Interval Finite Elements as a Basis for Generalized Models of Uncertainty in Engineering Analysis

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Outline

- Introduction
- Interval Arithmetic
- Interval Finite Elements
- Element-By-Element
- Examples
- Conclusions
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- Interval Arithmetic
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Introduction- Uncertainty

- Uncertainty is unavoidable in engineering system
  - structural mechanics entails uncertainties in material, geometry and load parameters (aleatory-epistemic)

- Probabilistic approach is the traditional approach
  - requires sufficient information to validate the probabilistic model
  - criticism of the credibility of probabilistic approach when data is insufficient (Elishakoff, 1995; Ferson and Ginzburg, 1996; Möller and Beer, 2007)
Introduction- Interval Approach

- Nonprobabilistic approach for uncertainty modeling when only range information (tolerance) is available
  \[ t = t_0 \pm \delta \]
- Represents an uncertain quantity by giving a range of possible values
  \[ t = [t_0 - \delta, t_0 + \delta] \]
- How to define bounds on the possible ranges of uncertainty?
  - experimental data, measurements, statistical analysis, expert knowledge
Introduction- Why Interval?

- Simple and elegant
- Conforms to practical tolerance concept
- Describes the uncertainty that can not be appropriately modeled by probabilistic approach
- Computational basis for other uncertainty approaches (e.g., fuzzy set, random set, imprecise probability)

- Provides guaranteed enclosures
Outline

- Introduction
- **Interval Arithmetic**
- Interval Finite Elements
- Element-By-Element
- Examples
- Conclusions
Interval arithmetic — Background

- Archimedes (287 – 212 B.C.)
  - A circle of radius one has an area equal to $\pi$
Interval arithmetic — Background

- Archimedes (287 – 212 B.C.)
  - A circle of radius one has an area equal to \( \pi \)
  - \( 2 < \pi < 4 \)

\[
\frac{10}{3} < \pi < \frac{1}{7} < \frac{1}{71} < \pi < 3\frac{1}{7}
\]

\( \pi = [3.14085, 3.14286] \)
Interval arithmetic — Background

- Modern interval arithmetic
  - Physical constants or measurements
    \[ g \in [9.8045, 9.8082] \]
  - Representation of numbers
    \[ 1/3 \approx 0.3333\ldots \quad \sqrt{2} \approx 1.4142\ldots \quad \pi \approx 3.1416\ldots \]
    \[ 1/3 \in [0.3333, 0.3334] \quad \sqrt{2} \in [1.4142, 1.4143] \quad \pi \in [3.1415, 3.1416] \]
  - Rounding errors
    \[ 1/0.12345 \approx 8.1004 \quad 1/0.12345 \in [8.1004, 8.1005] \]
Interval arithmetic

- Interval number represents a range of possible values within a closed set

\[ x \equiv [\underline{x}, \bar{x}] := \{ x \in R \mid \underline{x} \leq x \leq \bar{x} \} \]
Interval Operations

Let $x = [a, b]$ and $y = [c, d]$ be two interval numbers

1. Addition
   
   $x + y = [a, b] + [c, d] = [a + c, b + d]$

2. Subtraction
   
   $x - y = [a, b] - [c, d] = [a - d, b - c]$

3. Multiplication
   
   $xy = \left[ \min(ac, ad, bc, bd), \max(ac, ad, bc, bd) \right]$

4. Division
   
   $1 / x = \left[ \frac{1}{b}, \frac{1}{a} \right]$
Properties of Interval Arithmetic

Let $x, y$ and $z$ be interval numbers

1. Commutative Law
   \[ x + y = y + x \]
   \[ xy = yx \]

2. Associative Law
   \[ x + (y + z) = (x + y) + z \]
   \[ x(yz) = (xy)z \]

3. *Distributive Law does not always hold, but*
   \[ x(y + z) \subseteq xy + xz \]
The **DEPENDENCY** problem arises when one or several variables occur more than once in an interval expression

- $f(x) = x - x, \quad x = [1, 2]$
- $f(x) = [1 - 2, 2 - 1] = [-1, 1] \neq 0$
- $f(x, y) = \{ f(x, y) = x - y \mid x \in x, y \in y \}$

- $f(x) = x (1 - 1) \Rightarrow f(x) = 0$
- $f(x) = \{ f(x) = x - x \mid x \in x \}$
Sharp Results – Overestimation

- If $a$, $b$ and $c$ are interval numbers, then:
  \[ a (b \pm c) \subseteq ab \pm ac \]

- If we set
  \[ a = [-2, 2]; \quad b = [1, 2]; \quad c = [-2, 1], \]
  we get
  \[ a (b + c) = [-2, 2]([1, 2] + [-2, 1]) = [-2, 2] [-1, 3] = [-6, 6] \]

- However,
  \[ ab + ac = [-2, 2][1, 2] + [-2, 2][-2, 1] = [-4, 4] + [-4, 4] = [-8, 8] \]
Sharp Results – Overestimation

- Interval Vectors and Matrices
  - An interval matrix is such matrix that contains all real matrices whose elements are obtained from all possible values between the lower and upper bounds of its interval components

\[ A = \{ A \in \mathbb{R}^{m \times n} \mid A_{ij} \in A_{ij} \text{ for } i = 1, \ldots, m; \ j = 1, \ldots, n \} \]
Sharp Results – Overestimation

- Let $a$, $b$, $c$ and $d$ be independent variables, each with interval $[1, 3]$

\[ A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}, \quad A \times B = \begin{pmatrix} [-2, 2] & [-2, 2] \\ [-2, 2] & [-2, 2] \end{pmatrix} \]

\[ A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad B \text{phys} = \begin{pmatrix} b & -b \\ -b & b \end{pmatrix}, \quad A \times B \text{phys} = \begin{pmatrix} [b-b] & [b-b] \\ [b-b] & [b-b] \end{pmatrix} \]

\[ A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad B^\ast \text{phys} = b \times \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad A \times B^\ast \text{phys} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \]
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Finite Elements

Finite Element Method (FEM) is a numerical method that provides approximate solutions to differential equations (ODE and PDE)
Finite Elements - Uncertainty & Errors

- Mathematical model (validation)
- Discretization of the mathematical model into a computational framework
- Parameter uncertainty (loading, material properties)
- Rounding errors
Interval Finite Elements

Uncertain Data

Geometry
Materials
Loads

Interval Stiffness Matrix
\[ K = \int B^T CB \, dV \]

Interval Load Vector

Element Level

\[ K U = F \]
Interval Finite Elements

\[ K U = F \]

\[ K = \int B^T C B \, dV \quad = \text{Interval element stiffness matrix} \]

*\[ B = \text{Interval strain-displacement matrix} \]*

*\[ C = \text{Interval elasticity matrix} \]*

\[ F = [F_1, \ldots, F_i, \ldots, F_n] = \text{Interval element load vector (traction)} \]

\[ F_i = \int N_i \, t \, dA \]

*\[ N_i = \text{Shape function corresponding to the } i\text{-th DOF} \]*

*\[ t = \text{Surface traction} \]*
Interval Finite Elements (IFEM)

- Follows conventional FEM
- Loads, geometry and material property are expressed as interval quantities
- System response is a function of the interval variables and therefore varies in an interval
- Computing the exact response range is proven NP-hard
- The problem is to estimate the bounds on the unknown exact response range based on the bounds of the parameters
IFEM- Inner-Bound Methods

- Combinatorial method (Muhanna and Mullen 1995, Rao and Berke 1997)
- Sensitivity analysis method (Pownuk 2004)
- Perturbation (Mc William 2000)
- Monte Carlo sampling method

- Need for alternative methods that achieve
  - Rigorousness – guaranteed enclosure
  - Accuracy – sharp enclosure
  - Scalability – large scale problem
  - Efficiency
IFEM- Enclosure

- Linear static finite element
  - Popova 2003, and Kramer 2004
  - Neumaier and Pownuk 2004
  - Corliss, Foley, and Kearfott 2004

- Dynamic
  - Dessombz, 2000

- Free vibration-Buckling
  - Modares, Mullen 2004, and Billini and Muhanna 2005
Interval Finite Elements

- Interval Linear System of Equations

\[ A \mathbf{x} = \mathbf{b} \]

\[
\begin{pmatrix}
2 & [-1, 0] \\
[-1, 0] & 2
\end{pmatrix}
\times
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
= \begin{pmatrix} 1.2 \\ -1.2 \end{pmatrix}
\]

Then \( A \in A \) iff

\[
A: = \begin{pmatrix}
2 & -\alpha \\
-\beta & 2
\end{pmatrix}
\]

with \( \alpha, \beta \in [0, 1] \)
Interval Finite Elements

\[ A^H b = \Diamond S(A, b) = \begin{bmatrix} [0.3, 0.6] \\ [-0.6, -0.3] \end{bmatrix} \]

\[ A^{-1} b = \begin{bmatrix} [0.2, 0.8] \\ [-0.8, -0.2] \end{bmatrix} \neq \but \supset A^H b \]

\[ \Diamond S_{sym}(A, b) = \begin{bmatrix} [0.4, 0.6] \\ [-0.6, -0.4] \end{bmatrix} \neq A^H b \]

Points:

\[ P_1 = (0.3, -0.6) \]
\[ P_2 = (0.6, -0.6) \]
\[ P_3 = (0.6, -0.3) \]
\[ P_4 = (0.4, -0.4) \]
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**Naïve interval FEA**

\[
\begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \Rightarrow \begin{pmatrix} [2.85, 3.15] & [-2.1, -1.9] \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 1 \end{pmatrix}
\]

- exact solution: \( u_2 = [1.429, 1.579], \quad u_3 = [1.905, 2.105] \)
- naïve solution: \( u_2 = [-0.052, 3.052], \quad u_3 = [0.098, 3.902] \)
- interval arithmetic assumes that all coefficients are independent
- uncertainty in the response is severely overestimated (1900%)
Element-By-Element (EBE) technique

- elements are detached – no element coupling
- structure stiffness matrix is block-diagonal \((k_1, \ldots, k_{Ne})\)
- the size of the system is increased

\[
u = (u_1, \ldots, u_{Ne})^T
\]

- need to impose necessary constraints for compatibility and equilibrium

Element-By-Element model
Element-By-Element

Suppose the modulus of elasticity is interval:

\[ E = \hat{E}(1 + \delta) \]

\( \delta \): zero-midpoint interval

The element stiffness matrix can be split into two parts,

\[ k = \hat{k}(I + d) = \hat{k} + \hat{kd} \]

\( \hat{k} \): deterministic part, element stiffness matrix evaluated using \( \hat{E} \),

\( \hat{kd} \): interval part

\( d \): interval diagonal matrix, diag(\( \delta \),...,\( \delta \)).
Element-By-Element

- Element stiffness matrix: \( k = \hat{k}(I + d) \)
- Structure stiffness matrix:
  \[
  K = \hat{K}(I + D) = \hat{K} + \hat{K}D
  \]
  or
  \[
  K = \begin{pmatrix}
  k_1 & & \\
  & \ddots & \\
  & & k_{Ne}
  \end{pmatrix} = \begin{pmatrix}
  \hat{k}_1 & & \\
  & \ddots & \\
  & & \hat{k}_{Ne}
  \end{pmatrix} \begin{pmatrix}
  I + \\
  & \ddots \\
  & & d_{Ne}
  \end{pmatrix}
  \]
Constraints

Impose necessary constraints for compatibility and equilibrium
- Penalty method
- Lagrange multiplier method

Element-By-Element model
Constraints – penalty method

Constraint conditions: $cu = 0$

Using the penalty method:

$$(K + Q)u = p$$

$Q$: penalty matrix, $Q = c^T \eta c$

$\eta$: diagonal matrix of penalty number $\eta_i$

Requires a careful choice of the penalty number

A spring of large stiffness is added to force node 2 and node 3 to have the same displacement.
Constraints — Lagrange multiplier

Constraint conditions: \( cu = 0 \)

Using the Lagrange multiplier method:

\[
\begin{pmatrix}
K & c^T \\
c & 0
\end{pmatrix}
\begin{pmatrix}
u \\
\lambda
\end{pmatrix}
=
\begin{pmatrix}
p \\
0
\end{pmatrix}
\]

\( \lambda \): Lagrange multiplier vector, introduced as new unknowns
Load in EBE

Nodal load \( p_b \)

\[ p_b = (p_1, \ldots, p_{N_e})^T \]

where \( p_i = \int N^T \phi(x) dx \)

Suppose the surface traction \( \phi(x) \) is described by an interval function: 

\[ \phi(x) = \sum_{j=0}^{m} a_j x^j. \]

\( p_b \) can be rewritten as 

\[ p_b = WF \]

\( W \): deterministic matrix 

\( F \): interval vector containing the interval coefficients of the surface traction
Fixed point iteration

- For the interval equation $Ax = b$,
  - preconditioning: $RAx = Rb$, $R$ is the preconditioning matrix
  - transform it into $g(x^*) = x^*$:
    \[ Rb - RAx_0 + (I - RA)x^* = x^*, \quad x = x^* + x_0 \]
- **Theorem** (Rump, 1990): for some interval vector $x^*$,
  - if $g(x^*) \subseteq \text{int}(x^*)$
  - then $A^Hb \subseteq x^* + x_0$
- Iteration algorithm:
  - iterate: $x^{*(l+1)} = z + G(e \cdot x^{*(l)})$
  - where $z = Rb - RAx_0$, $G = I - RA$, $R = \hat{A}^{-1}$, $\hat{A}x_0 = \hat{b}$
- No dependency handling
Fixed point iteration

- Interval FEA calls for a modified method which exploits the special form of the structure equations

\[(K + Q)u = p \text{ with } K = \hat{K} + \hat{K}D\]

- Choose \( R = (\hat{K} + Q)^{-1} \), construct iterations:

\[
u^{*(l+1)} = Rp - R(K + Q)u_0 + (I - R(K + Q))(\varepsilon \cdot u^{*(l)})
\]

\[= Rp - u_0 - \hat{K}D(u_0 + \varepsilon \cdot u^{*(l)})\]

\[= Rp - u_0 - \hat{K}M^{(l)}\Delta\]

if \( u^{*(l+1)} \subseteq \text{int}(u^{*(l)}) \), then \( u = u^{*(l+1)} + u_0 = Rp - \hat{K}M^{(l)}\Delta \)

\( \Delta \): interval vector, \( \Delta = (\delta_1, ..., \delta_{N_e})^T \)

The interval variables \( \delta_1, ..., \delta_{N_e} \) appear only once in each iteration.
Convergence of fixed point

- The algorithm converges if and only if $\rho(|G|) < 1$.
  - Smaller $\rho(|G|) \Rightarrow$ less iterations required, and less overestimation in results.
- To minimize $\rho(|G|)$:
  - Choose $R = \hat{A}^{-1}$ so that $G = I - RA$ has a small spectral radius.
  - Reduce the overestimation in $G$:
    $$G = I - RA = I - (\hat{K} + Q)^{-1}(\hat{K} + Q + \hat{K}D) = -R\hat{K}D$$
Stress calculation

- Conventional method:
  \[ \sigma = CBu_e, \text{ (severe overestimation)} \]
  \( C \): elasticity matrix, \( B \): strain-displacement matrix

- Present method:
  \[ E = (1 + \delta)\hat{E}, \quad C = (1 + \delta)\hat{C} \]
  \[ \sigma = CBLu \]
  \[ = CBL(Rp - R\hat{C}M^{(l)}\Delta) \]
  \[ = (1 + \delta)(\hat{CBLR}p - \hat{CBLR}\hat{K}M^{(l)}\Delta) \]

\( L \): Boolean matrix, \( Lu = u_e \)
Element nodal force calculation

- Conventional method:
  \[ f = T_e (k u_e - p_e), \quad \text{(severe overestimation)} \]

- Present method:
  \[
  \begin{align*}
  \left( T_e \right)_1 (k_1 u_e)_1 & - (p_e)_1 \\
  \vdots \\
  \left( T_e \right)_{N_e} (k_{N_e} u_e)_{N_e} & - (p_e)_{N_e}
  \end{align*}
  \]
  in the EBE model, \( T(Ku - p_b) = \)
  \[
  \begin{align*}
  \left( T_e \right)_1 (k_1 u_e)_1 & - (p_e)_1 \\
  \vdots \\
  \left( T_e \right)_{N_e} (k_{N_e} u_e)_{N_e} & - (p_e)_{N_e}
  \end{align*}
  \]
  from \((K + Q)u = p_c + p_b \Rightarrow T(Ku - p_b) = T(p_c - Qu)\)
  Calculate \( T(p_c - Qu) \) to obtain the element nodal forces for all elements.
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Numerical example

- Examine the rigorousness, accuracy, scalability, and efficiency of the present method
- Comparison with the alternative methods
  - the combinatorial method, sensitivity analysis method, and Monte Carlo sampling method
  - these alternative methods give inner estimation

\[ x : \text{exact solution, } x_i : \text{inner bound, } x_o : \text{outer bound} \]
Examples – Load Uncertainty

- Four-bay forty-story frame
Examples – Load Uncertainty

- Four-bay forty-story frame

Loading A  Loading B  Loading C  Loading D
Examples – Load Uncertainty

- Four-bay forty-story frame

Total number of floor load patterns

\[2^{160} = 1.46 \times 10^{48}\]

If one were able to calculate

10,000 \textit{patterns} / \text{s}

there has not been sufficient time since
the creation of the universe (4-8) billion
years? to solve all load patterns for this
simple structure

Material \textit{A36}, Beams \textit{W24 x 55},
Columns \textit{W14 x 398}
Examples – Load Uncertainty

- Four-bay forty-story frame

Four bay forty floor frame - Interval solutions for shear force and bending moment of first floor columns

<table>
<thead>
<tr>
<th>Elements</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nodes</td>
<td>1</td>
<td>6</td>
<td>2</td>
</tr>
</tbody>
</table>
| Combination solution | Total number of required combinations = $1.461501637 \times 10^{48}$
| Interval  | Axial force (kN) | [-2034.5, 185.7] | [-2161.7, 0.0] | [-2226.7, 0.0] |
| solution  | Shear force (kN) | [-5.1, 0.9] | [-5.8, 5.0] | [-5.0, 5.0] |
| Moment (kN m) | [-10.3, 4.5] | [-15.3, 5.4] | [-10.6, 9.3] | [-17, 15.2] | [-8.9, 8.9] | [-16, 16] |
Examples — Load Uncertainty

> Ten-bay truss

\[ A = 0.006 \text{ m}^2 \]
\[ E = 2.0 \times 10^8 \text{ kPa} \]

\[ F = [-4.28, 28.3] \text{ kN} \]

\[ F_{\text{min}} = -(0.062 + 0.139 + 0.113) \times 20 = -4.28 \text{ kN} \]

\[ F_{\text{max}} = (0.464 + 0.309 + 0.258 + 0.192 + 0.128 + 0.064) \times 20 = 28.3 \text{ kN} \]
Examples – Load Uncertainty

➢ Three-Span Beam

![Diagram of a three-span beam with load distribution and bending moments.](image)
Truss structure

$A_1, A_2, A_3, A_{13}, A_{14}, A_{15} : [9.95, 10.05] \text{ cm}^2 (1\% \text{ uncertainty})$

cross-sectional area

of all other elements: $[5.97, 6.03] \text{ cm}^2 (1\% \text{ uncertainty})$

modulus of elasticity of all elements: 200,000 MPa

$p_1 = [190, 210] \text{ kN}, p_2 = [95, 105] \text{ kN}$

$p_3 = [95, 105] \text{ kN}, p_4 = [85.5, 94.5] \text{ kN} (10\% \text{ uncertainty})$
Truss structure - results

Table: results of selected responses

<table>
<thead>
<tr>
<th>Method</th>
<th>$u_5$(LB)</th>
<th>$u_5$(UB)</th>
<th>$N_7$(LB)</th>
<th>$N_7$(UB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Combinatorial</td>
<td>0.017676</td>
<td>0.019756</td>
<td>273.562</td>
<td>303.584</td>
</tr>
<tr>
<td>Naïve IFEA</td>
<td>$-0.011216$</td>
<td>0.048636</td>
<td>$-717.152$</td>
<td>1297.124</td>
</tr>
<tr>
<td>$\delta$</td>
<td>163.45%</td>
<td>146.18%</td>
<td>362%</td>
<td>327%</td>
</tr>
<tr>
<td>Present IFEA</td>
<td>0.017642</td>
<td>0.019778</td>
<td>273.049</td>
<td>304.037</td>
</tr>
<tr>
<td>$\delta$</td>
<td>0.19%</td>
<td>0.11%</td>
<td>0.19%</td>
<td>0.15%</td>
</tr>
</tbody>
</table>

unit: $u_5$ (m), $N_7$ (kN). LB: lower bound; UB: upper bound.
Truss structure – results

- for moderate uncertainty (≤ 5%), very sharp bounds are obtained
- for relatively large uncertainty, reasonable bounds are obtained

in the case of 10% uncertainty:
Comb.: $\mathbf{u}_5 = [0.017711, 0.019811]$, IFEM: $\mathbf{u}_5 = [0.017252, 0.020168]$
(relative difference: 2.59%, 1.80% for LB, UB, respectively)
Truss with a large number of interval variables

\[ A_i = [0.995, 1.005] A_0, \]
\[ E_i = [0.995, 1.005] E_0 \] for \( i = 1, \ldots, N_e \)

<table>
<thead>
<tr>
<th>story ( \times ) bay</th>
<th>( N_e )</th>
<th>( N_v )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3 ( \times ) 10</td>
<td>123</td>
<td>246</td>
</tr>
<tr>
<td>4 ( \times ) 12</td>
<td>196</td>
<td>392</td>
</tr>
<tr>
<td>4 ( \times ) 20</td>
<td>324</td>
<td>648</td>
</tr>
<tr>
<td>5 ( \times ) 22</td>
<td>445</td>
<td>890</td>
</tr>
<tr>
<td>5 ( \times ) 30</td>
<td>605</td>
<td>1210</td>
</tr>
<tr>
<td>6 ( \times ) 30</td>
<td>726</td>
<td>1452</td>
</tr>
<tr>
<td>6 ( \times ) 35</td>
<td>846</td>
<td>1692</td>
</tr>
<tr>
<td>6 ( \times ) 40</td>
<td>966</td>
<td>1932</td>
</tr>
<tr>
<td>7 ( \times ) 40</td>
<td>1127</td>
<td>2254</td>
</tr>
<tr>
<td>8 ( \times ) 40</td>
<td>1288</td>
<td>2576</td>
</tr>
</tbody>
</table>
## Scalability study

vertical displacement at right upper corner (node D): $v_D = a \frac{PL}{E_0 A_0}$

### Table: displacement at node D

<table>
<thead>
<tr>
<th>Story $\times$ bay</th>
<th>Sensitivity Analysis</th>
<th>Present IFEA</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>LB*</td>
<td>UB*</td>
</tr>
<tr>
<td>3 $\times$ 10</td>
<td>2.5143</td>
<td>2.5756</td>
</tr>
<tr>
<td>4 $\times$ 20</td>
<td>3.2592</td>
<td>3.3418</td>
</tr>
<tr>
<td>5 $\times$ 30</td>
<td>4.0486</td>
<td>4.1532</td>
</tr>
<tr>
<td>6 $\times$ 35</td>
<td>4.8482</td>
<td>4.9751</td>
</tr>
<tr>
<td>7 $\times$ 40</td>
<td>5.6461</td>
<td>5.7954</td>
</tr>
<tr>
<td>8 $\times$ 40</td>
<td>6.4570</td>
<td>6.6289</td>
</tr>
</tbody>
</table>

$\delta_{LB} = \frac{|LB - LB^*|}{LB^*}$, $\delta_{LB} = \frac{|UB - UB^*|}{UB^*}$, $\delta_{LB} = \frac{(LB - LB^*)}{LB^*}$
### Efficiency study

Table: CPU time for the analyses with the present method (unit: seconds)

<table>
<thead>
<tr>
<th>Story × bay</th>
<th>( N_v )</th>
<th>Iteration</th>
<th>( t_i )</th>
<th>( t_r )</th>
<th>( t )</th>
<th>( t_i/t )</th>
<th>( t_r/t )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3 × 10</td>
<td>246</td>
<td>4</td>
<td>0.14</td>
<td>0.56</td>
<td>0.72</td>
<td>19.5%</td>
<td>78.4%</td>
</tr>
<tr>
<td>4 × 20</td>
<td>648</td>
<td>5</td>
<td>1.27</td>
<td>8.80</td>
<td>10.17</td>
<td>12.4%</td>
<td>80.5%</td>
</tr>
<tr>
<td>5 × 30</td>
<td>1210</td>
<td>6</td>
<td>6.09</td>
<td>53.17</td>
<td>59.70</td>
<td>10.2%</td>
<td>89.1%</td>
</tr>
<tr>
<td>6 × 35</td>
<td>1692</td>
<td>6</td>
<td>15.11</td>
<td>140.23</td>
<td>156.27</td>
<td>9.7%</td>
<td>89.7%</td>
</tr>
<tr>
<td>7 × 40</td>
<td>2254</td>
<td>6</td>
<td>32.53</td>
<td>323.14</td>
<td>358.76</td>
<td>9.1%</td>
<td>90.1%</td>
</tr>
<tr>
<td>8 × 40</td>
<td>2576</td>
<td>7</td>
<td>48.454</td>
<td>475.72</td>
<td>528.45</td>
<td>9.2%</td>
<td>90.0%</td>
</tr>
</tbody>
</table>

\( t_i \): iteration time, \( t_r \): CPU time for matrix inversion, \( t \): total comp. CPU time

- majority of time is spent on matrix inversion
Efficiency study

Computational time: a comparison of the sensitivity analysis method and the present method

<table>
<thead>
<tr>
<th>$N_v$</th>
<th>Sens.</th>
<th>Present</th>
</tr>
</thead>
<tbody>
<tr>
<td>246</td>
<td>1.06</td>
<td>0.72</td>
</tr>
<tr>
<td>648</td>
<td>64.05</td>
<td>10.17</td>
</tr>
<tr>
<td>1210</td>
<td>965.86</td>
<td>59.7</td>
</tr>
<tr>
<td>1692</td>
<td>4100</td>
<td>156.3</td>
</tr>
<tr>
<td>2254</td>
<td>14450</td>
<td>358.8</td>
</tr>
<tr>
<td>2576</td>
<td>32402</td>
<td>528.45</td>
</tr>
</tbody>
</table>

0 500 1000 1500 2000 2500

CPU time (sec)

0 5000 10000 15000 20000 25000 30000 35000

Number of interval variables

9 hr 9 min
Plate with quarter-circle cutout

- thickness: 0.005m
- Possion ratio: 0.3
- load: 100kN/m
- modulus of elasticity: $E = [199, 201]$ GPa

- number of element: 352
- element type: six-node isoparametric quadratic triangle
- results presented: $u_A$, $v_E$, $\sigma_{xx}$ and $\sigma_{yy}$ at node F
Plate with quarter-circle cutout

Case 1: the modulus of elasticity for each element varies independently in the interval [199, 201] GPa.

<table>
<thead>
<tr>
<th>Response</th>
<th>Monte Carlo sampling*</th>
<th>Present IFEA</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>LB</td>
<td>UB</td>
</tr>
<tr>
<td>$u_A$ $(10^{-5} \text{ m})$</td>
<td>1.19094</td>
<td>1.20081</td>
</tr>
<tr>
<td>$v_E$ $(10^{-5} \text{ m})$</td>
<td>$-0.42638$</td>
<td>$-0.42238$</td>
</tr>
<tr>
<td>$\sigma_{xx}$ (MPa)</td>
<td>13.164</td>
<td>13.223</td>
</tr>
<tr>
<td>$\sigma_{yy}$ (MPa)</td>
<td>1.803</td>
<td>1.882</td>
</tr>
</tbody>
</table>

*10^6 samples are made.
Outline

- Introduction
- Interval Arithmetic
- Interval Finite Elements
- Element-By-Element
- Examples
- Conclusions
Conclusions

- Development and implementation of IFEM
  - uncertain material, geometry and load parameters are described by interval variables
  - interval arithmetic is used to guarantee an enclosure of response

- Enhanced dependence problem control
  - use Element-By-Element technique
  - use the penalty method or Lagrange multiplier method to impose constraints
  - modify and enhance fixed point iteration to take into account the dependence problem
  - develop special algorithms to calculate stress and element nodal force
Conclusions

- The method is generally applicable to linear static FEM, regardless of element type
- Evaluation of the present method
  - Rigorousness: in all the examples, the results obtained by the present method enclose those from the alternative methods
  - Accuracy: sharp results are obtained for moderate parameter uncertainty (no more than 5%); reasonable results are obtained for relatively large parameter uncertainty (5%~10%)
Conclusions

- Scalability: the accuracy of the method remains at the same level with increase of the problem scale
- Efficiency: the present method is significantly superior to the conventional methods such as the combinatorial, Monte Carlo sampling, and sensitivity analysis method
- The present IFEM represents an efficient method to handle uncertainty in engineering applications
Center for Reliable Engineering Computing (REC)

We handle computations with care
Frame structure

<table>
<thead>
<tr>
<th>Member</th>
<th>Shape</th>
</tr>
</thead>
<tbody>
<tr>
<td>C₁</td>
<td>W12×19</td>
</tr>
<tr>
<td>C₂</td>
<td>W14×132</td>
</tr>
<tr>
<td>C₃</td>
<td>W14×109</td>
</tr>
<tr>
<td>C₄</td>
<td>W10×12</td>
</tr>
<tr>
<td>C₅</td>
<td>W14×109</td>
</tr>
<tr>
<td>C₆</td>
<td>W14×109</td>
</tr>
<tr>
<td>B₁</td>
<td>W27×84</td>
</tr>
<tr>
<td>B₂</td>
<td>W36×135</td>
</tr>
<tr>
<td>B₃</td>
<td>W18×40</td>
</tr>
<tr>
<td>B₄</td>
<td>W27×94</td>
</tr>
</tbody>
</table>

results listed: nodal forces at the left node of member B₂
Frame structure – case 1

Case 1: load uncertainty
\[ w_1 = [105.8, 113.1] \text{kN/m}, \quad w_2 = [105.8, 113.1] \text{kN/m}, \]
\[ w_3 = [49.255, 52.905] \text{kN/m}, \quad w_4 = [49.255, 52.905] \text{kN/m}, \]

Table: Nodal forces at the left node of member B_2

<table>
<thead>
<tr>
<th>Nodal force</th>
<th>Combinatorial</th>
<th>Present IFEA</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>LB</td>
<td>UB</td>
</tr>
<tr>
<td>Axial (kN)</td>
<td>219.60</td>
<td>239.37</td>
</tr>
<tr>
<td>Shear (kN)</td>
<td>833.61</td>
<td>891.90</td>
</tr>
<tr>
<td>Moment (kN·m)</td>
<td>1847.21</td>
<td>1974.95</td>
</tr>
</tbody>
</table>

- exact solution is obtained in the case of load uncertainty
Frame structure – case 2

Case 2: stiffness uncertainty and load uncertainty
1% uncertainty introduced to $A$, $I$, and $E$ of each element.
Number of interval variables: 34.

Table: Nodal forces at the left node of member $B_2$

<table>
<thead>
<tr>
<th>Nodal force</th>
<th>Monte Carlo sampling*</th>
<th>Present IFEA</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>LB</td>
<td>UB</td>
</tr>
<tr>
<td>Axial (kN)</td>
<td>218.23</td>
<td>240.98</td>
</tr>
<tr>
<td>Shear (kN)</td>
<td>833.34</td>
<td>892.24</td>
</tr>
<tr>
<td>Moment (kN.m)</td>
<td>1842.86</td>
<td>1979.32</td>
</tr>
</tbody>
</table>

*10^6 samples are made.