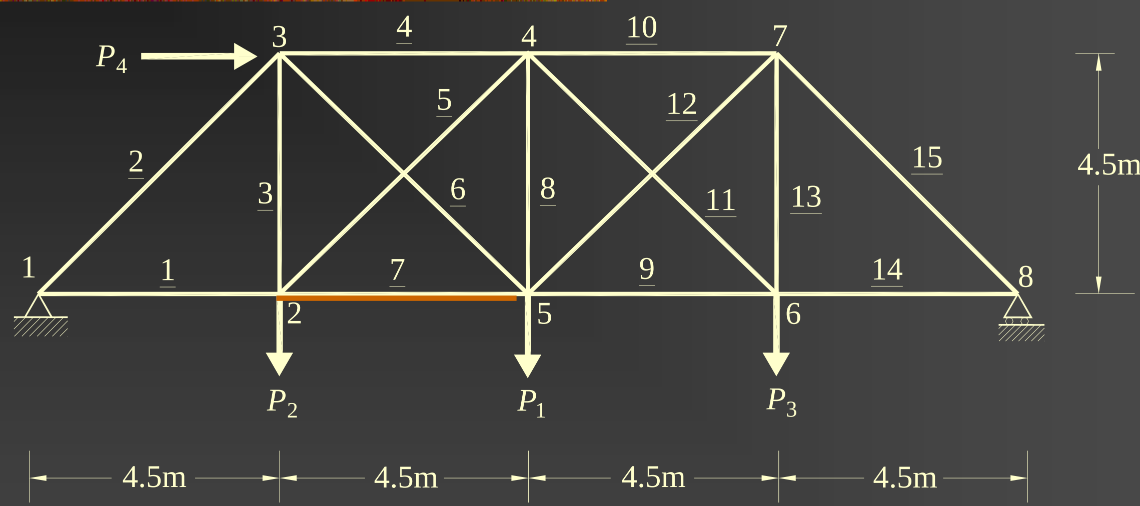


Contour values of maximum σ_{yy} (kPa)



Contour values of minimum σ_{yy} (kPa)

Truss structure



$$\boldsymbol{A}_1, \boldsymbol{A}_2, \boldsymbol{A}_3, \boldsymbol{A}_{13}, \boldsymbol{A}_{14}, \boldsymbol{A}_{15} : \ [9.95,\ 10.05]\ \text{cm}^2 (1\%\ \text{uncertainty})$$

cross-sectional area

of all other elements: $[5.97,\ 6.03]\ \text{cm}^2 (1\%\ \text{uncertainty})$

modulus of elasticity of all elements: 200,000 MPa

$$\boldsymbol{p}_1 = [190, 210]\ \text{kN},\ \boldsymbol{p}_2 = [95,105]\ \text{kN}$$

$$\boldsymbol{p}_3 = [95,105]\ \text{kN},\ \boldsymbol{p}_4 = [85.5, 94.5]\ \text{kN}\ (10\%\ \text{uncertainty})$$

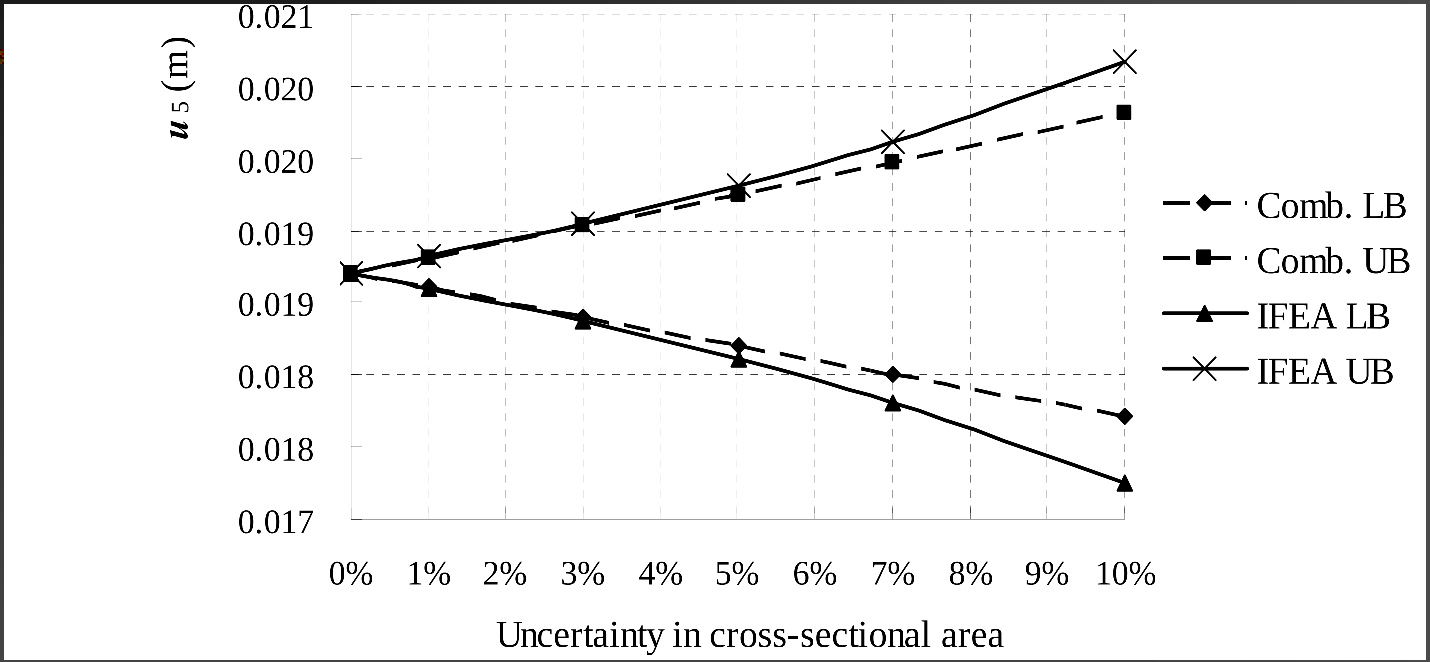
Truss structure - results

Table: results of selected responses

Method	u_5(LB)	u_5(UB)	N_7(LB)	N_7(UB)
Combinatorial	0.017676	0.019756	273.562	303.584
Naïve IFEA	– 0.011216	0.048636	– 717.152	1297.124
δ	163.45%	146.18%	362%	327%
Present IFEA	0.017642	0.019778	273.049	304.037
δ	0.19%	0.11%	0.19%	0.15%

unit: u_5 (m), N_7 (kN). LB: lower bound; UB: upper bound.

Truss structure – results



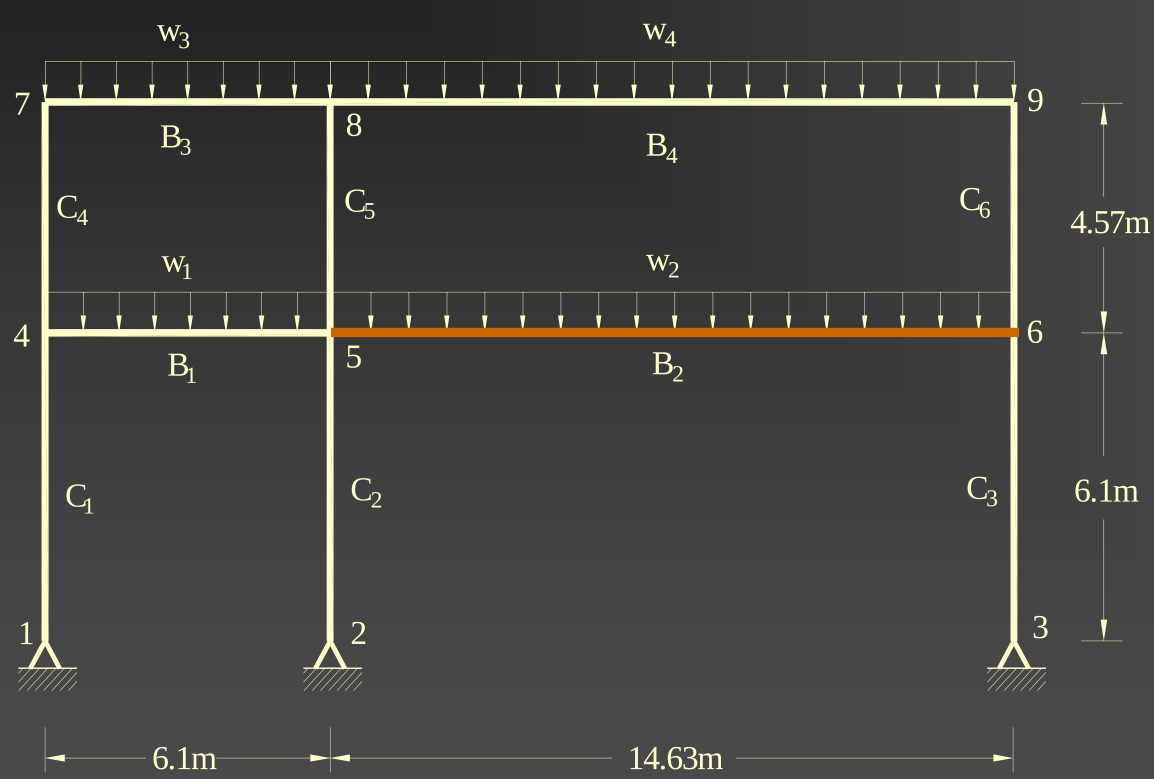
- for moderate uncertainty ($\leq$ 5%), very sharp bounds are obtained
- for relatively large uncertainty, reasonable bounds are obtained

  in the case of 10% uncertainty:

  Comb.: $\boldsymbol{u}_5$ = [0.017711,0.019811], IFEM: $\boldsymbol{u}_5$ = [0.017252,0.020168]

  (relative difference: 2.59%, 1.80% for LB, UB, respectively)

Frame structure



Member	Shape
C_1	W12×19
C_2	W14×132
C_3	W14×109
C_4	W10×12
C_5	W14×109
C_6	W14×109
B_1	W27×84
B_2	W36×135
B_3	W18×40
B_4	W27×94

results listed: nodal forces at the left node of member B_2

Frame structure – case 1

Case 1: load uncertainty

$\mathbf{w}_1 = [105.8, 113.1]$ kN/m, $\mathbf{w}_2 = [105.8, 113.1]$ kN/m,

$\mathbf{w}_3 = [49.255, 52.905]$ kN/m, $\mathbf{w}_4 = [49.255, 52.905]$ kN/m,

Table: Nodal forces at the left node of member B_2

Nodal force	Combinatorial		Present IFEA	
	LB	UB	LB	UB
Axial (kN)	219.60	239.37	219.60	239.37
Shear (kN)	833.61	891.90	833.61	891.90
Moment (kN·m)	1847.21	1974.95	1847.21	1974.95

- exact solution is obtained in the case of load uncertainty

Frame structure – case 2

Case 2: stiffness uncertainty and load uncertainty

1% uncertainty introduced to $\boldsymbol{A}$, $\boldsymbol{I}$, and $\boldsymbol{E}$ of each element.

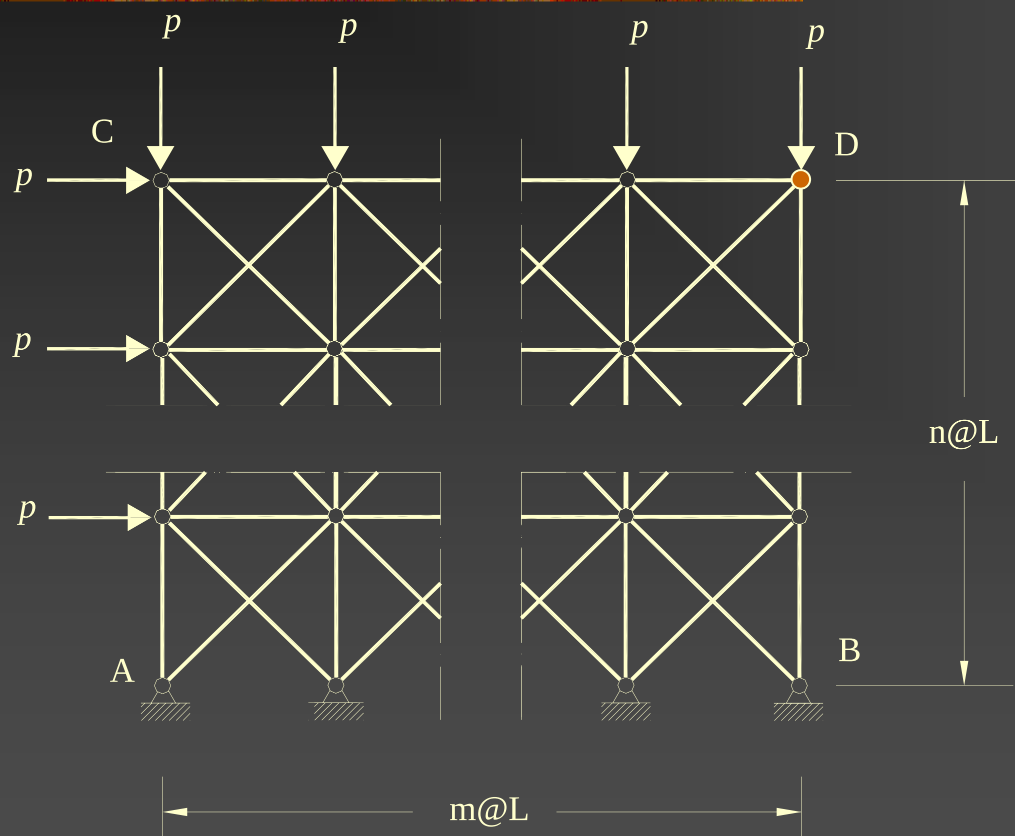
Number of interval variables: 34.

Table: Nodal forces at the left node of member B_2

Nodal force	Monte Carlo sampling*		Present IFEA	
	LB	UB	LB	UB
Axial (kN)	218.23	240.98	219.35	242.67
Shear (kN)	833.34	892.24	832.96	892.47
Moment (kN.m)	1842.86	1979.32	1839.01	1982.63

*Using 10^6 realization.

Truss with a large number of interval variables



$$A_i = [0.995, 1.005]A_0,$$
$$E_i = [0.995, 1.005]E_0 \quad \text{for } i = 1, \ldots, N_e$$

story×bay	N_e	N_v
3×10	123	246
4×12	196	392
4×20	324	648
5×22	445	890
5×30	605	1210
6×30	726	1452
6×35	846	1692
6×40	966	1932
7×40	1127	2254
8×40	1288	2576

Efficiency study

Table: CPU time for the analyses with the present method (unit: seconds)

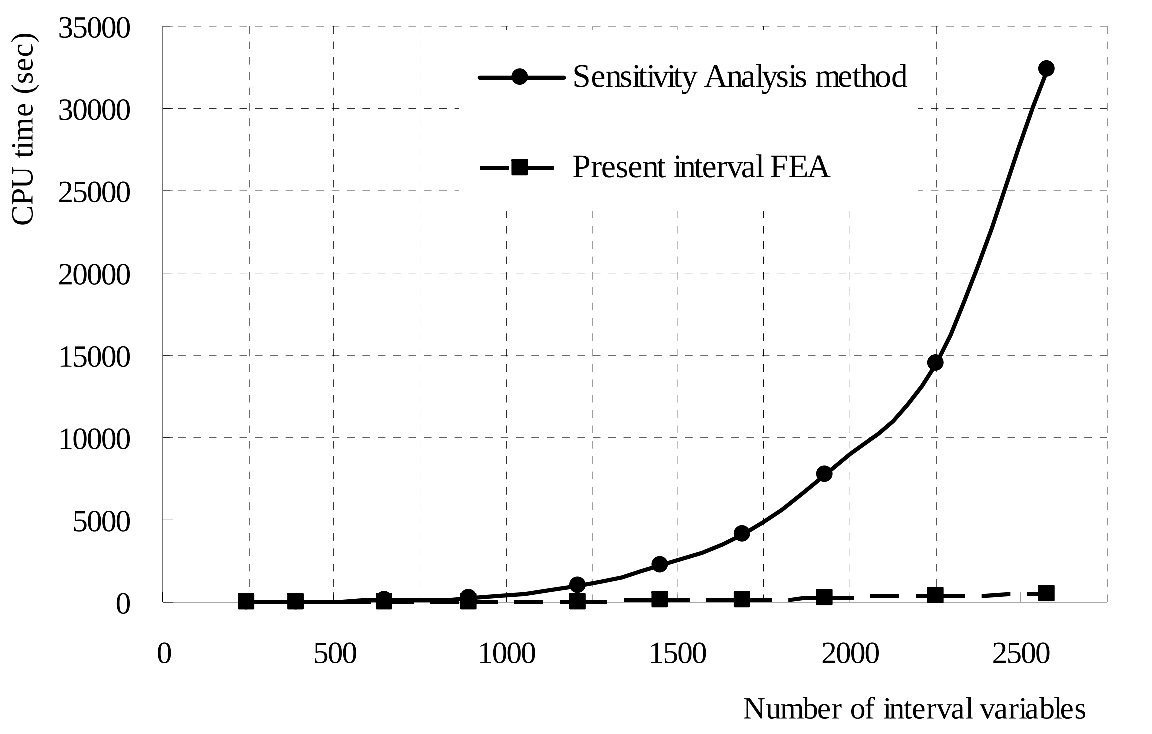
Story×bay	N_v	Iteration	t_i	t_r	t	t_i/t	t_r/t
3×10	246	4	0.14	0.56	0.72	19.5%	78.4%
4×20	648	5	1.27	8.80	10.17	12.4%	80.5%
5×30	1210	6	6.09	53.17	59.70	10.2%	89.1%
6×35	1692	6	15.11	140.2	156.27	9.7%	89.7%
7×40	2254	6	32.53	323.1	358.76	9.1%	90.1%
8×40	2576	7	48.45	475.7	528.45	9.2%	90.%

t_i : iteration time, t_r : CPU time for matrix inversion, t : total comp. CPU time

- majority of time is spent on matrix inversion

Efficiency study

Computational time: a comparison of the senstitivity analysis method and the present method



Computational time (seconds)

N_v	Sens.	Present
246	1.06	0.72
648	64.05	10.17
1210	965.86	59.7
1692	4100	156.3
2254	14450	358.8
2576	32402	528.45

Outline

- COMPUTING ON SETS: Random Sets, Fuzzy Sets, Convex Sets, Clouds
- Interval Operations
- Examples of progress in Reliable Engineering Analysis
- Validation Errors:
  Errors in representing the physics
  Errors in parameters used in model
  Interval Finite Elements
- **Verification Errors**
  Errors in discretization of model
  Errors in computations (truncation errors)
  Interval Boundary Elements
- Conclusions

Boundary Element Analysis of Laplace Equation

The Laplace equation is:

$$\nabla^2 u = 0 \qquad in\ \Omega$$

$$u = \hat{u} \qquad on\ \Gamma_1$$

$$\frac{\partial u}{\partial n} = q = \hat{q} \qquad on\ \Gamma_2$$

(Ω) is the domain of the system

(Γ) is the boundary of the system

$(\hat{u})$, $(\hat{q})$ are the values at the boundary

Boundary Element Analysis of Laplace Equation (Cont.)

Orthogonalization of Laplace equation with respect to a test function (w) is performed to minimize the error due to approximation of the exact solution of $(u)$and (q) :

$$\int_{\Omega}(\nabla^2 u)wd\Omega = \int_{\Gamma_2}(q-\hat{q})wd\Gamma - \int_{\Gamma_1}(u-\hat{u})\frac{\partial w}{\partial n}d\Gamma$$

Boundary Element Analysis of Laplace Equation (Cont.)

Twice integrating by parts on the left side and considering $u^* = w$ and $q^* = \partial u^* / \partial n$ yields:

$$u(\xi) + \int_{\Gamma_2} u q^* d\Gamma + \int_{\Gamma_1} \hat{u} q^* d\Gamma = \int_{\Gamma_2} \hat{q} u^* d\Gamma + \int_{\Gamma_1} q u^* d\Gamma,$$

(ξ) is a source point

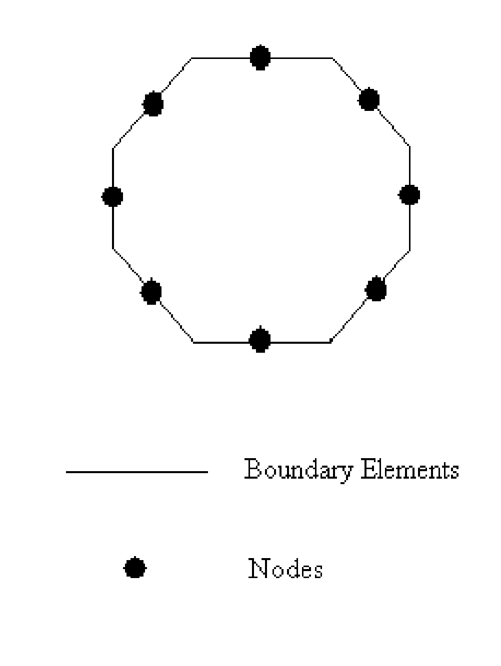
Boundary Element Analysis of Laplace Equation (Cont.)

The boundary integral equation is integrated such that the source point (ξ) is included on the circular boundary of radius (ε), as $\varepsilon \to 0$:

$$\frac{1}{2}u(\xi) + \int_{\Gamma_x} q^*(x,\xi)u(x)d\Gamma_x = \int_{\Gamma_x} u^*(x,\xi)q(x)d\Gamma_x, \qquad \xi \in \Gamma$$

Constant Boundary Element Discretization

Any boundary (Γ) can be discretized into boundary elements (Γ_i) consisting of nodes, at which a value of either (u) or (q) is known, and also consisting of assumed polynomial shape functions between nodes.



Constant Boundary Element Discretization (Cont.)

In this work one-noded boundary elements with constant shape functions are used, leading to the following discretization:

$$u(x) = [\Phi(x)][u_i]$$

$$q(x) = [\Phi(x)][q_i]$$

$[u_i]$ and $[q_i]$ are vectors of nodal values

$[\Phi(x)]$ is a vector of constant shape functions

Constant Boundary Element Discretization (Cont.)

The discretized integral equation is written as:

$$\frac{1}{2}\{u_i\}+\sum_{Elements}\int_{\Gamma_x} q^*(x,\xi)[\Phi(x)]d\Gamma_x[u_i]=\sum_{Elements}\int_{\Gamma_x} u^*(x,\xi)[\Phi(x)]d\Gamma_x[q_i]$$

or in matrix form: $[H][u]=[G][q]$

Applying the boundary conditions, the system of linear equations is rearranged as: $[A][x]=[b]$

Discretization Error

In the analysis of the discretiztion error, the interval bounded unknown functions must satisfy the continuous problem.

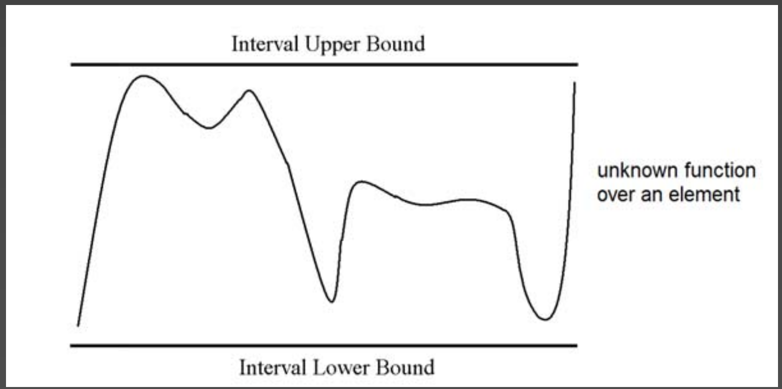
$$\frac{1}{2}u(\xi)+\int_{\Gamma} q^{*}(x,\xi)u(x)d\Gamma \;=\int_{\Gamma} u^{*}(x,\xi)q(x)d\Gamma, \qquad \xi\in\Gamma$$

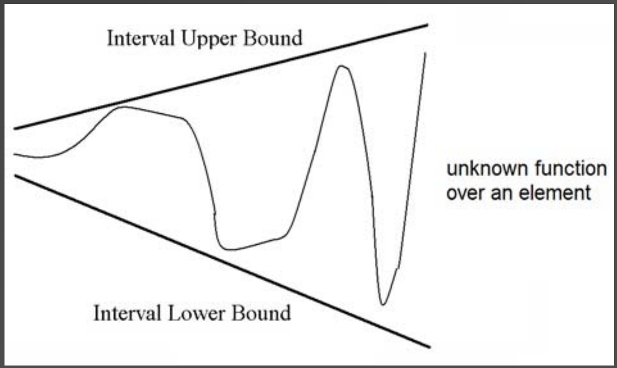
Discretization Error Bounds

The boundary (Γ) is subdivided into elements and for each element, the interval values ($\tilde{q}$) and ($\tilde{u}$) are found that bound the functions (u) and (q) over an element (i) such that:

$$\forall j \in \{1,2,\ldots,n\} \; Assume \; \underline{u_i} \le u_i \le \overline{u_i}, \; \underline{q_i} \le q_i \le \overline{q_i} \; is \; known \; \forall i \ne j. \; Find \; \underline{u_j} \le u_j \le \overline{u_j}, \; \underline{q_j} \le q_j \le \overline{q_j}$$

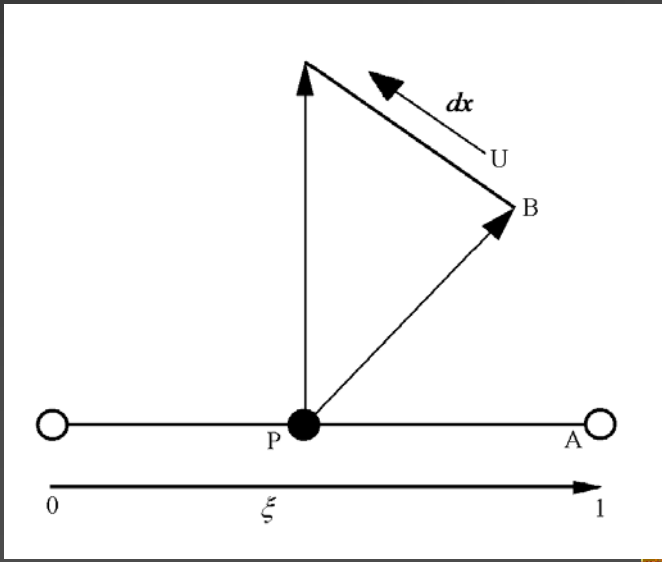
$$\forall \xi_j \left| \; \frac{1}{2}u(\xi_j) + \sum_{i=1}^{n} \int_{\Gamma_i} q^*(x,\xi_j) u_i(x) d\Gamma_i = \sum_{i=1}^{n} \int_{\Gamma_i} u^*(x,\xi_j) q_i(x) d\Gamma_i \right.$$





Discretization Error (Cont.)

Each term of the summation is represented graphically



Discretization Error (Cont.)

The kernel splitting techniques have been used to bound the integral equations in which the left hand side is deterministic (Dobner). The boundary integral equations have an interval left hand side and therefore a new approach is developed.

$$b(\xi) = \int_{\Gamma} k(x, \tilde{\xi}) u(x) d\Gamma$$

Discretization Error (Cont.)

The integral of the product of two functions is bounded as:

$$\tilde{b} = \int_{\Gamma} a(x,\tilde{\xi})u(x)d\Gamma \subset \int_{\Gamma} a(x,\tilde{\xi})\tilde{u}d\Gamma$$

The right hand side is expressed as a sum of the integrals:

$$\int_{\Gamma} a(x,\tilde{\xi})\tilde{u}d\Gamma = \int_{\Gamma_1} a(x,\tilde{\xi})\tilde{u}d\Gamma_1 + \int_{\Gamma_2} a(x,\tilde{\xi})\tilde{u}d\Gamma_2$$

$a(x,\tilde{\xi}) > 0$ or $a(x,\tilde{\xi}) < 0 \quad on \ \Gamma_1$

$$a(x,\tilde{\xi}) \in 0 \quad on \ \Gamma_2$$

Discretization Error (Cont.)

The interval kernel is of the same sign on (Γ_1), thus $(\tilde{u})$ can be taken out of the integral on (Γ_1):

$$\int_{\Gamma} a(x,\tilde{\xi})\tilde{u}\,d\Gamma = \int_{\Gamma_1} a(x,\tilde{\xi})\,d\Gamma_1 \tilde{u} + \int_{\Gamma_2} a(x,\tilde{\xi})\tilde{u}\,d\Gamma_2$$

$(\tilde{u})$ cannot be taken out of the integral on (Γ_2) due to subdistributivity property.

Discretization Error (Cont.)

The interval kernel is bounded by its limits:

$$\int_{\Gamma_2} a(x,\tilde{\xi})\tilde{u}\,d\Gamma_2 \subset \int_{\Gamma_2} \tilde{a}\tilde{u}\,d\Gamma_2$$

$$\tilde{a} = [\min\{a(x+\tilde{\varepsilon},\tilde{\xi})\}, \max\{a(x+\tilde{\varepsilon},\tilde{\xi})\}]$$

where $\tilde{\varepsilon} = [-\varepsilon, \varepsilon]$

Discretization Error (Cont.)

$$\int_{\Gamma_2} \tilde{a}\tilde{u}\,d\Gamma_2 = \lim_{\Delta x \to 0} \sum_{i=1}^{n} (\Delta x \tilde{a}\tilde{u})\Big|_{\Gamma_2} = \lim_{\Delta x \to 0} (n\Delta x \tilde{a}\tilde{u})\big|_{\Gamma_2} = \lim_{\Delta x \to 0} (n\Delta x \tilde{a})\tilde{u}\big|_{\Gamma_2} = \lim_{\Delta x \to 0} \sum_{i=1}^{n} (\Delta x \tilde{a})\Big|_{\Gamma_2} \tilde{u} = \int_{\Gamma_2} \tilde{a}\,d\Gamma_2 \tilde{u}$$

$(\tilde{u})$ can be taken out of the integral and the integral equation becomes:

$$\tilde{b} \subset \int_{\Gamma_1} a(x,\tilde{\xi})\,d\Gamma_1 \tilde{u} + \int_{\Gamma_2} \tilde{a}\,d\Gamma_2 \tilde{u}$$

The kernels are bounded for all the elements resulting in the system of equations:

$$[\tilde{A}_1][\tilde{u}] + [\tilde{A}_2][\tilde{u}] = [\tilde{c}] \supset [\tilde{b}]$$

Interval Equation Solver

Considering an interval linear system of equations:

$$[\tilde{A}_1][\tilde{u}] + [\tilde{A}_2][\tilde{u}] = [\tilde{b}]$$

The system is preconditioned by $[A_1]^{-1}$

$$[A_1]^{-1}[\tilde{A}_1][\tilde{u}] + [A_1]^{-1}[\tilde{A}_2][\tilde{u}] = [A_1]^{-1}[\tilde{b}]$$

Let $\quad [A_1]^{-1}[\tilde{A}_1] = [\tilde{I}_1] \qquad [A_1]^{-1}[\tilde{A}_2] = [\tilde{A}_3] \qquad [A_1]^{-1}[\tilde{b}] = [\tilde{b}_1]$

$$[\tilde{I}_1][\tilde{u}] + [\tilde{A}_3][\tilde{u}] = [\tilde{b}_1]$$

Parameterized Interval Equation Solver

- The interval bounds obtained by the solver are not sharp since the dependency of the location of the source point has not been considered.
- The uniqueness of the problem is not preserved since two source points are allowed to have the same location at one time resulting in rectangular matrices.
- The parameterization considers each source point to have a unique location and allows for sharper interval bounds.

Parameterized Interval Equation Solver (Cont.)

The system of equations is rearranged:

$$[A_1][\tilde{u}] + [A_2][\tilde{u}] = [\tilde{b}]$$

Preconditioning and substitution as described above lead to: $[\tilde{A}][\tilde{u}] = [\tilde{b}_1]$

The parameterization is incorporated into the solver:

$$[\tilde{u}_0] = [A]^{-1} \bigcup_{i=1}^{n} [\tilde{b}_1(\tilde{\xi}_i)]$$

$[A]^{-1}$ is computed when $\xi = 0.5$

Parameterized Interval Equation Solver (Cont.)

$$[\tilde{I}_d] = [I] - \bigcup_{i=1}^{n} [A]^{-1} [\tilde{A}(\tilde{\xi}_i)]$$

The difference between the solution and the initial guess is computed and pre-multiplied by the preconditioning matrix $[A]^{-1}$:

$$[\tilde{\delta}] = \bigcup_{i=1}^{n} \left([\tilde{b}(\tilde{\xi}_i)] - [\tilde{A}_1(\tilde{\xi}_i)][\tilde{u}_0] - [\tilde{A}_2(\tilde{\xi}_i)][\tilde{u}_0] \right)$$

$$[\tilde{\delta}_1] = [\tilde{\delta}]$$

Parameterized Interval Equation Solver (Cont.)

The iteration follows as:

$$[\tilde{del}] = [\tilde{\delta}_1]$$

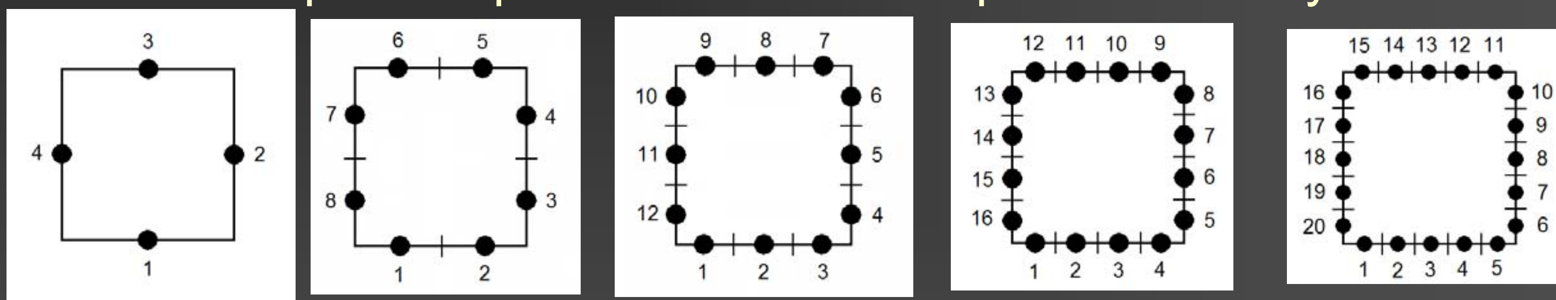
$$[\tilde{\delta}_1] = [\tilde{\delta}] + [\tilde{I}_d][\tilde{del}]$$

If $$[\tilde{\delta}_1] \supset [\tilde{\delta}]$$

$$[\tilde{u}] = [\tilde{u}_0] + [\tilde{\delta}_1]$$

Example 1

The first example obtains the bounds on discretization error for the BEA of the Laplace equation for the unit square boundary.

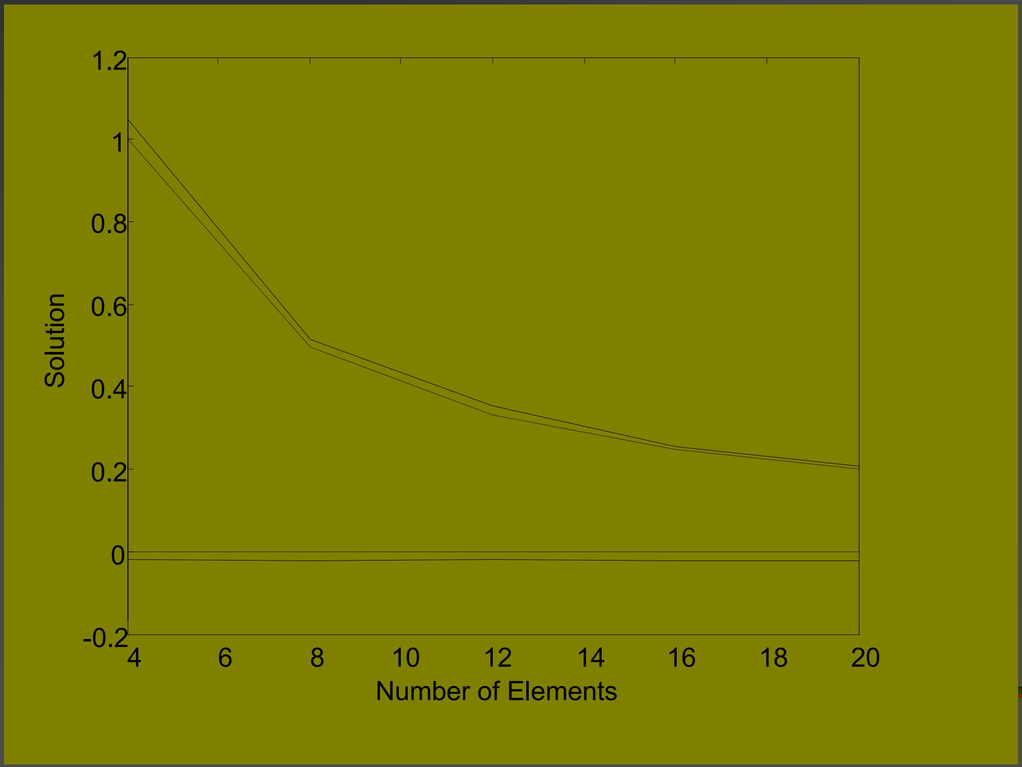


Boundary Conditions: $u_{bottom}=0$, $q_{sides}=0$, $u_{top}=1$

$$Effectivity\ \ index = \frac{Computed\ \ width}{True\ \ width}$$

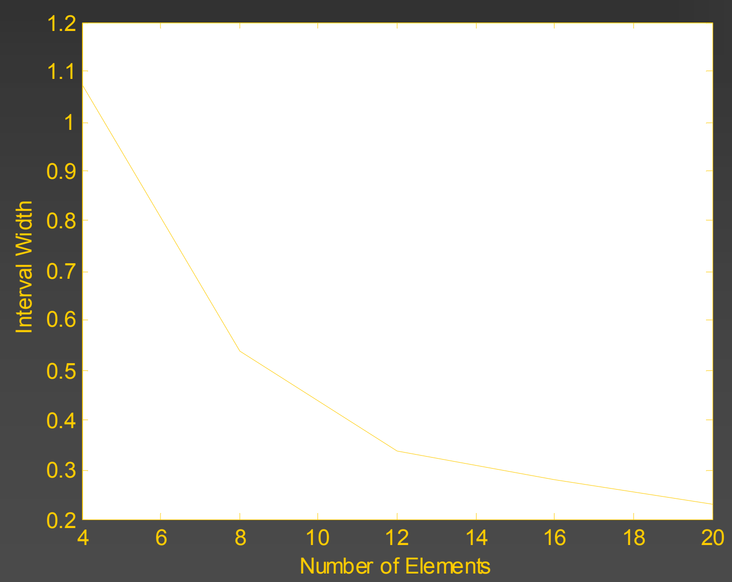
Example 1 (Cont.)

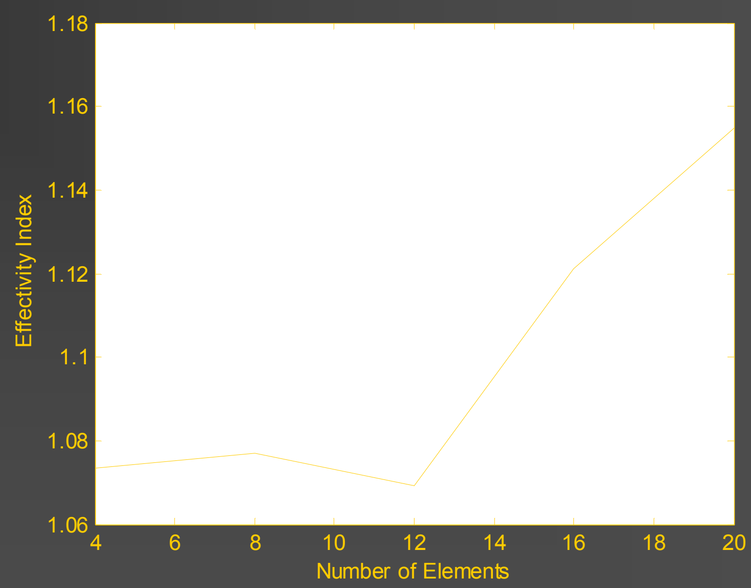
The figure shows the interval bounds (solid line) on the true solution (dashed line) for the potential on the right lower corner nodes, nodes 2, 3, 4, 5, and 6 respectively, for the five meshes.



Example 1 (Cont.)

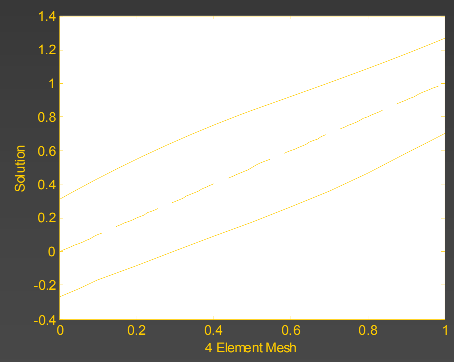
The effect of the mesh refinement on the solution width (left) and the effectivity index (right) is shown for these nodes.

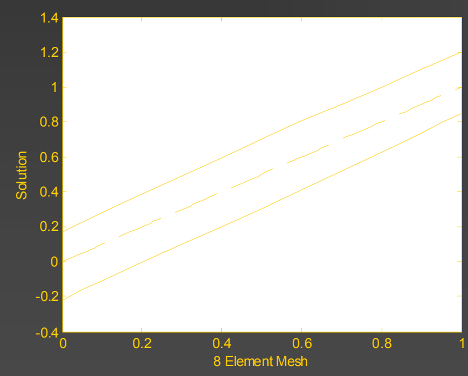


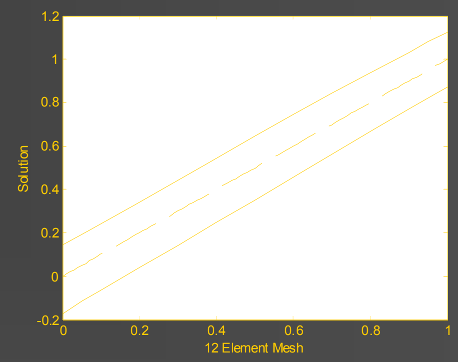


Example 1 (Cont.)

The interval bounds for the interior potential on the x=0.5 plane is shown (solid line) and compared with the true solution (dashed line) for three uniform meshes composed of 4, 8, and 12 elements.

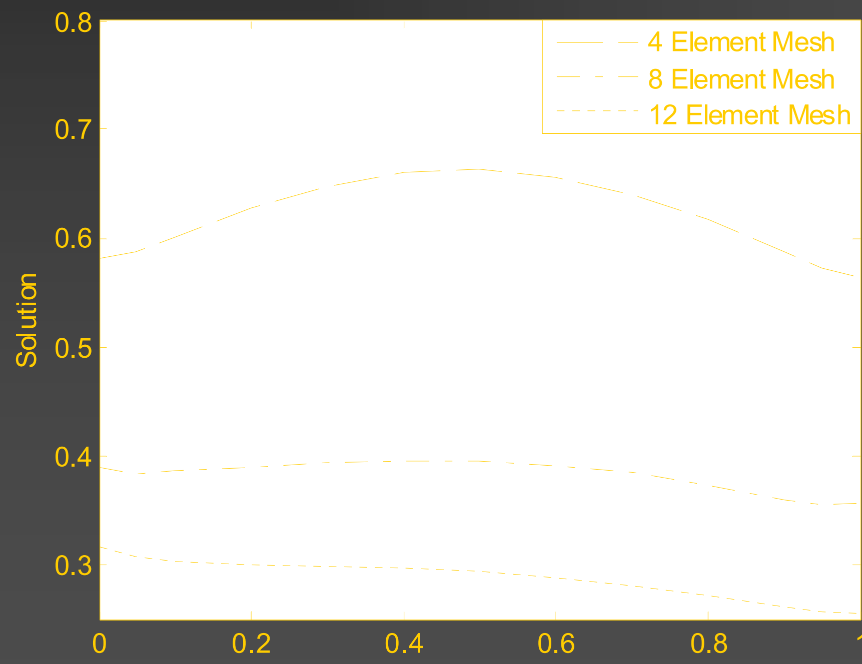






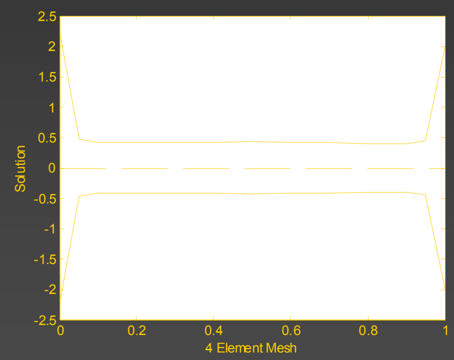
Example 1 (Cont.)

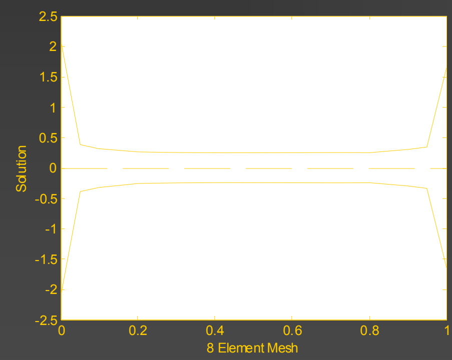
The effectivity index for the previous solution is shown for the three uniform meshes.

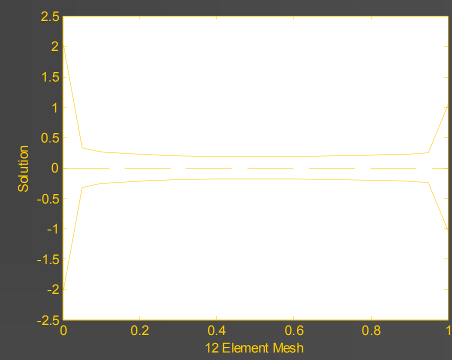


Example 1 (Cont.)

The interval bounds for the interior x-direction flux on the x=0.5 plane is shown (solid line) and compared with the true solution (dashed line) for three uniform meshes composed of 4, 8, and 12 elements.

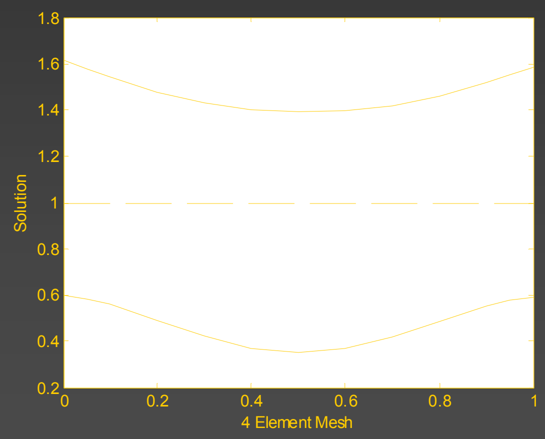


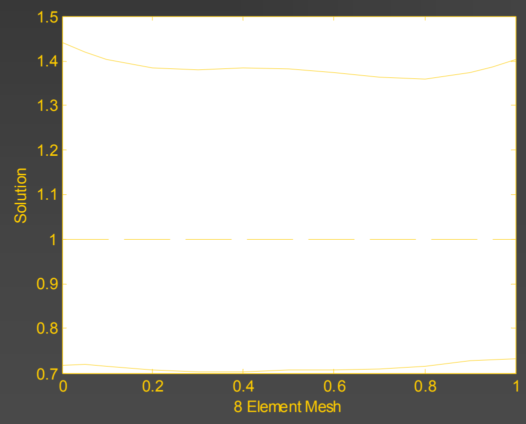


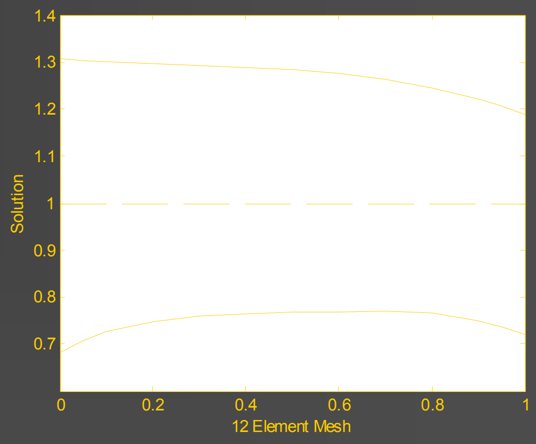


Example 1 (Cont.)

The interval bounds for the interior y-direction flux on the x=0.5 plane is shown (solid line) and compared with the true solution (dashed line) for three uniform meshes composed of 4, 8, and 12 elements.

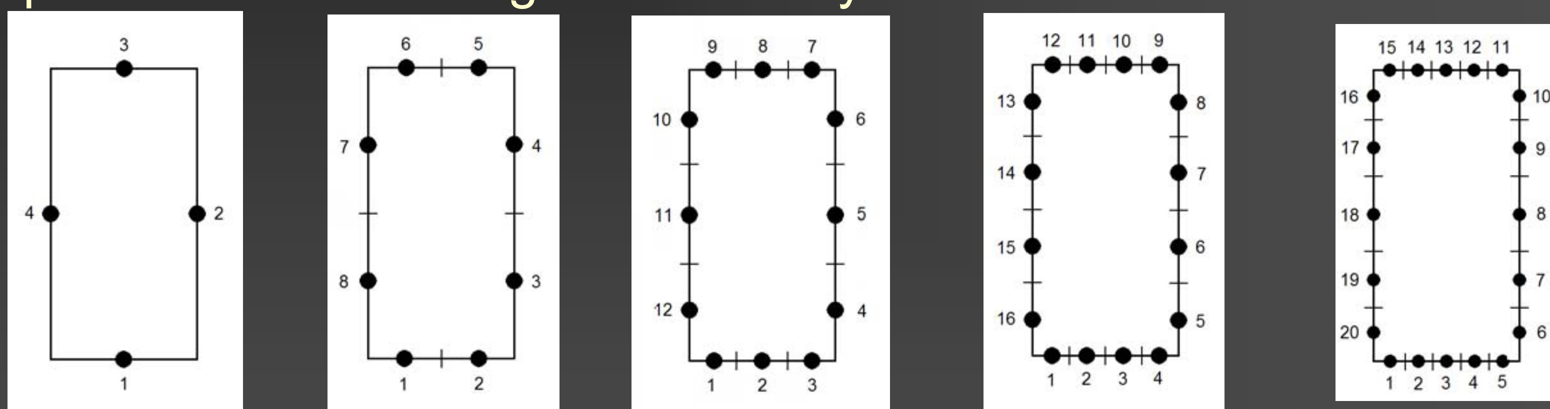






Example 2

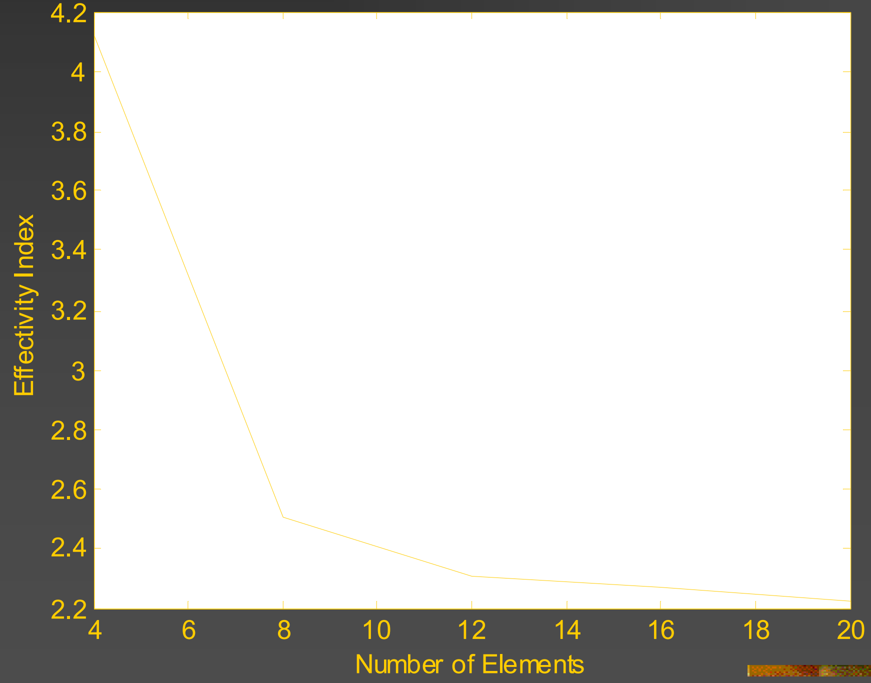
The second example shows the behavior of effectivity index in the presence of significant discretization error in the BEA of the Laplace equation for the rectangular boundary with ratio 1:2.



Boundary Conditions: $u_{bottom}=0$, $q_{sides}=0$, $u_{top}=1$

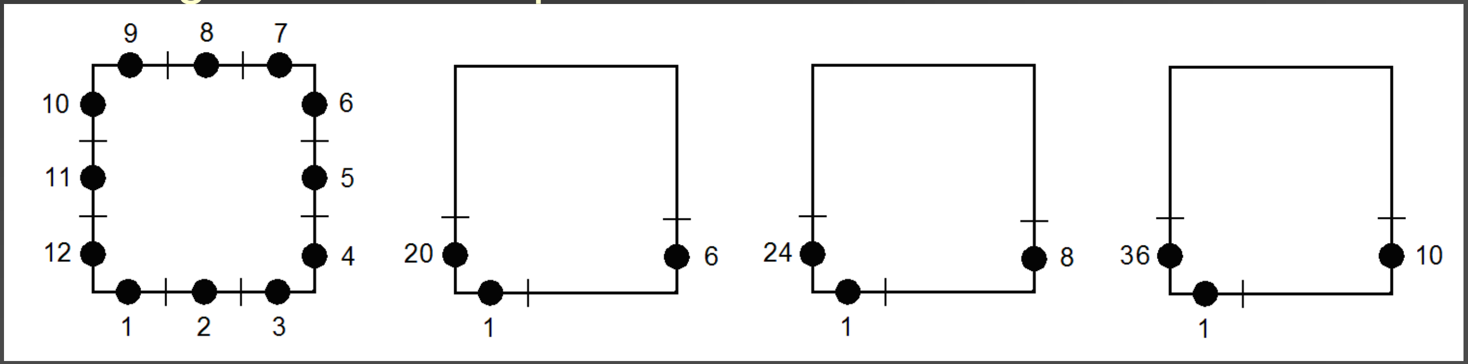
Example 2 (Cont.)

The figure shows the behavior of effectivity index of the potential solution for the right lower corner nodes, 2, 3, 4, 5, and 6 for the five respective meshes.



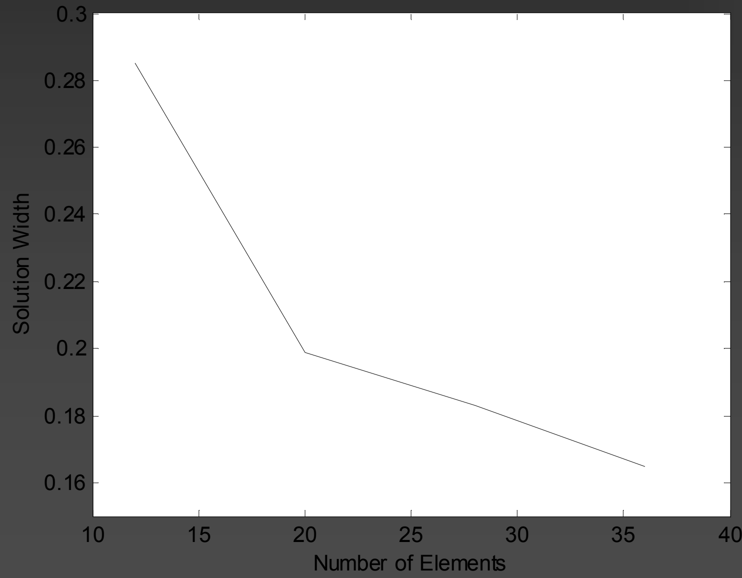
Example 3

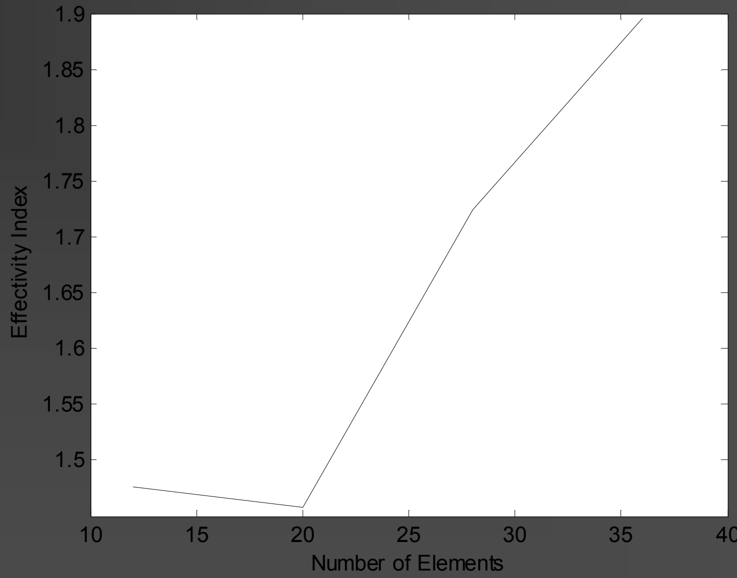
The third example is a boundary element analysis of the Torsion problem stated in terms of the Laplacian of the warping function. Four uniform constant element meshes are analyzed. Neumann boundary conditions are applied at all boundaries and an exact Dirichlet boundary condition is applied on the middle element at the bottom edge for each respective mesh.



Example 3 (Cont.)

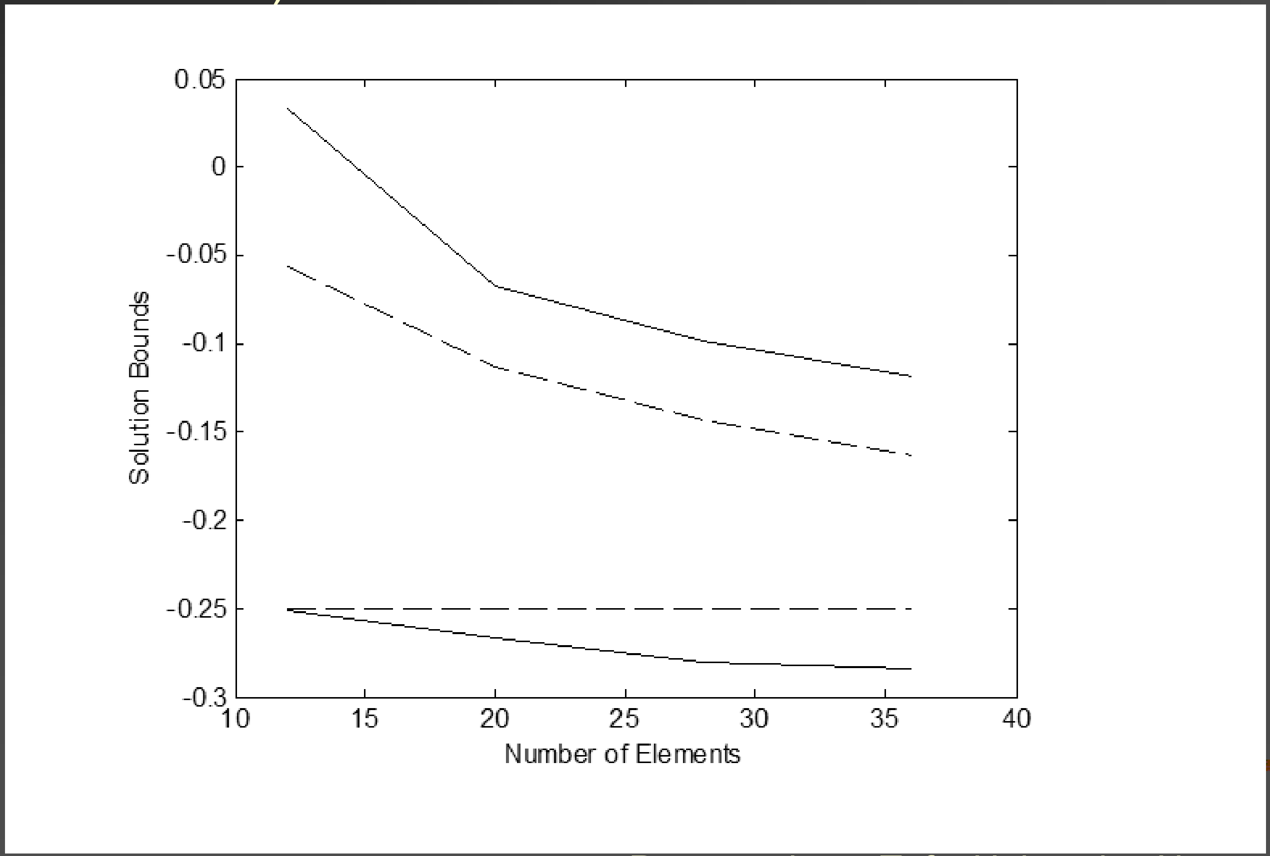
The warping function solution width (left) and the effectivity index (right) is shown for the bottom right corner nodes, 4, 6, 8, and 10, for the four respective meshes.





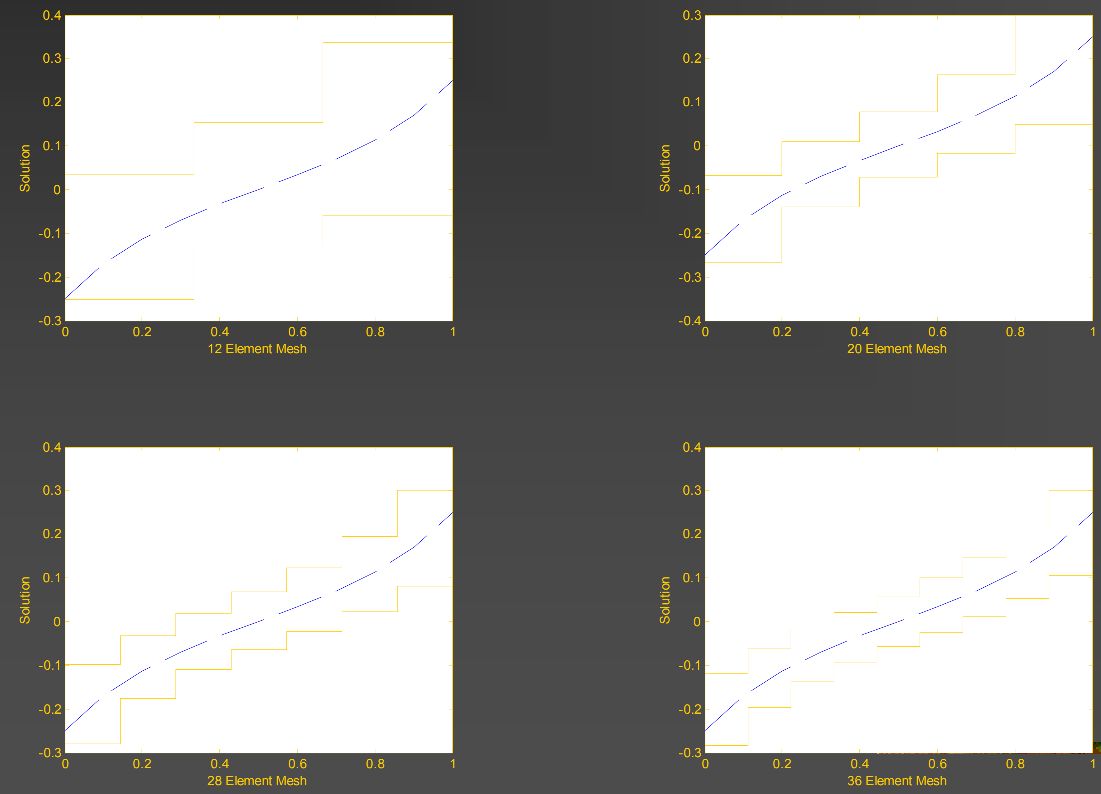
Example 3 (Cont.)

The interval bounds (solid line) are shown enclosing the true solution (dashed line) for these nodes.



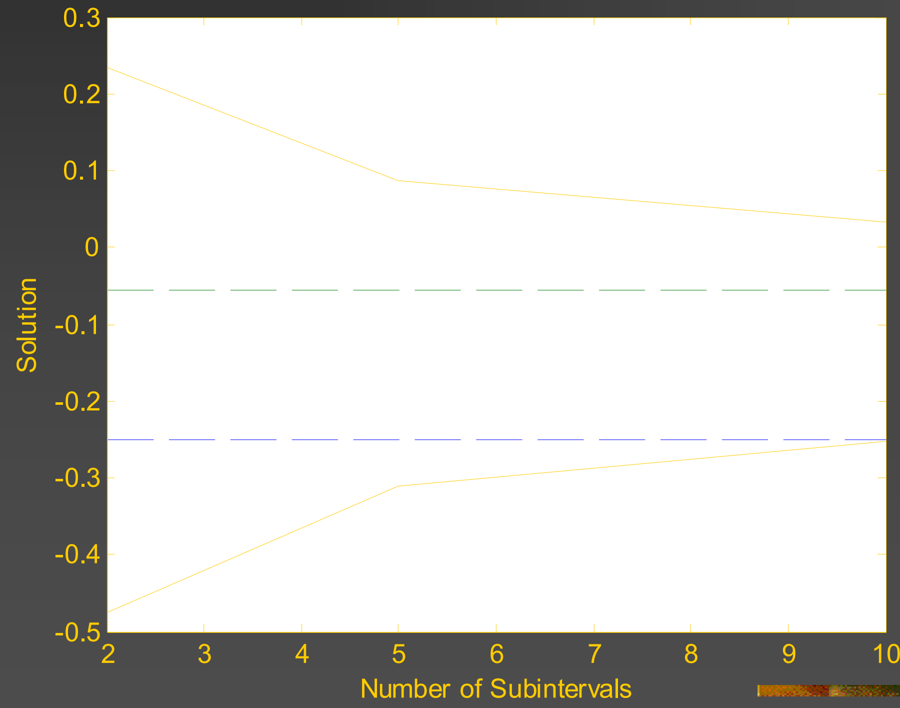
Example 3 (Cont.)

The true solution (dashed line) of the warping function is enclosed by the interval bounds (solid line) for the right edge for four uniform meshes.



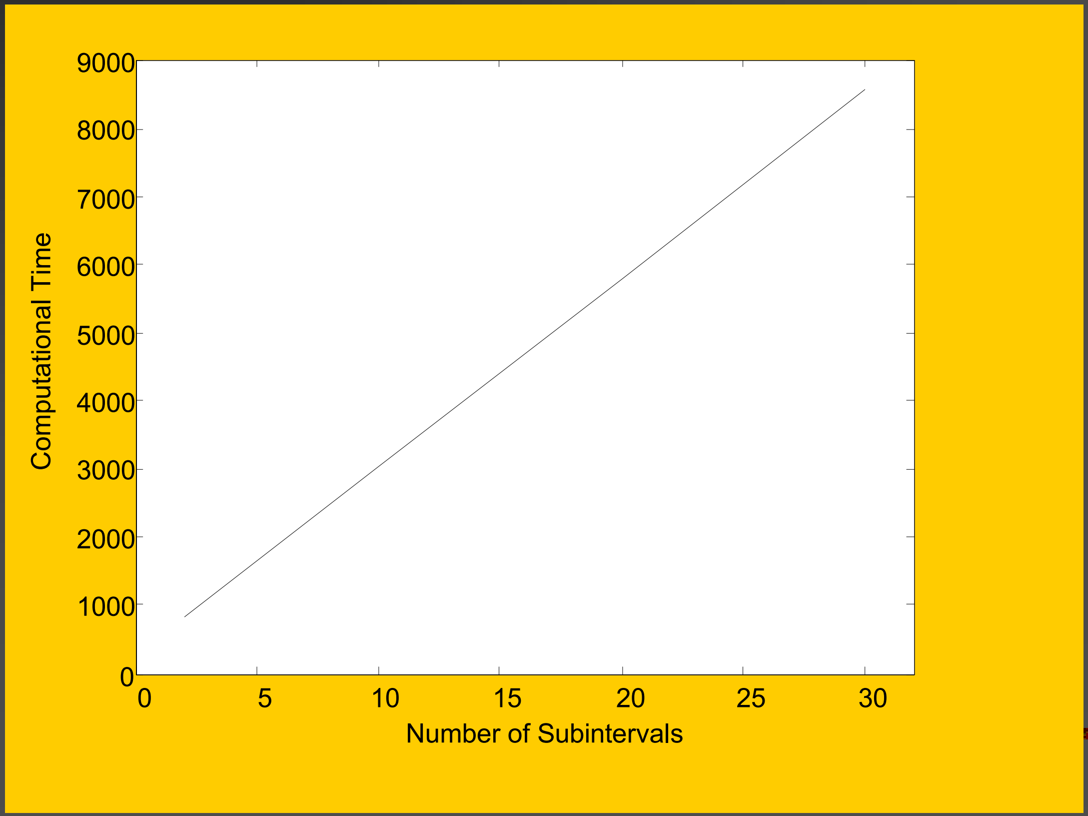
Example 3 (Cont.)

The behavior of the interval solution bounds (solid line) for the warping function with parameterization is shown and compared with the true solution (dashed line) for node 4 in the 12 element mesh.



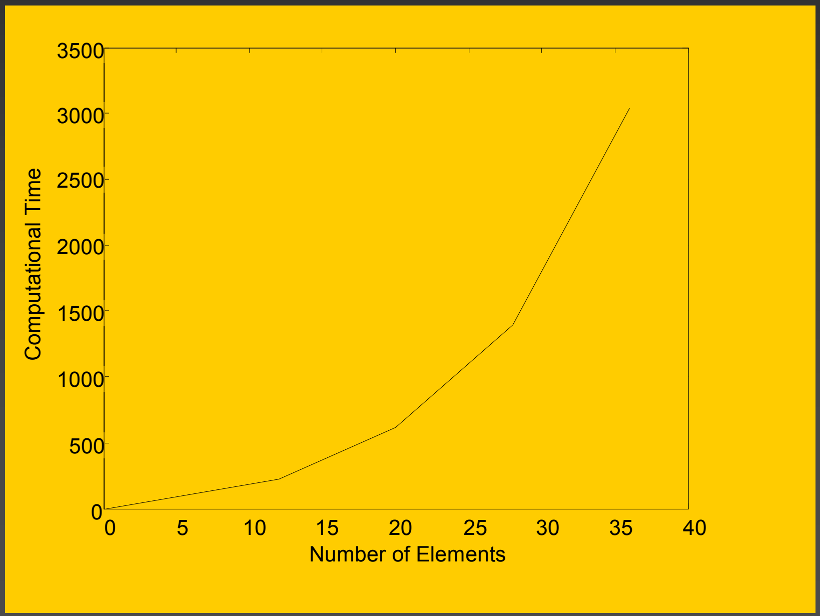
Example 3 (Cont.)

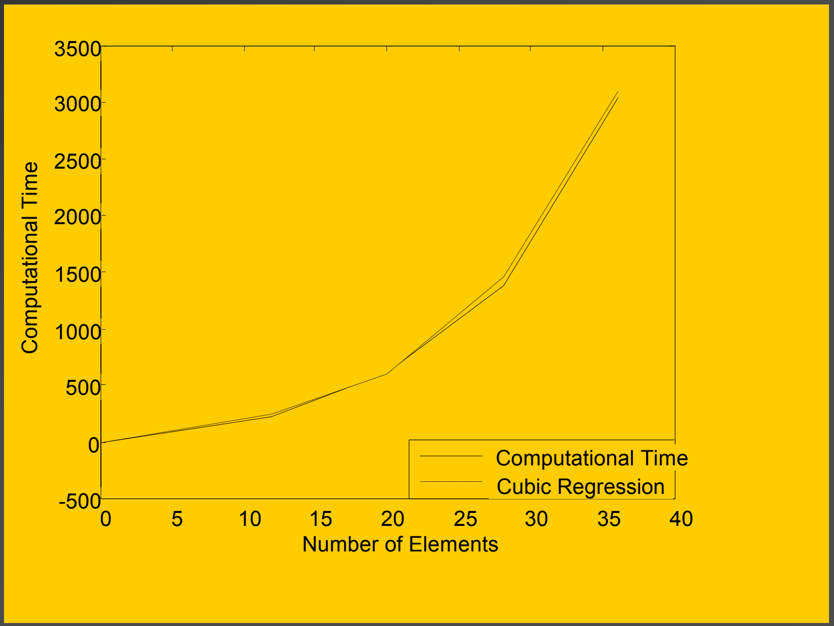
The computational cost with increased parameterization is shown for the 36 element mesh.



Example 3 (Cont.)

The computational cost with mesh refinement is shown (left). The computational time varies approximately cubically (right).



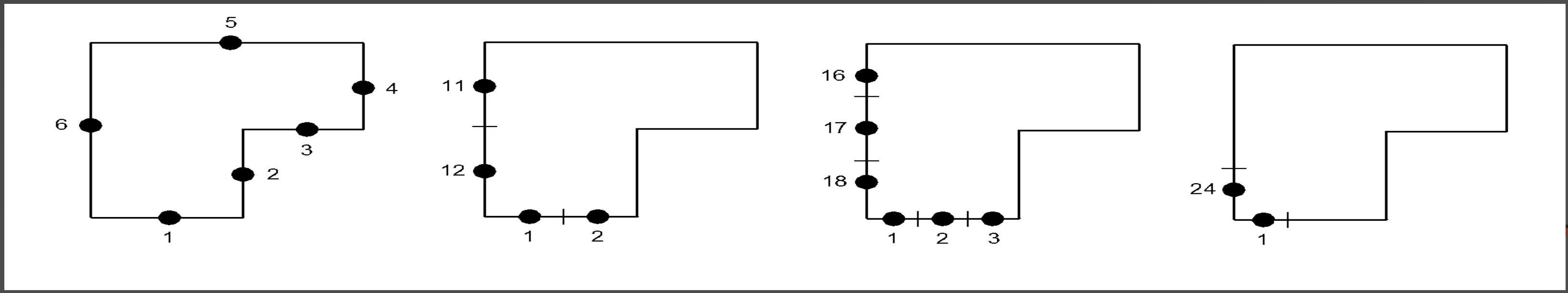


Example 4

The example obtains the solution for the Laplace equation whose true solution is:

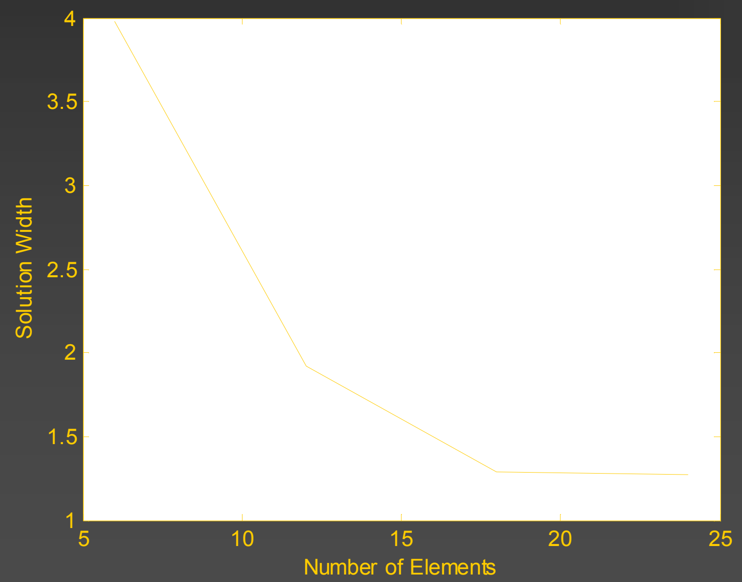
$$\left.\begin{array}{c} \nabla^2 u = 0 \quad in\ \Omega \\ \\ u = \sinh(x)\sin(y) \quad on\ \Gamma \end{array}\right\}$$

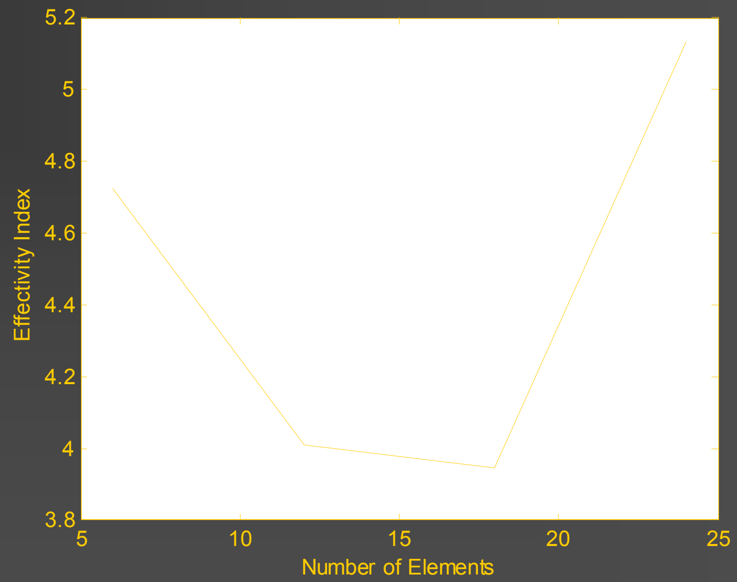
for the L-shaped domain. Four uniform constant element meshes are studied. The potential boundary conditions are applied at all edges.



Example 4 (Cont.)

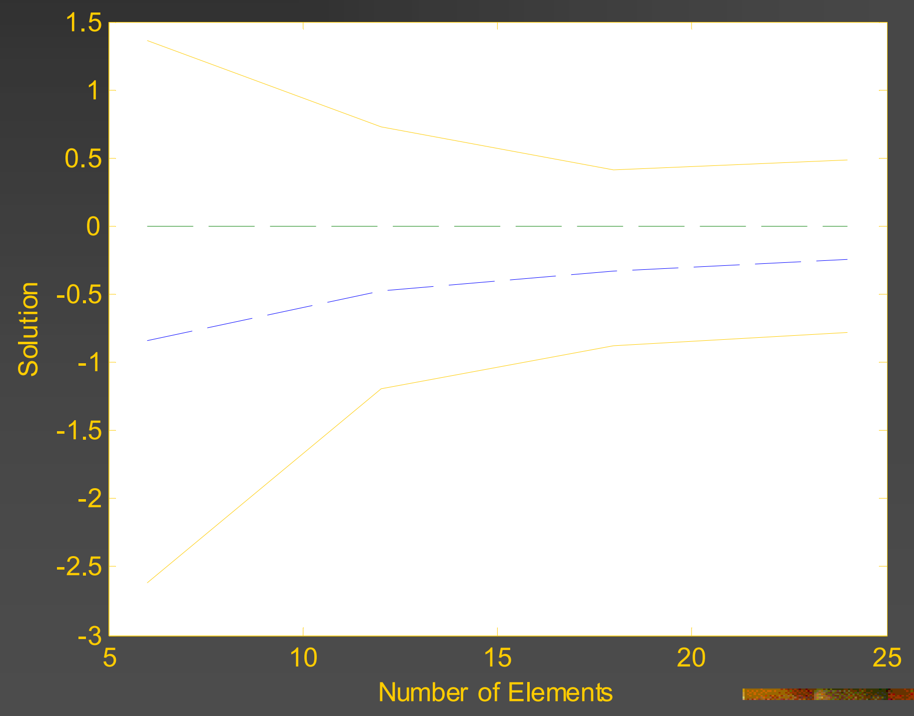
The interval solution width for the boundary flux (left) and the effectivity index (right) for the left lower corner elements, 6, 12, 18, and 24, for the four respective meshes are shown.





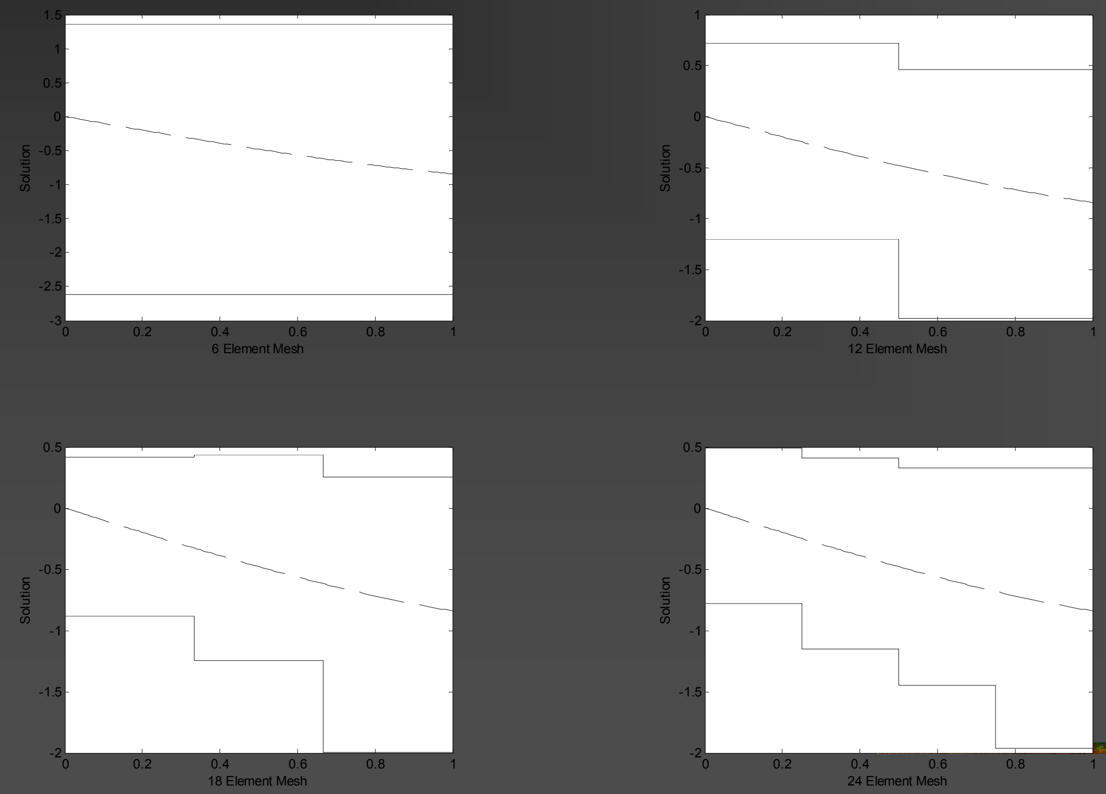
Example 4 (Cont.)

The true solution (dashed line) is bounded by the interval solution (solid line) for these nodes.



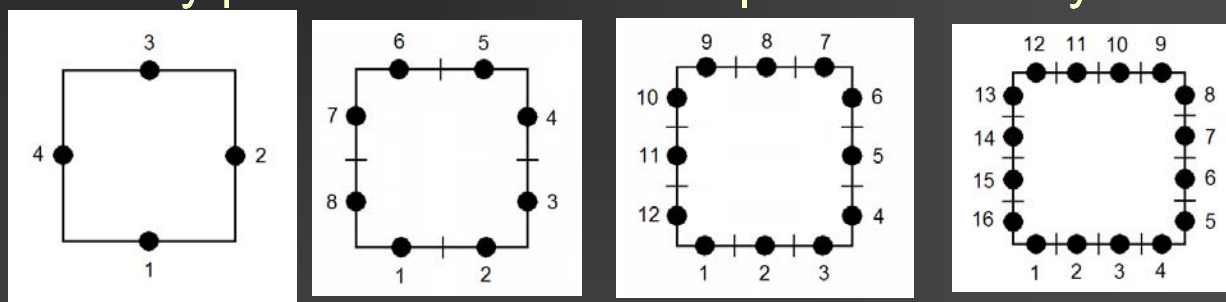
Example 4 (Cont.)

The interval bounds (solid line) enclose the true solution (dashed line) for the left edge for the four meshes.



Example 5

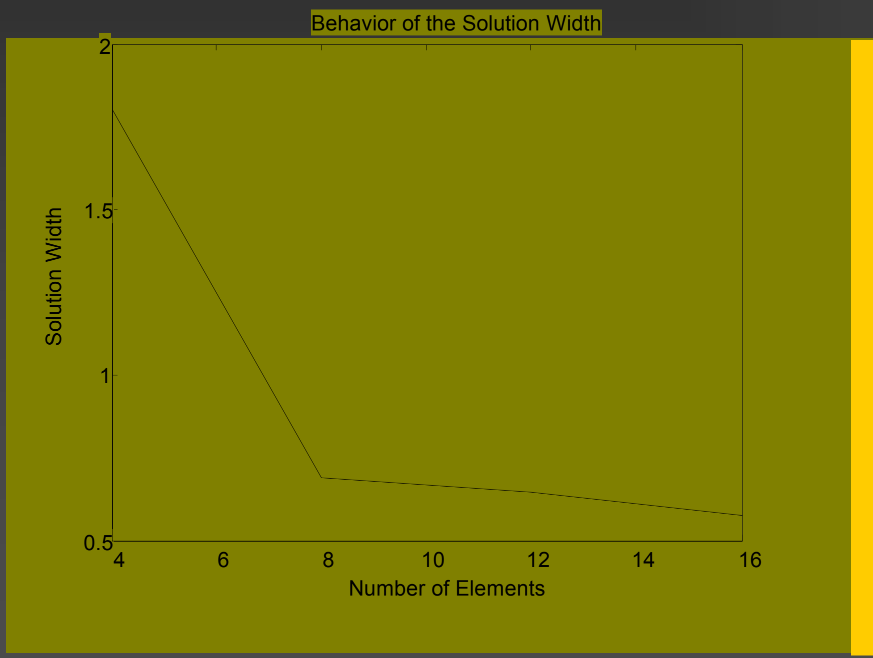
This example obtains the bounds on discretization error for the BEA of the Elasticity problem for the unit square boundary.

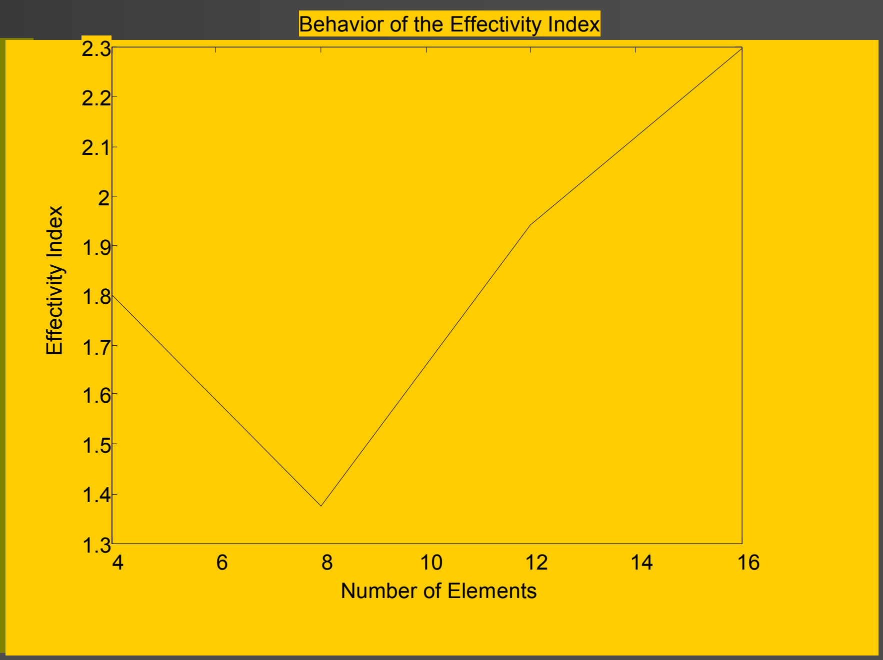


Boundary Conditions: $u_{bottom}=0$, $t_{sides}=0$, $u_{y\ top}=1$, $t_{x\ top}=0$

Example 5 (Cont.)

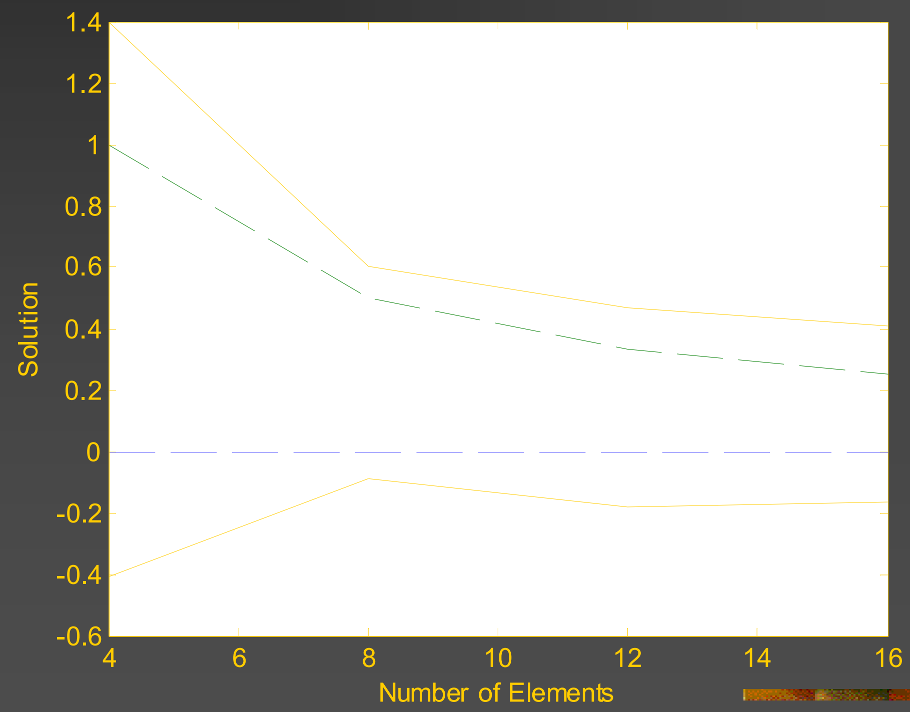
The interval solution width for the y-direction displacement (left) and the effectivity index (right) for the right lower corner elements, 2, 3, 4, and 5, for the four respective meshes are shown.





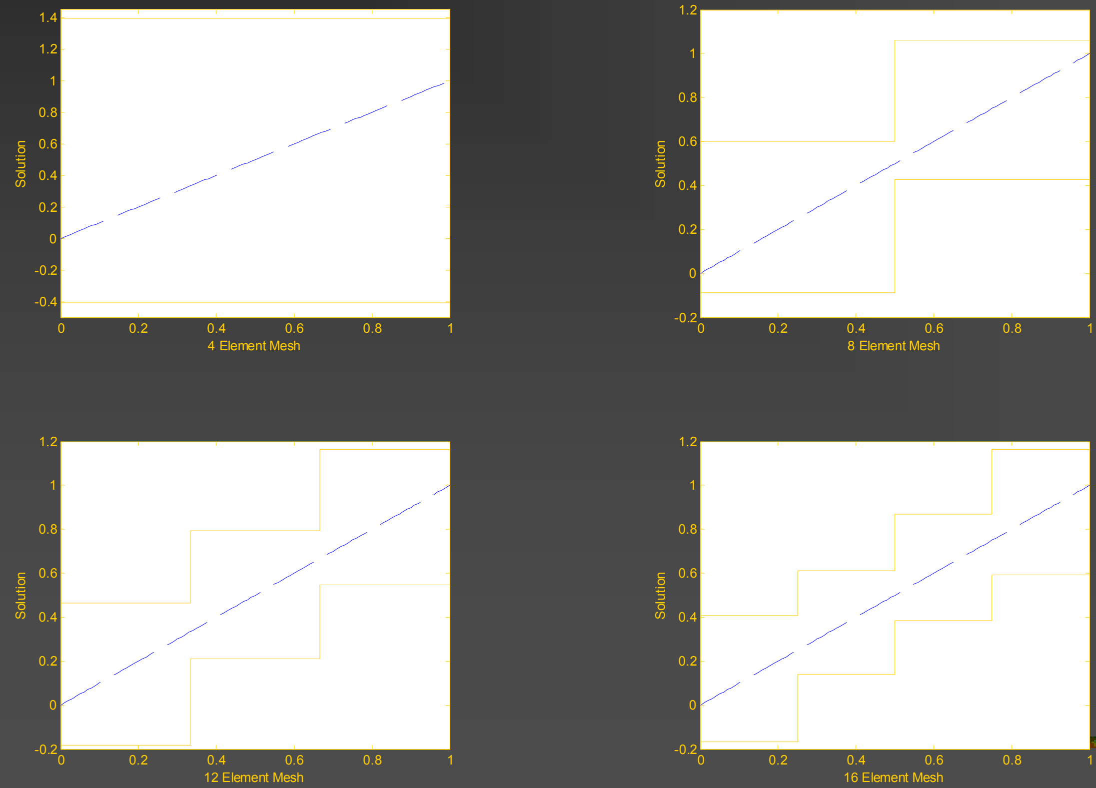
Example 5 (Cont.)

The interval bounds (solid line) are shown enclosing the true solution (dashed line) for these nodes.



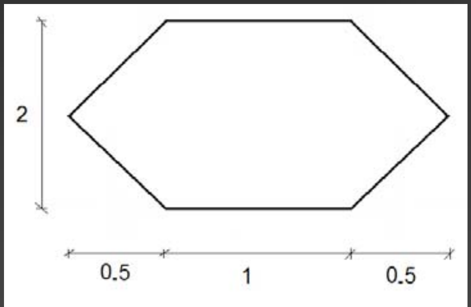
Example 5 (Cont.)

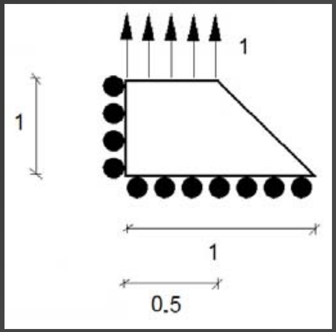
The interval bounds (solid line) enclose the true solution (dashed line) of the y-direction displacement for the right edge for the four meshes.

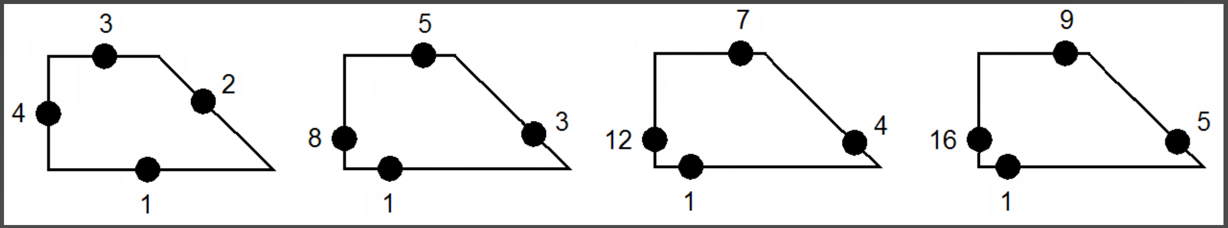


Example 6

The second example shows the behavior of the solution for a hexagonal plate in tension. A symmetry model is considered to decrease computational time. Four uniform meshes are analyzed.

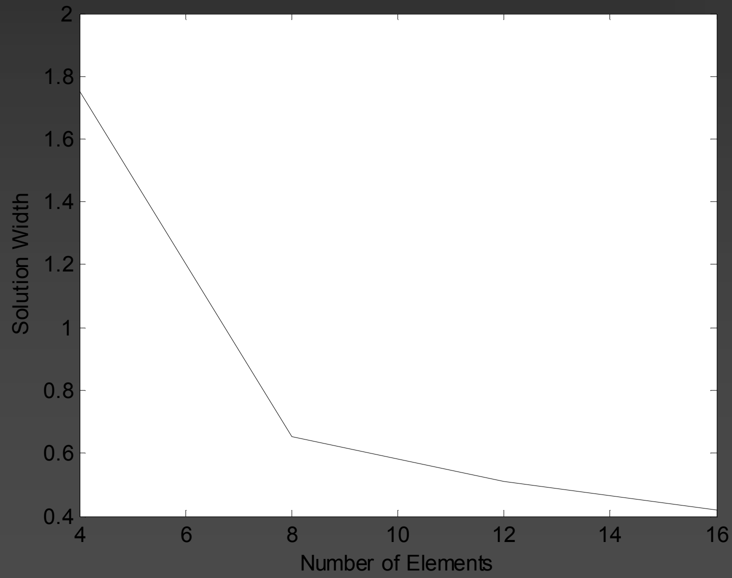


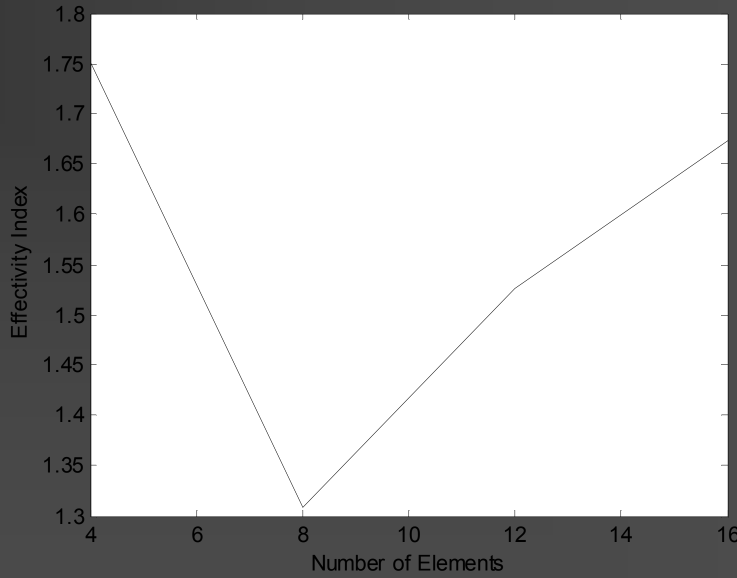




Example 6 (Cont.)

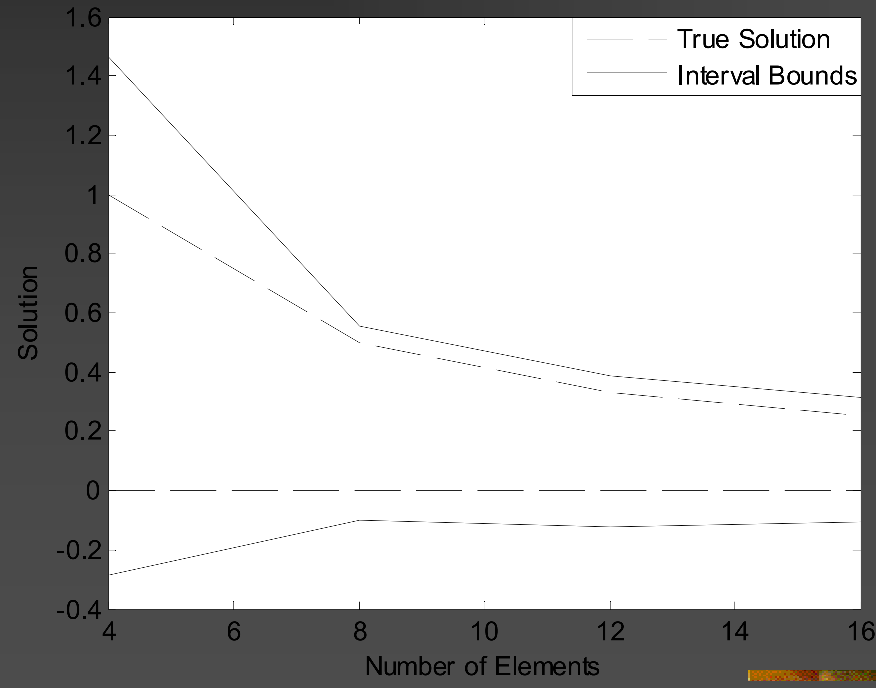
The interval solution width for the y-direction displacement (left) and the effectivity index (right) for the left lower corner elements, 4, 8, 12, and 16, for the four respective meshes are shown.





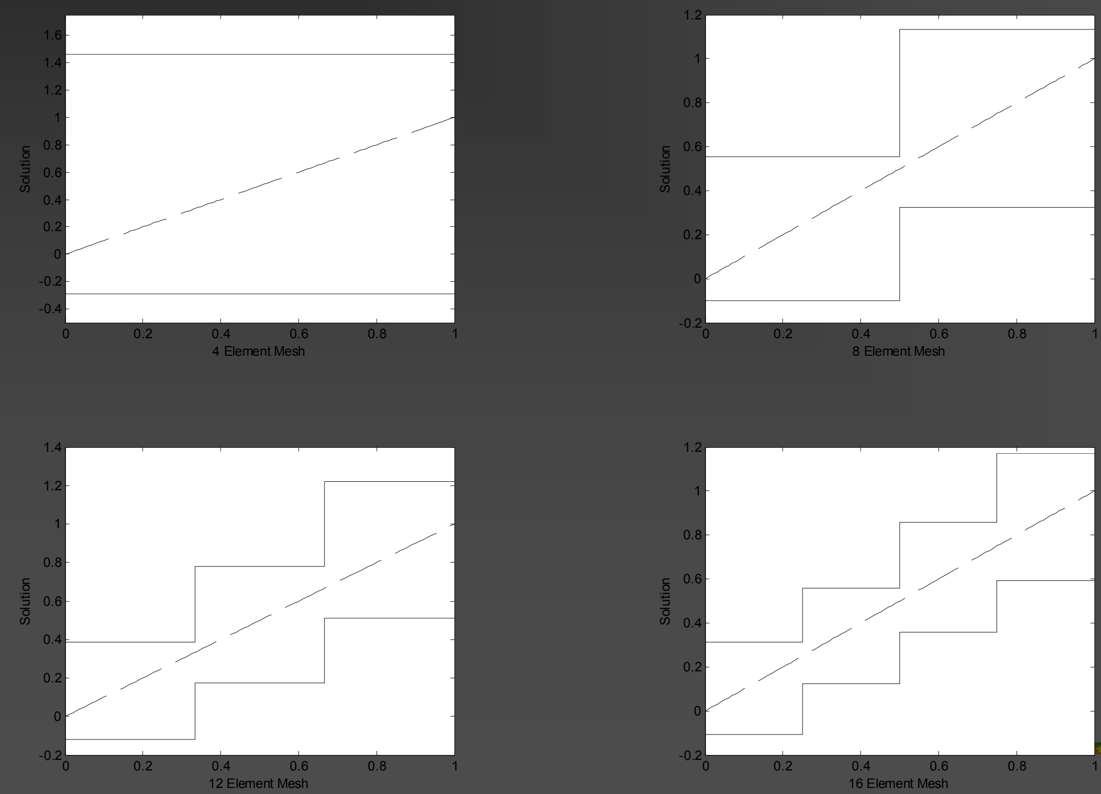
Example 6 (Cont.)

The interval bounds (solid line) are shown enclosing the true solution (dashed line) for these nodes.



Example 6 (Cont.)

The interval bounds (solid line) enclose the true solution (dashed line) of the y-direction displacement for the left edge for the four meshes.

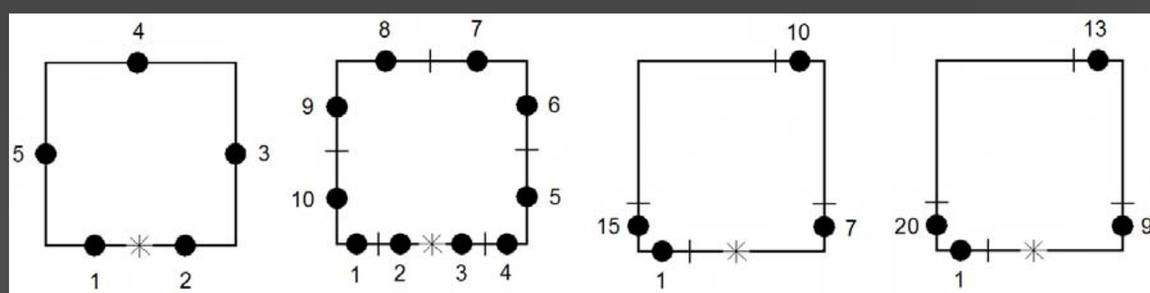


Example 7

The following example solves a Laplace equation whose exact solution is:

$$\left.\begin{aligned} u &= \sinh(x)\sin(y) \quad on\ \Gamma_1 \\ q &= \frac{0.5\cos\left[0.5\arctan\left(\dfrac{y}{x-0.5}\right)\right]}{\sqrt{\sqrt{(x-0.5)^2+y^2}}} \quad on\ \Gamma_2 \end{aligned}\right\}$$

using four uniform constant element mesh.

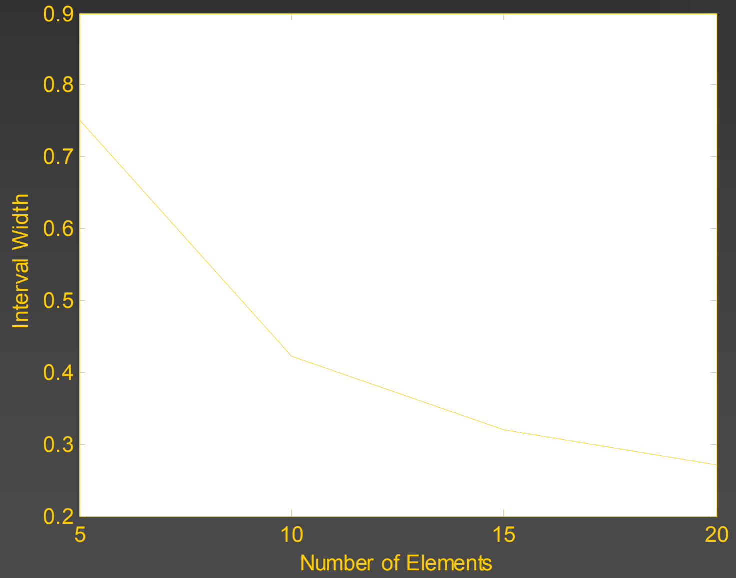


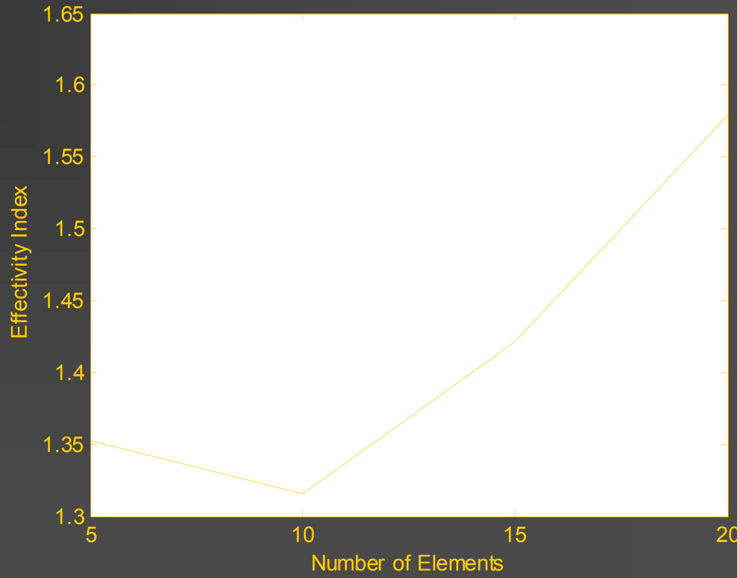
Example 7 (Cont.)

The potential boundary conditions are applied on the top edge and on the right part of the bottom edge from the singularity. The flux boundary conditions were applied on the left and right edges and on the left part of the bottom edge from the singularity. Twenty subintervals were used to obtain nearly sharp interval bounds.

Example 7 (Cont.)

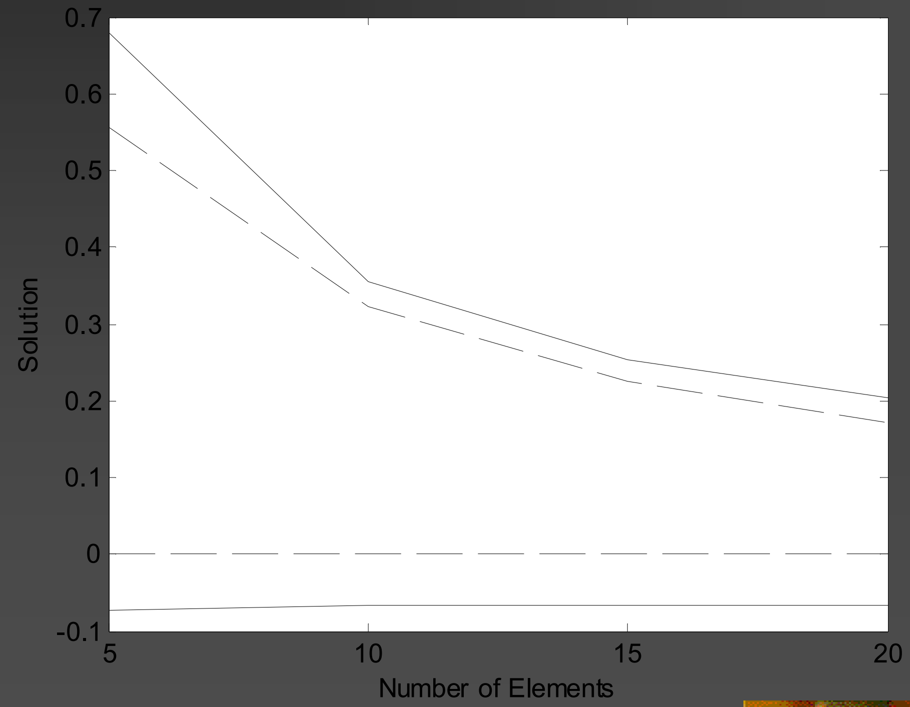
The interval solution width for the potential (left) and the effectivity index (right) for the right lower corner elements, 3, 5, 7, and 9, for the four respective meshes are shown.





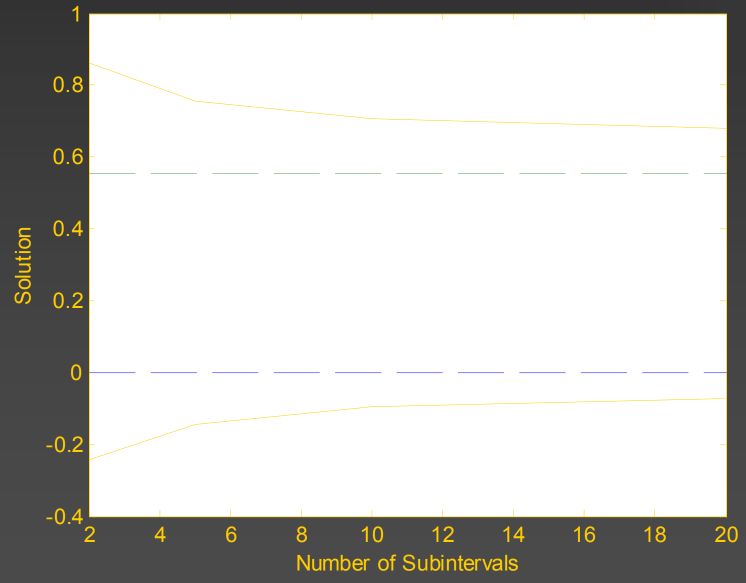
Example 7 (Cont.)

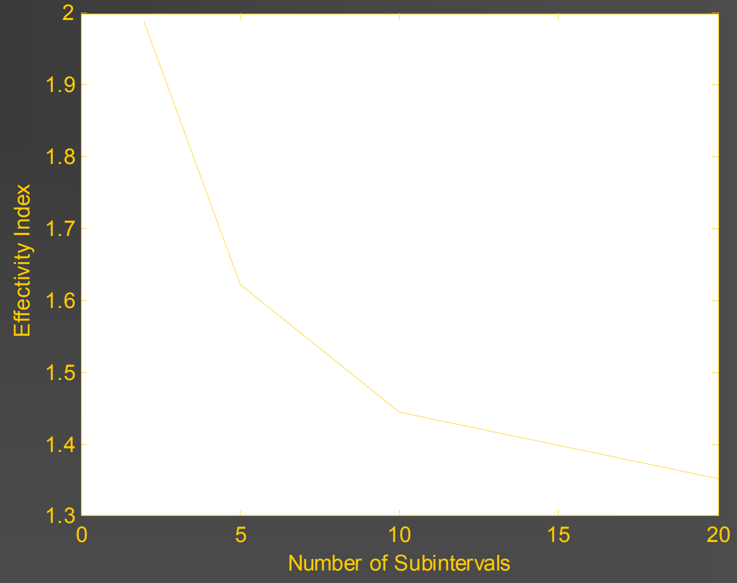
- The interval bounds (solid line) are shown enclosing the true solution (dashed line) for these nodes.



Example 7 (Cont.)

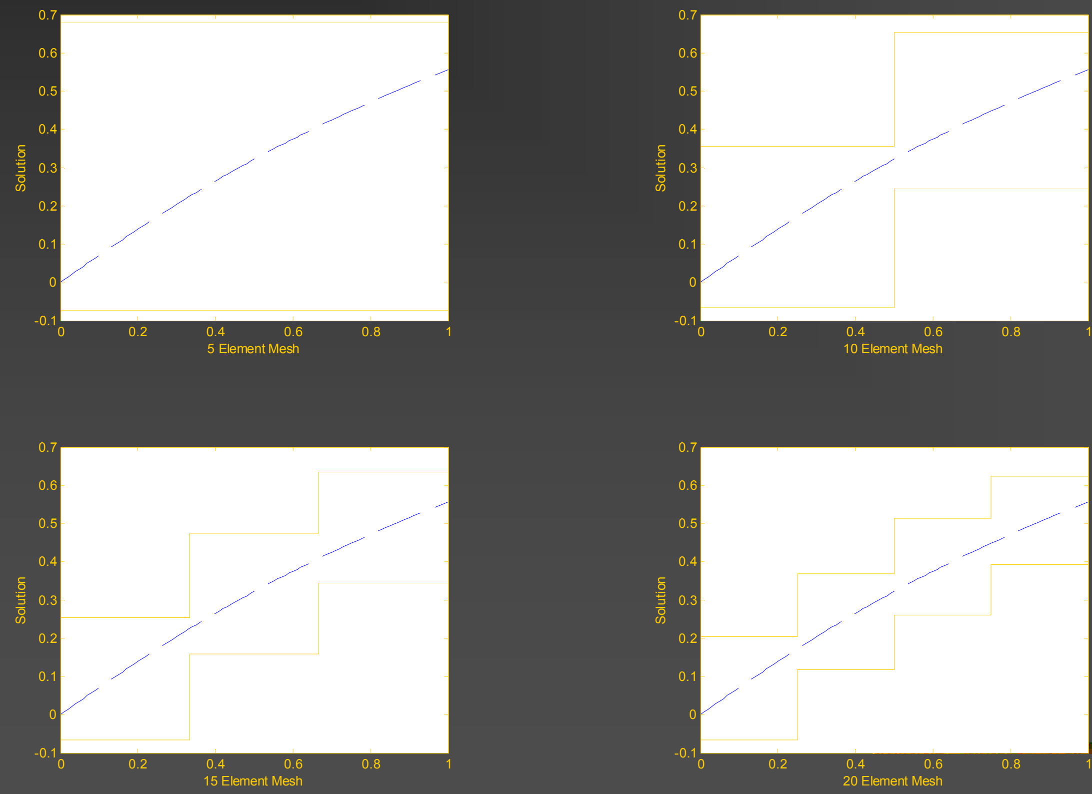
- The behavior of the interval solution bounds (solid line) for the potential with parameterization is shown and compared with the true solution (dashed line) for node 3 in the 5 element mesh.





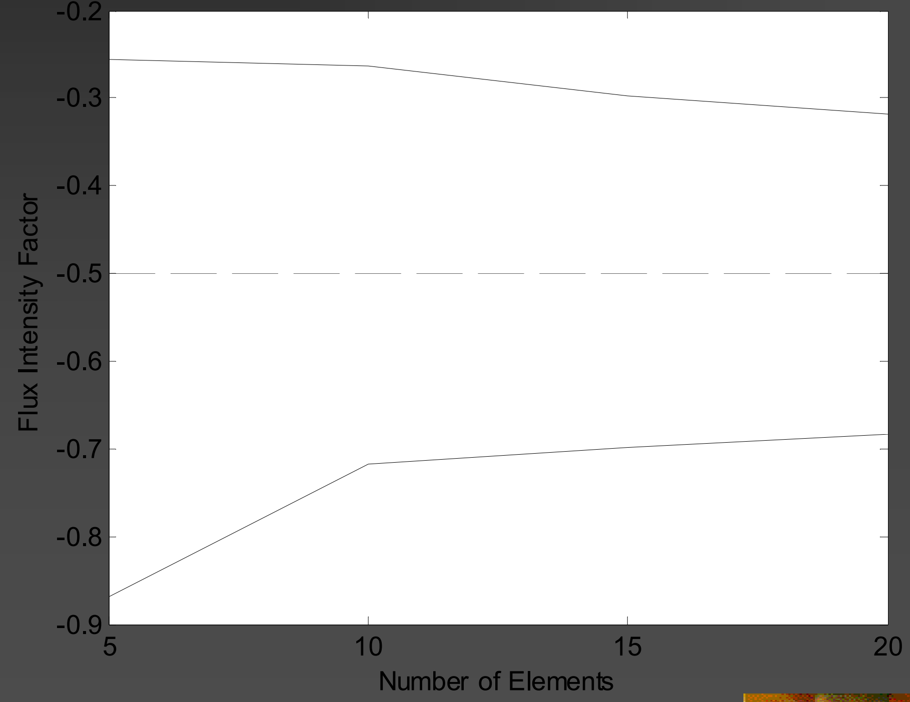
Example 7 (Cont.)

The interval bounds (solid line) enclose the true solution (dashed line) for the right edge for the four meshes.



Example 7 (Cont.)

The bounds on the flux intensity factor (solid line) enclosing the true solution (dashed line) are shown for elements 2, 3, 4, and 5 for the sour respective meshes.



Outline

- COMPUTING ON SETS: Random Sets, Fuzzy Sets, Measures on Borel Sets, Convex Sets, Clouds
- Interval Operations
- Examples of progress in Reliable Engineering Analysis
- Validation Errors:

  Errors in representing the physics

  Errors in parameters used in model

  Interval Finite Elements
- Verification Errors

  Errors in discretization of model

  Errors in computations (truncation errors)

  Interval Boundary Elements
- **Conclusions**

Conclusions

- Interval finite element methods (IFEM) can provide sharp bounds on system response for static and dynamic problems.
- Interval boundary elements can provide effective worse case bounds on discretization errors as well as truncation errors and boundary uncertainty.

Conclusions

Interval methods can provide a foundation for computationally efficient methods for treating uncertainties in engineering calculation.

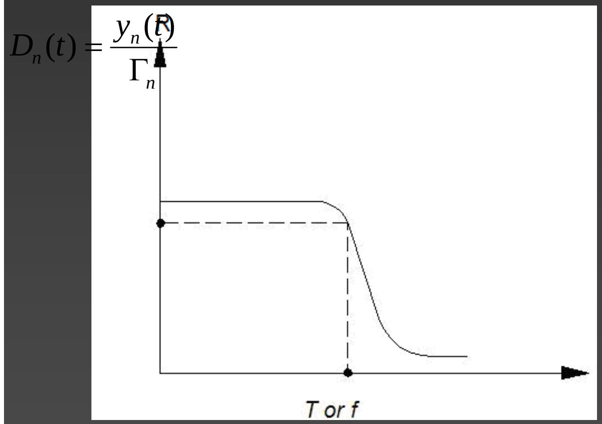
Quantification of the uncertainty in a calculation is essential to measure the information provided by the analysis.

Thank you

Rene Magritte, Clairvoyance, 1936

Structural Dynamics, Stability …

- Response Spectrum Analysis



$$\{U_{n,\max}\} = (D_{n,\max})(\Gamma_n)\{\varphi_n\}$$

$$\Gamma_n = \frac{\{\varphi_n\}^T\{P\}}{M_n} = \frac{\{\varphi_n\}^T\{P\}}{\{\varphi_n\}^T[M]\{\varphi_n\}}$$

$$D_n(t) = \frac{y_n(t)}{\Gamma_n}$$

Examples (Cont.)

Results:

Node Value	Lower Bound	Exact Lower Bound	Exact Upper Bound	Upper Bound	Middle Value	Width	Effective Width	Mid-point Node Solution
q1	-1.4814	-1	-1	-0.7383	-1.1099	0.7431	N.A.	-1.1746
u2	-0.0221	0	1	1.0515	0.5147	1.0736	1.0736	0.5000
q3	0.7657	1	1	1.4371	1.1014	0.6714	N.A.	1.1746
u4	-0.0221	0	1	1.0515	0.5147	1.0736	1.0736	0.5000

Examples (Cont.)

Results:

Node Value	Lower Bound	Exact Lower Bound	Exact Upper Bound	Upper Bound	Middle Value	Width	Effective Width	Mid-point Node Solution
q1	-1.4793	-1	-1	-0.7709	-1.1251	0.7084	N.A.	-1.0586
q2	-1.4807	-1	-1	-0.7730	-1.1269	0.7077	N.A.	-1.0586
u3	-0.0244	0	0.5	0.5140	0.2448	0.5384	1.0769	0.2414
u4	0.4749	0.5	1	1.0653	0.7701	0.5904	1.1808	0.7586
q5	0.8354	1	1	1.3893	1.1123	0.5539	N.A.	1.0586
q6	0.8350	1	1	1.3897	1.1123	0.5546	N.A.	1.0586
u7	0.4747	0.5	1	1.0655	0.7701	0.5908	1.1816	0.7586
u8	-0.0253	0	0.5	0.5150	0.2448	0.5403	1.0806	0.2414

Examples (Cont.)

Results:

Node Value	Lower Bound	Exact Lower Bound	Exact Upper Bound	Upper Bound	Middle Value	Width	Effective Width	Mid-point Node Solution
q1	-1.4525	-1	-1	-0.8603	-1.1564	0.5922	N.A.	-1.0591
q2	-1.3434	-1	-1	-0.6169	-0.9801	0.7265	N.A.	-0.9752
q3	-1.4525	-1	-1	-0.8603	-1.1564	0.5922	N.A.	-1.0591
u4	-0.0213	0	1/3	0.3351	0.1569	0.3564	1.0693	0.1591
u5	0.3003	1/3	2/3	0.7054	0.5028	0.4051	1.2153	0.5000
u6	0.6397	2/3	1	1.0623	0.8510	0.4226	1.2678	0.8409
q7	0.8971	1	1	1.3790	1.1380	0.4819	N.A.	1.0591
q8	0.7218	1	1	1.2187	0.9703	0.4968	N.A.	0.9752
q9	0.8971	1	1	1.3790	1.1380	0.4819	N.A.	1.0591
u10	0.6397	2/3	1	1.0623	0.8510	0.4226	1.2678	0.8409
u11	0.3003	1/3	2/3	0.7054	0.5028	0.4051	1.2153	0.5000
u12	-0.0213	0	1/3	0.3351	0.1569	0.3564	1.0693	0.1591

Examples (Cont.)

Results:

Node Value	Lower Bound	Exact Lower Bound	Exact Upper Bound	Upper Bound	Middle Value	Width	Effective Width	Mid-point Node Solution
q1	-1.4510	-1	-1	-0.8578	-1.1544	0.5932	N.A.	-1.0552
q2	-1.3739	-1	-1	-0.5905	-0.9822	0.7834	N.A.	-0.9842
q3	-1.3738	-1	-1	-0.5907	-0.9822	0.7831	N.A.	-0.9842
q4	-1.4584	-1	-1	-0.8864	-1.1724	0.5720	N.A.	-1.0552
u5	-0.0232	0	0.25	0.2571	0.1170	0.2803	1.1210	0.1188
u6	0.2243	0.25	0.5	0.5221	0.3732	0.2979	1.1914	0.3732
u7	0.4687	0.5	0.75	0.7919	0.6303	0.3232	1.2929	0.6268
u8	0.7191	0.75	1	1.0610	0.8901	0.3419	1.3678	0.8812
q9	0.9282	1	1	1.3660	1.1471	0.4379	N.A.	1.0552
q10	0.7288	1	1	1.2321	0.9804	0.5033	N.A.	0.9842
q11	0.7288	1	1	1.2321	0.9804	0.5033	N.A.	0.9842
q12	0.9282	1	1	1.3660	1.1471	0.4379	N.A.	1.0552
u13	0.7191	0.75	1	1.0610	0.8901	0.3419	1.3676	0.8812
u14	0.4687	0.5	0.75	0.7919	0.6303	0.3232	1.2929	0.6268
u15	0.2391	0.25	0.5	0.5320	0.3856	0.2929	1.1716	0.3732
u16	-0.0233	0	0.25	0.2572	0.1170	0.2805	1.1219	0.1188

Examples (Cont.)

Results This is way too small to read. You should show a key point(s) and the associated values

Node Value	Lower Bound	Exact Lower Bound	Exact Upper Bound	Upper Bound	Middle Value	Width	Effective Width	Mid-point Node Solution
q1	-1.4574	-1	-1	-0.6554	-1.0564	0.8020	N.A.	-1.0532
q2	-1.3879	-1	-1	-0.5738	-0.9809	0.8140	N.A.	-0.9838
q3	-1.3554	-1	-1	-0.6445	-0.9999	0.7110	N.A.	-0.9949
q4	-1.3879	-1	-1	-0.5738	-0.9809	0.8140	N.A.	-0.9838
q5	-1.4574	-1	-1	-0.6554	-1.0564	0.8020	N.A.	-1.0532
u6	-0.0233	0	0.2	0.2076	0.0921	0.2310	1.1548	0.0948
u7	0.1767	0.2	0.4	0.4171	0.2969	0.2404	1.2019	0.2979
u8	0.3748	0.4	0.6	0.6267	0.5007	0.2518	1.2591	0.5000
u9	0.5697	0.6	0.8	0.8415	0.7056	0.2718	1.3589	0.7021
u10	0.7674	0.8	1	1.0601	0.9138	0.2927	1.4635	0.9052
q11	0.7221	1	1	1.3577	1.0399	0.6356	N.A.	1.0532
q12	0.7266	1	1	1.2352	0.9809	0.5086	N.A.	0.9838
q13	0.7428	1	1	1.2420	0.9924	0.4992	N.A.	0.9949
q14	0.7266	1	1	1.2352	0.9809	0.5086	N.A.	0.9838
q15	0.7222	1	1	1.3577	1.0399	0.6355	N.A.	1.0532
u16	0.7674	0.8	1	1.0601	0.9138	0.2927	1.4635	0.9052
u17	0.5697	0.6	0.8	0.8415	0.7056	0.2718	1.3589	0.7021
u18	0.3748	0.4	0.6	0.6267	0.5007	0.2518	1.2591	0.5000
u19	0.1767	0.2	0.4	0.4171	0.2969	0.2404	1.2019	0.2979
u20	-0.0233	0	0.2	0.2076	0.0921	0.2310	1.1548	0.0948