Interval Finite Elements as a Basis for Generalized Models of Uncertainty in Engineering Analysis

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German National Science Foundation (DFG)
TU Dresden, Germany, June 19, 2007

Outline

- Introduction
- Interval Arithmetic
- Interval Finite Elements
- Element-By-Element
- Examples
- Conclusions

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Introduction- Uncertainty

- Uncertainty is unavoidable in engineering system
  - structural mechanics entails uncertainties in material, geometry and load parameters (aleatory-epistemic)
- Probabilistic approach is the traditional approach
  - requires sufficient information to validate the probabilistic model
  - criticism of the credibility of probabilistic approach when data is insufficient (Elishakoff, 1995; Ferson and Ginzburg, 1996; Möller and Beer, 2007)

Introduction- Interval Approach

- Nonprobabilistic approach for uncertainty modeling when only range information (tolerance) is available

$$t = t_0 \pm \delta$$

- Represents an uncertain quantity by giving a range of possible values

$$t = [t_0 - \delta,\ t_0 + \delta]$$

- How to define bounds on the possible ranges of uncertainty?
  - experimental data, measurements, statistical analysis, expert knowledge

Introduction- Why Interval?

- Simple and elegant
- Conforms to practical tolerance concept
- Describes the uncertainty that can not be appropriately modeled by probabilistic approach
- Computational basis for other uncertainty approaches (e.g., fuzzy set, random set, imprecise probability)

- **Provides guaranteed enclosures**

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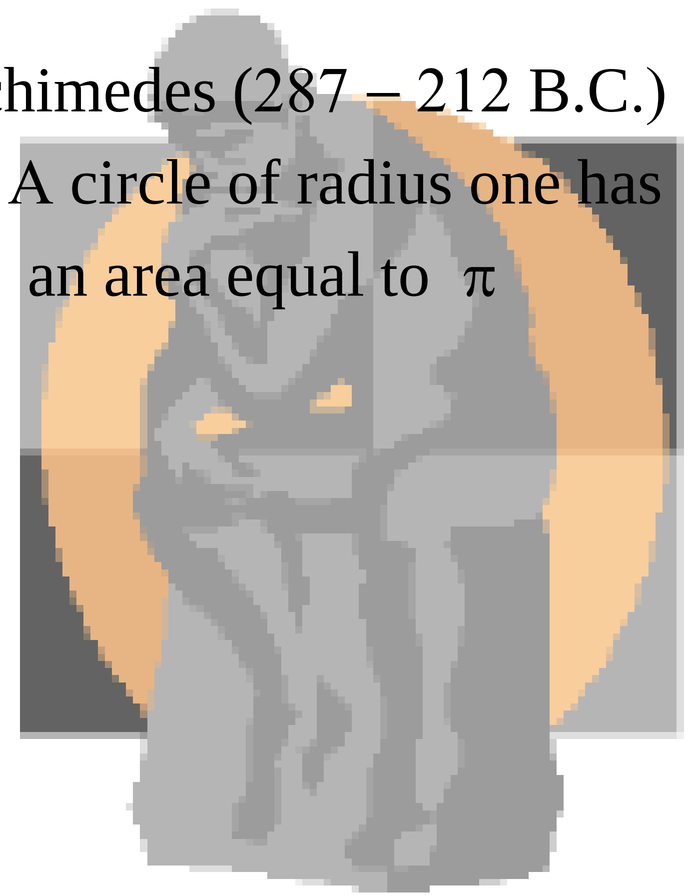
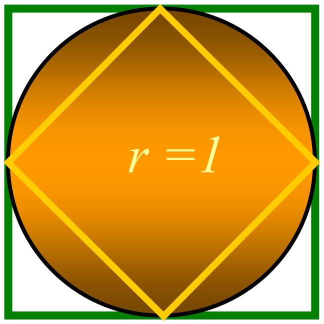
Interval arithmetic – Background



- Archimedes (287 – 212 B.C.)
  - A circle of radius one has an area equal to π

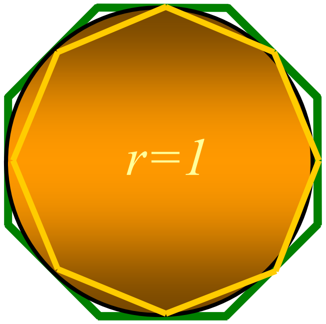
Interval arithmetic – Background

- Archimedes (287 – 212 B.C.)
  - A circle of radius one has an area equal to π
  - $2 < \pi < 4$



$$3\frac{10}{71} < \pi < 3\frac{1}{7}$$

$\pi = [3.14085, 3.14286]$



Interval arithmetic – Background

- Modern interval arithmetic
  - Physical constants or measurements

    $g \in [9.8045, 9.8082]$

  - Representation of numbers

    $1/3 \approx 0.3333\cdots$ $\sqrt{2} \approx 1.4142\cdots$ $\pi \approx 3.1416\cdots$

    $1/3 \in [0.3333, 0.3334]$ $\sqrt{2} \in [1.4142, 1.4143]$ $\pi \in [3.1415, 3.1416]$

  - Rounding errors

    $1/0.12345 \approx 8.1004$ $1/0.12345 \in [8.1004, 8.1005]$

Interval arithmetic

- Interval number represents a range of possible values within a closed set

$$\boldsymbol{x} \equiv [\underline{x}, \overline{x}] := \{x \in R \mid \underline{x} \le x \le \overline{x}\}$$

Interval Operations

Let $x = [a, b]$ and $y = [c, d]$ be two interval numbers

1. Addition

$$x + y = [a, b] + [c, d] = [a + c, b + d]$$

2. Subtraction

$$x - y = [a, b] - [c, d] = [a - d, b - c]$$

3. Multiplication

$$xy = [\min(ac, ad, bc, bd), \max(ac, ad, bc, bd)]$$

4. Division

$$1/x = [1/b, 1/a]$$

Properties of Interval Arithmetic

Let ***x**, **y*** and ***z*** be interval numbers

1. Commutative Law

$$x + y = y + x$$

$$xy = yx$$

2. Associative Law

$$x + (y + z) = (x + y) + z$$

$$x(yz) = (xy)z$$

3. ***Distributive Law does not always hold,*** *but*

$$x(y + z) \subseteq xy + xz$$

Sharp Results – Overestimation

- The ***DEPENDENCY*** problem arises when one or several variables occur more than once in an interval expression

  - $f(\boldsymbol{x}) = \boldsymbol{x} - \boldsymbol{x}\,, \qquad \boldsymbol{x} = [1, 2]$
  - $f(\boldsymbol{x}) = [1 - 2, 2 - 1] = [-1, 1] \neq 0$
  - ~~$f(\boldsymbol{x}, \boldsymbol{y}) = \{ f(x, y) = x - y \mid x \in \boldsymbol{x}, y \in \boldsymbol{y}\}$~~
  - $f(\boldsymbol{x}) = \boldsymbol{x}\,(1 - 1) \quad \Rightarrow \quad f(\boldsymbol{x}) = 0$
  - $f(\boldsymbol{x}) = \{ f(x) = x - x \mid x \in \boldsymbol{x}\}$

Sharp Results – Overestimation

- If a, b and c are interval numbers, then:

  $a\,(b \pm c) \subseteq ab \pm ac$

- If we set

  $a = [-2, 2]$; $b = [1, 2]$; $c = [-2, 1]$, we get

  $a\,(b + c) = [-2, 2]([1, 2] + [-2, 1]) = [-2, 2]\,[-1, 3] = [-6, 6]$

- However,

  $ab + ac = [-2, 2][1, 2] + [-2, 2][-2, 1] = [-4, 4] + [-4, 4] = [-8, 8]$

Sharp Results – Overestimation

- Interval Vectors and Matrices
  - An interval matrix is such matrix that contains all real matrices whose elements are obtained from all possible values between the lower and upper bounds of its interval components

$$\boldsymbol{A} = \{A \in R^{m \times n} \mid A_{ij} \in \boldsymbol{A}_{ij} \quad for\ i = 1,\ldots,m;\ \ j = 1,\ldots,n\}$$

Sharp Results – Overestimation

- Let ***a, b, c*** and ***d*** be independent variables, each with interval [1, 3]

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix} \boldsymbol{a} & -\boldsymbol{b} \\ -\boldsymbol{c} & \boldsymbol{d} \end{pmatrix}, \quad A \times \boldsymbol{B} = \begin{pmatrix} [-2, 2] & [-2, 2] \\ [-2, 2] & [-2, 2] \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \boldsymbol{B}_{phys} = \begin{pmatrix} \boldsymbol{b} & -\boldsymbol{b} \\ -\boldsymbol{b} & \boldsymbol{b} \end{pmatrix}, \quad A \times \boldsymbol{B}_{phys} = \begin{pmatrix} [\boldsymbol{b}-\boldsymbol{b}] & [\boldsymbol{b}-\boldsymbol{b}] \\ [\boldsymbol{b}-\boldsymbol{b}] & [\boldsymbol{b}-\boldsymbol{b}] \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad B^*_{phys} = \boldsymbol{b} \times \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad A \times \boldsymbol{B}^*_{phys} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

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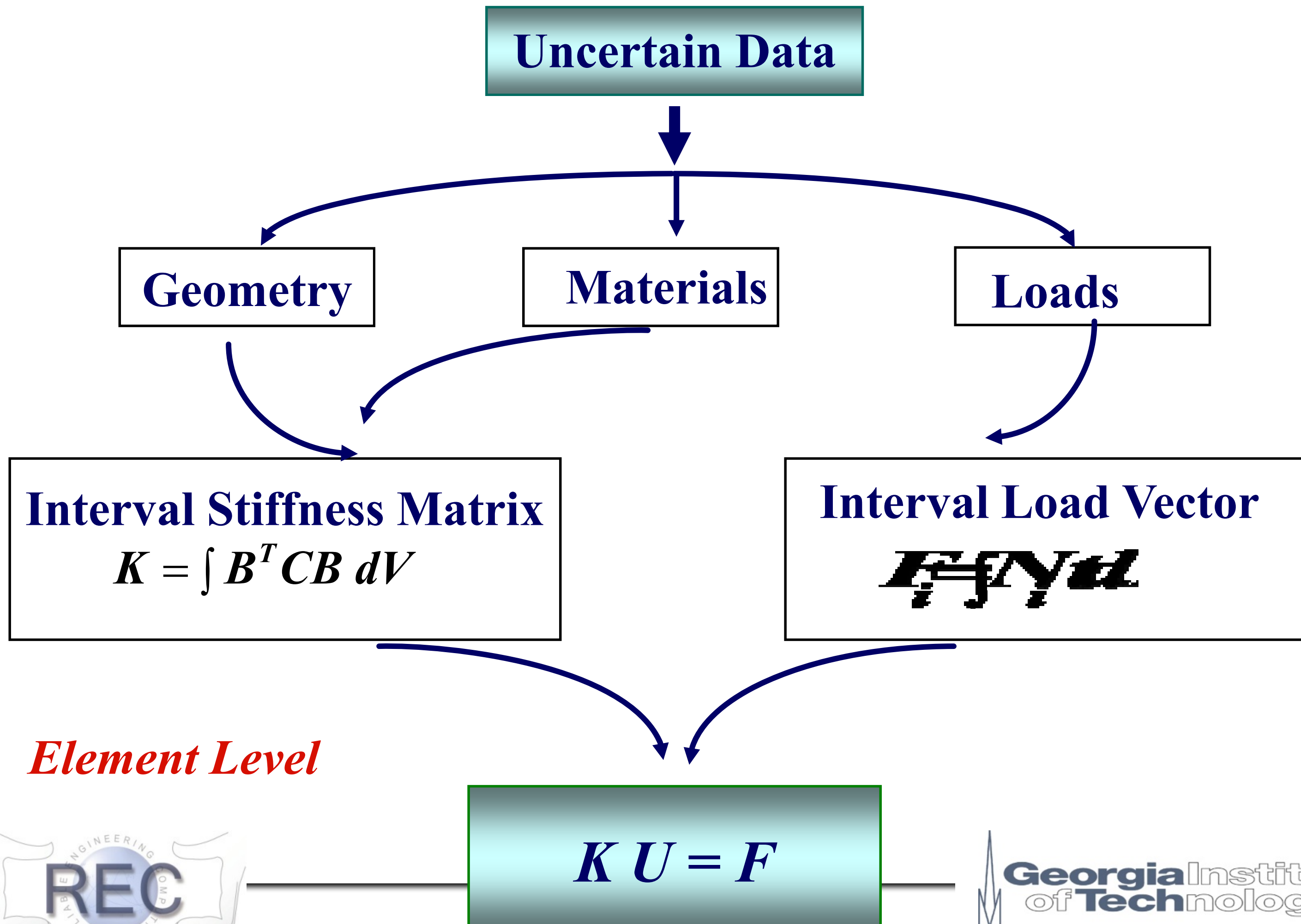
Finite Elements

Finite Element Method (FEM) is a numerical method that provides approximate solutions to differential equations (ODE and PDE)

Finite Elements- Uncertainty& Errors

- Mathematical model (validation)
- Discretization of the mathematical model into a computational framework
- Parameter uncertainty (loading, material properties)
- Rounding errors

Interval Finite Elements



Interval Finite Elements

$\boldsymbol{K\,U = F}$

$\boldsymbol{K = \int B^T\ C\ B\ dV}$ = Interval element stiffness matrix

$\boldsymbol{B}$ = Interval strain-displacement matrix

$\boldsymbol{C}$ = Interval elasticity matrix

$\boldsymbol{F}$ = $[F_1, \ldots F_i, \ldots F_n]$ = Interval element load vector (traction)

$\boldsymbol{F_i = \int N_i\ t\ dA}$

$\boldsymbol{N_i}$ = Shape function corresponding to the *i-th* DOF

$\boldsymbol{t}$ = Surface traction

Interval Finite Elements (IFEM)

- Follows conventional FEM
- Loads, geometry and material property are expressed as interval quantities
- System response is a function of the interval variables and therefore varies in an interval
- Computing the exact response range is proven NP-hard
- The problem is to estimate the bounds on the unknown exact response range based on the bounds of the parameters

IFEM- Inner-Bound Methods

- Combinatorial method (Muhanna and Mullen 1995, Rao and Berke 1997)
- Sensitivity analysis method (Pownuk 2004)
- Perturbation (Mc William 2000)
- Monte Carlo sampling method
- Need for alternative methods that achieve
  - Rigorousness – guaranteed enclosure
  - Accuracy – sharp enclosure
  - Scalability – large scale problem
  - Efficiency

IFEM- Enclosure

- Linear static finite element
  - Muhanna, Mullen, 1995, 1999, 2001,and Zhang 2004
  - Popova 2003, and Kramer 2004
  - Neumaier and Pownuk 2004
  - Corliss, Foley, and Kearfott 2004
- Dynamic
  - Dessombz, 2000
- Free vibration-Buckling
  - Modares, Mullen 2004, and Billini and Muhanna 2005

Interval Finite Elements

- Interval Linear System of Equations

$$\boldsymbol{A}\,\boldsymbol{x} = \boldsymbol{b}$$

$$\begin{pmatrix} 2 & [-1,0] \\ [-1,0] & 2 \end{pmatrix} \times \begin{pmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{pmatrix} = \begin{pmatrix} 1.2 \\ -1.2 \end{pmatrix}$$

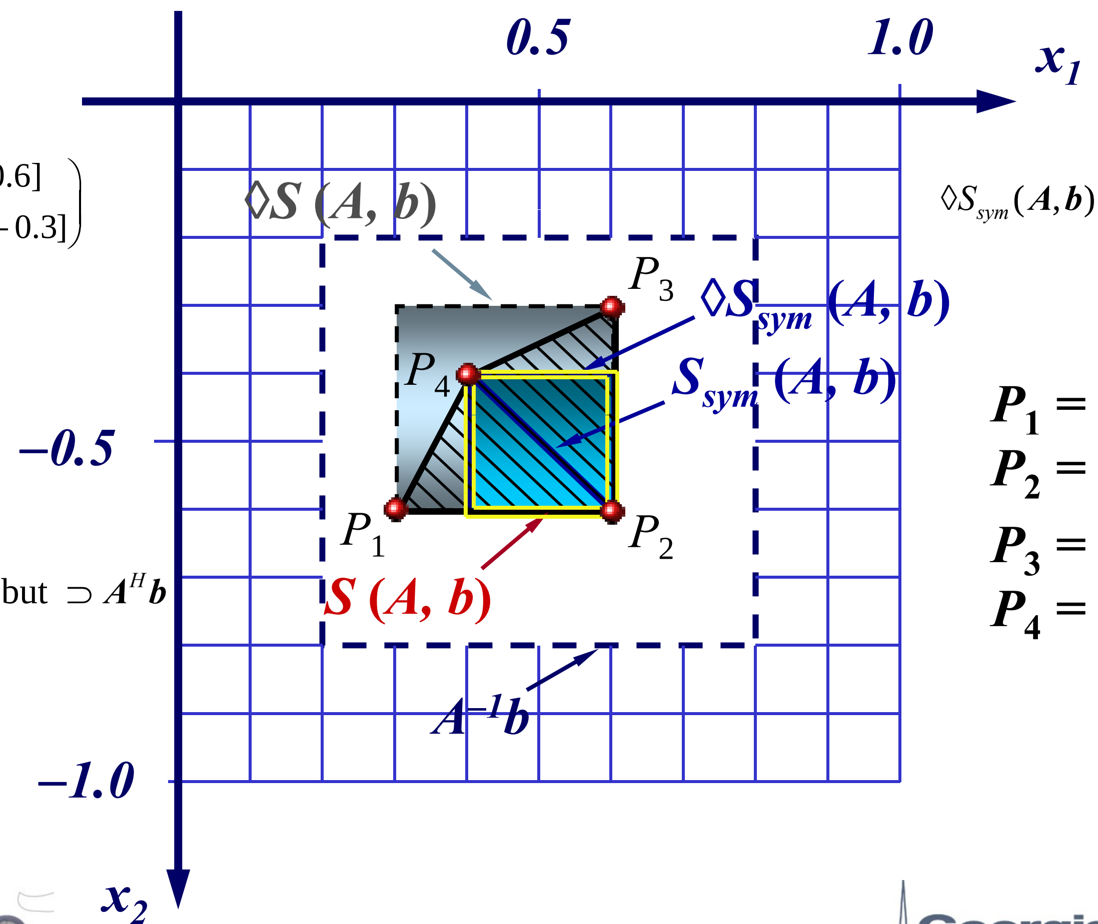
$$Then \quad A \in \boldsymbol{A} \quad iff$$

$$A := \begin{pmatrix} 2 & -\boldsymbol{\alpha} \\ -\boldsymbol{\beta} & 2 \end{pmatrix} \qquad with \quad \boldsymbol{\alpha}, \boldsymbol{\beta} \in [0,1]$$

Interval Finite Elements

$$A^H \boldsymbol{b} = \lozenge S(\boldsymbol{A},\boldsymbol{b}) = \begin{pmatrix} [0.3,\ 0.6] \\ [-0.6,\ -0.3] \end{pmatrix}$$

$$\lozenge S_{sym}(\boldsymbol{A},\boldsymbol{b}) = \begin{pmatrix} [0.4,\ 0.6] \\ [-0.6,\ -0.4] \end{pmatrix} \neq A^H \boldsymbol{b}$$



$$A^{-1} \boldsymbol{b} = \begin{pmatrix} [0.2,\ 0.8] \\ [-0.8,\ -0.2] \end{pmatrix} \neq \text{ but } \supset A^H \boldsymbol{b}$$

$P_1 = (0.3, -0.6)$
$P_2 = (0.6, -0.6)$
$P_3 = (0.6, -0.3)$
$P_4 = (0.4, -0.4)$

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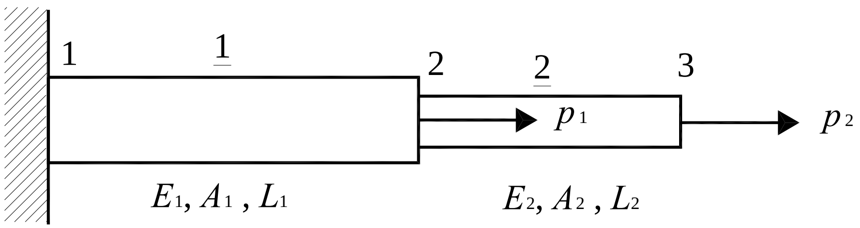
Naïve interval FEA



$E_1A_1/L_1 = \boldsymbol{k}_1 = [0.95,\ 1.05],$

$E_2A_2/L_2 = \boldsymbol{k}_2 = [1.9,\ 2.1],$

$p_1 = 0.5, \quad p_2 = 1$

$$\begin{pmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \Rightarrow \begin{pmatrix} [2.85,\ 3.15] & [-2.1,\ -1.9] \\ [-2.1,\ -1.9] & [1.9,\ 2.1] \end{pmatrix}\begin{pmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 1 \end{pmatrix}$$

- exact solution: $\boldsymbol{u}_2 = [1.429, 1.579]$, $\boldsymbol{u}_3 = [1.905, 2.105]$
- naïve solution: $\boldsymbol{u}_2 = [-0.052, 3.052]$, $\boldsymbol{u}_3 = [0.098, 3.902]$
- interval arithmetic assumes that all coefficients are independent
- uncertainty in the response is severely overestimated (1900%)

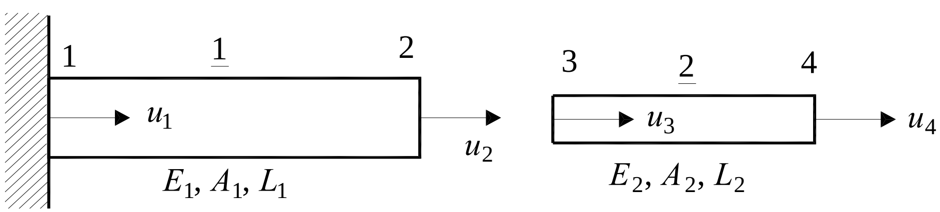
Element-By-Element

Element-By-Element (EBE) technique

- elements are detached – no element coupling
- structure stiffness matrix is block-diagonal (k_1 ,…, k_{Ne})
- the size of the system is increased

$$u = (u_1, \ldots, u_{Ne})^T$$

- need to impose necessary constraints for compatibility and equilibrium



Element-By-Element model

Element-By-Element

Suppose the modulus of elasticity is interval:

$$\boldsymbol{E} = \hat{E}(1+\boldsymbol{\delta})$$

$\boldsymbol{\delta}$: zero-midpoint interval

The element stiffness matrix can be split into two parts,

$$\boldsymbol{k} = \hat{k}(I+\boldsymbol{d}) = \hat{k} + \hat{k}\boldsymbol{d}$$

$\hat{k}$: deterministic part, element stiffness matrix evalued using $\hat{E}$,

$\hat{k}\boldsymbol{d}$: interval part

$\boldsymbol{d}$: interval diagonal matrix, diag($\boldsymbol{\delta}$,...,$\boldsymbol{\delta}$).

Element-By-Element

- Element stiffness matrix: $\boldsymbol{k} = \hat{k}(I + \boldsymbol{d})$

- Structure stiffness matrix:

$$\boldsymbol{K} = \hat{K}(I + \boldsymbol{D}) = \hat{K} + \hat{K}\boldsymbol{D}$$

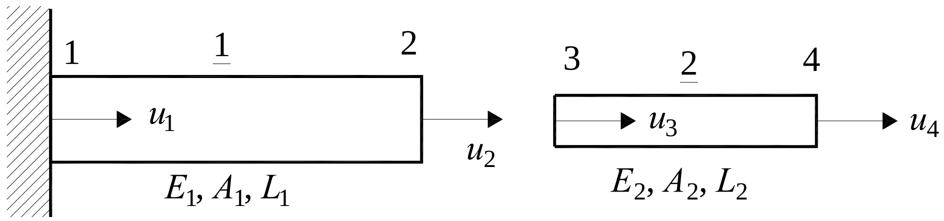
or

$$\boldsymbol{K} = \begin{pmatrix} \boldsymbol{k}_1 & & \\ & \ddots & \\ & & \boldsymbol{k}_{N_e} \end{pmatrix} = \begin{pmatrix} \hat{k}_1 & & \\ & \ddots & \\ & & \hat{k}_{N_e} \end{pmatrix} \left(I + \begin{pmatrix} \boldsymbol{d}_1 & & \\ & \ddots & \\ & & \boldsymbol{d}_{N_e} \end{pmatrix} \right)$$

Constraints

Impose necessary constraints for compatibility and equilibrium

- Penalty method
- Lagrange multiplier method



Element-By-Element model

Constraints – penalty method

Constraint conditions: $c\boldsymbol{u} = 0$

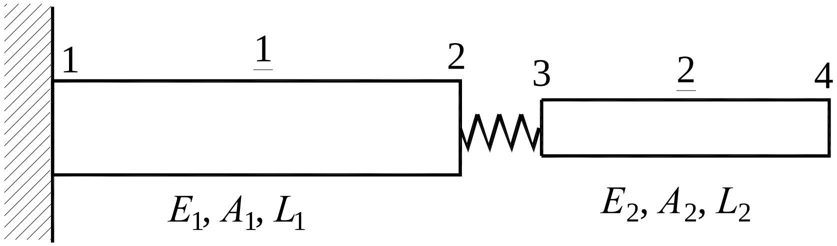
Using the penalty method:

$$(\boldsymbol{K} + Q)\boldsymbol{u} = \boldsymbol{p}$$

Q : penalty matrix, $Q = c^T \eta c$

η : diagonal matrix of penalty number η_i

Requires a careful choice of the penatly number



A spring of large stiffness is added to force node 2 and node 3 to have the same displacement.

Constraints – Lagrange multiplier

Constraint conditions: $c\boldsymbol{u} = 0$

Using the Lagrange multiplier method:

$$\begin{pmatrix} \boldsymbol{K} & c^T \\ c & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \boldsymbol{p} \\ 0 \end{pmatrix}$$

$\boldsymbol{\lambda}$: Lagrange multiplier vector, introdued as new unknowns

Load in EBE

Nodal load $\boldsymbol{p}_b$

$$\boldsymbol{p}_b = (\boldsymbol{p}_1, \ldots, \boldsymbol{p}_{N_e})^T$$

where $\boldsymbol{p}_i = \int N^T \boldsymbol{\varphi}(x) dx$

Suppose the surface traction $\phi(x)$ is described by

an interval function: $\boldsymbol{\varphi}(x) = \sum_{j=0}^{m} \boldsymbol{a}_j x^j$.

$\boldsymbol{p}_b$ can be rewritten as

$$\boldsymbol{p}_b = W\boldsymbol{F}$$

W: deterministic matrix

$\boldsymbol{F}$: interval vector containing the interval coefficients of the surface tractiton

Fixed point iteration

- For the interval equation $\boldsymbol{Ax} = \boldsymbol{b}$,
  - preconditioning: $R\boldsymbol{Ax} = R\boldsymbol{b}$, R is the preconditioning matrix
  - transform it into $\boldsymbol{g}(\boldsymbol{x}^*) = \boldsymbol{x}^*$:

$$R\,\boldsymbol{b} - R\boldsymbol{A}\,x_0 + (I - R\boldsymbol{A})\,\boldsymbol{x}^* = \boldsymbol{x}^*, \qquad \boldsymbol{x} = \boldsymbol{x}^* + x_0$$

- **Theorem** (Rump, 1990): for some interval vector $\boldsymbol{x}^*$,

if $\quad \boldsymbol{g}(\boldsymbol{x}^*) \subseteq \text{int}(\boldsymbol{x}^*)$

then $\quad \boldsymbol{A}^H \boldsymbol{b} \subseteq \boldsymbol{x}^* + x_0$

- Iteration algorithm:

iterate: $\boldsymbol{x}^{*(l+1)} = \boldsymbol{z} + \boldsymbol{G}(\boldsymbol{\varepsilon} \cdot \boldsymbol{x}^{*(l)})$

where $\boldsymbol{z} = R\boldsymbol{b} - R\boldsymbol{A}x_0,\ \boldsymbol{G} = I - R\boldsymbol{A},\ R = \hat{A}^{-1},\ \hat{A}x_0 = \hat{b}$

- No dependency handling

Fixed point iteration

- Interval FEA calls for a modified method which exploits the special form of the structure equations

$$(\boldsymbol{K} + Q)\boldsymbol{u} = \boldsymbol{p} \text{ with } \boldsymbol{K} = \hat{K} + \hat{K}\boldsymbol{D}$$

- Choose $R = (\hat{K} + Q)^{-1}$, construct iterations:

$$\begin{aligned}\boldsymbol{u}^{*(l+1)} &= R\boldsymbol{p} - R(\boldsymbol{K} + Q)u_0 + (I - R(\boldsymbol{K} + Q))(\boldsymbol{\varepsilon} \cdot \boldsymbol{u}^{*(l)}) \\ &= R\boldsymbol{p} - u_0 - R\hat{K}\boldsymbol{D}(u_0 + \boldsymbol{\varepsilon} \cdot \boldsymbol{u}^{*(l)}) \\ &= R\boldsymbol{p} - u_0 - R\hat{K}\boldsymbol{M}^{(l)}\boldsymbol{\Delta}\end{aligned}$$

if $\boldsymbol{u}^{*(l+1)} \subseteq \text{int}(\boldsymbol{u}^{*(l)})$, then $\boldsymbol{u} = \boldsymbol{u}^{*(l+1)} + u_0 = R\boldsymbol{p} - R\check{K}\boldsymbol{M}^{(l)}\boldsymbol{\Delta}$

$\boldsymbol{\Delta}$: interval vector, $\boldsymbol{\Delta} = (\delta_1, \ldots, \delta_{N_e})^T$

The interval variables $\delta_1, \ldots, \delta_{N_e}$ appear only once in each iteration.

Convergence of fixed point

- The algorithm converges if and only if $\rho(|\boldsymbol{G}|) < 1$

  smaller $\rho(|\boldsymbol{G}|) \Rightarrow$ less iterations required, and less overestimation in results
- To minimize $\rho(|\boldsymbol{G}|)$:
  - choose $R = \hat{A}^{-1}$ so that $\boldsymbol{G} = I - R\boldsymbol{A}$ has a small spectral radius
  - reduce the overestimation in $\boldsymbol{G}$

$$\boldsymbol{G} = I - R\boldsymbol{A} = \boldsymbol{I} - (\hat{K} + Q)^{-1}(\hat{K} + Q + \hat{K}\boldsymbol{D}) = -R\hat{K}\boldsymbol{D}$$

Stress calculation

- Conventional method:

$$\boldsymbol{\sigma} = \boldsymbol{C}B\boldsymbol{u}_e, \text{ (severe overestimation)}$$

$\boldsymbol{C}$: elasticity matrix, B: strain-displacement matrix

- Present method:

$$\boldsymbol{E} = (1+\boldsymbol{\delta})\hat{E}, \quad \boldsymbol{C} = (1+\boldsymbol{\delta})\hat{C}$$

$$\begin{aligned}\boldsymbol{\sigma} &= \boldsymbol{C}BL\boldsymbol{u} \\ &= \boldsymbol{C}BL(R\boldsymbol{p} - R\hat{C}\boldsymbol{M}^{(l)}\boldsymbol{\Delta}) \\ &= (1+\boldsymbol{\delta})(\hat{C}BLR\boldsymbol{p} - \hat{C}BLR\hat{K}\boldsymbol{M}^{(l)}\boldsymbol{\Delta})\end{aligned}$$

L: Boolean matrix, $L\boldsymbol{u} = \boldsymbol{u}_e$

Element nodal force calculation

- Conventional method:

$$\boldsymbol{f} = T_e(\boldsymbol{k}\boldsymbol{u}_e - \boldsymbol{p}_e), \quad \text{(severe overestimation)}$$

- Present method:

in the EBE model, $$T(\boldsymbol{K}\boldsymbol{u} - \boldsymbol{p}_b) = \begin{pmatrix} (\boldsymbol{T}_e)_1(\boldsymbol{k}_1(\boldsymbol{u}_e)_1 - (\boldsymbol{p}_e)_1) \\ \vdots \\ (\boldsymbol{T}_e)_{N_e}(\boldsymbol{k}_{N_e}(\boldsymbol{u}_e)_{N_e} - (\boldsymbol{p}_e)_{N_e}) \end{pmatrix}$$

from $(\boldsymbol{K} + Q)\boldsymbol{u} = \boldsymbol{p}_c + \boldsymbol{p}_b \Rightarrow T(\boldsymbol{K}\boldsymbol{u} - \boldsymbol{p}_b) = T(\boldsymbol{p}_c - Q\boldsymbol{u})$

Calculate $T(\boldsymbol{p}_c - Q\boldsymbol{u})$ to obtain the element nodal forces for all elements.

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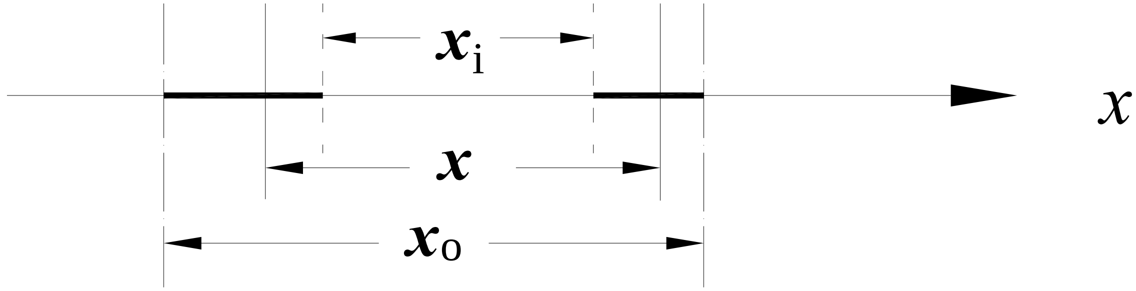
Numerical example

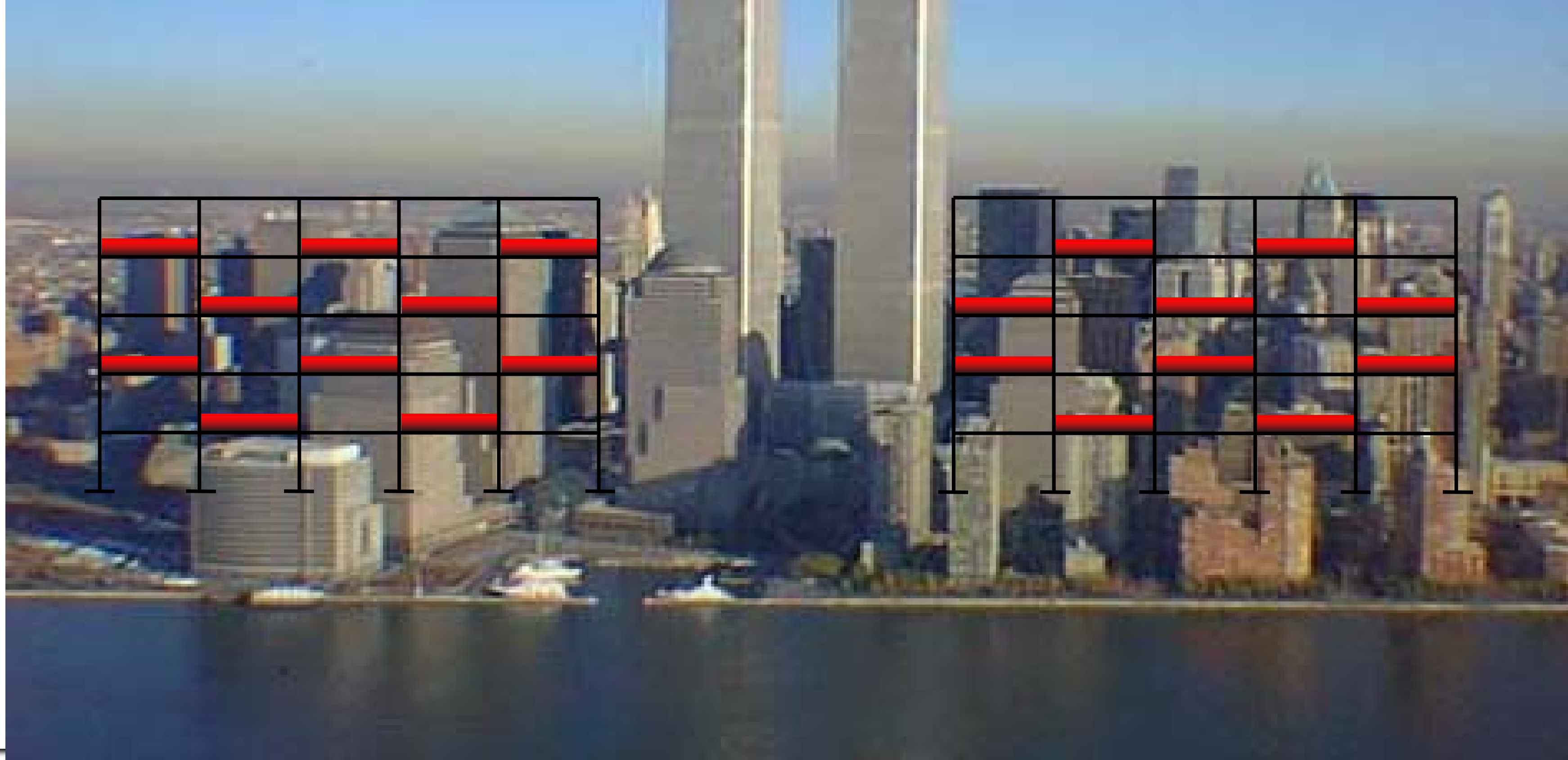
- Examine the rigorousness, accuracy, scalability, and efficiency of the present method
- Comparison with the alternative methods
  - the combinatorial method, sensitivity analysis method, and Monte Carlo sampling method
  - these alternative methods give inner estimation



$\boldsymbol{x}$: exact solution, $\boldsymbol{x}_i$: inner bound, $\boldsymbol{x}_o$: outer bound

Examples – Load Uncertainty

- Four-bay forty-story frame



Examples – Load Uncertainty

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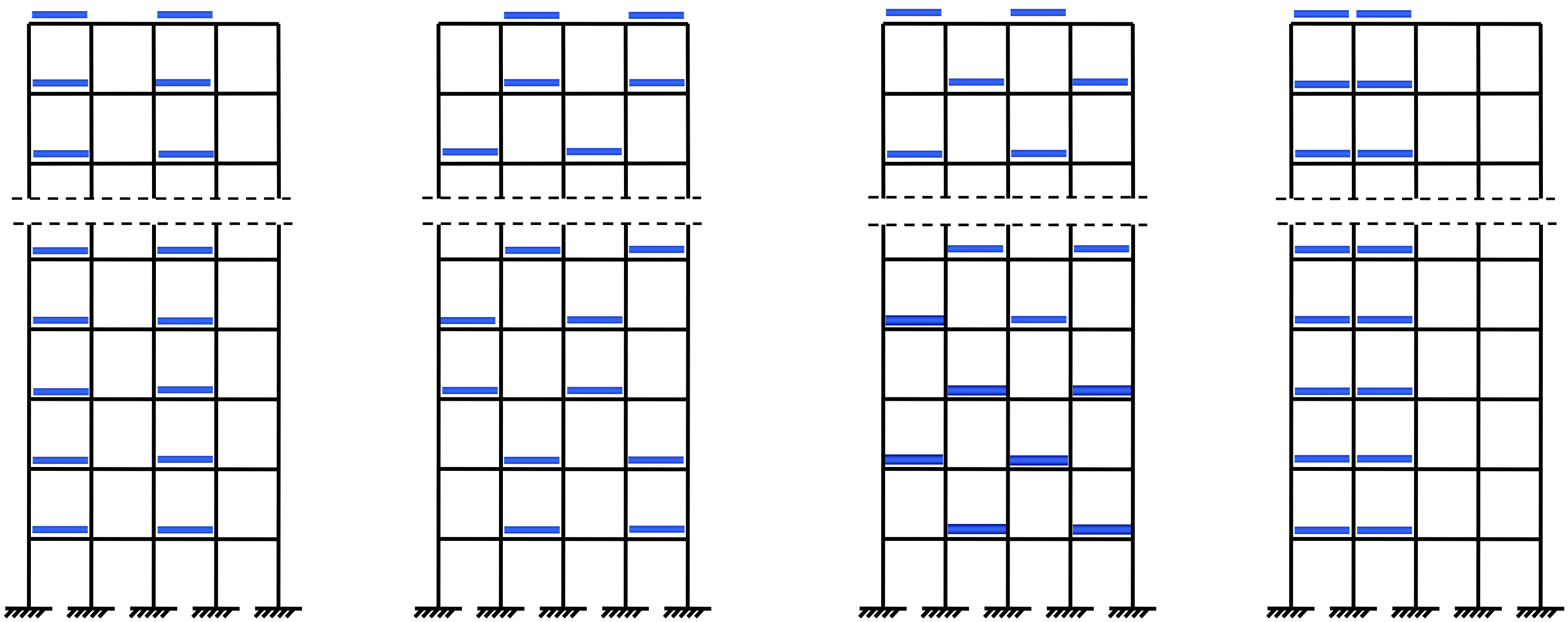
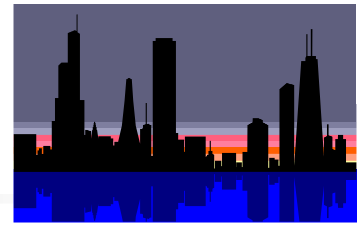


Loading A   Loading B   Loading C   Loading D

Examples – Load Uncertainty



- **Four-bay forty-story frame**

Total number of floor load patterns

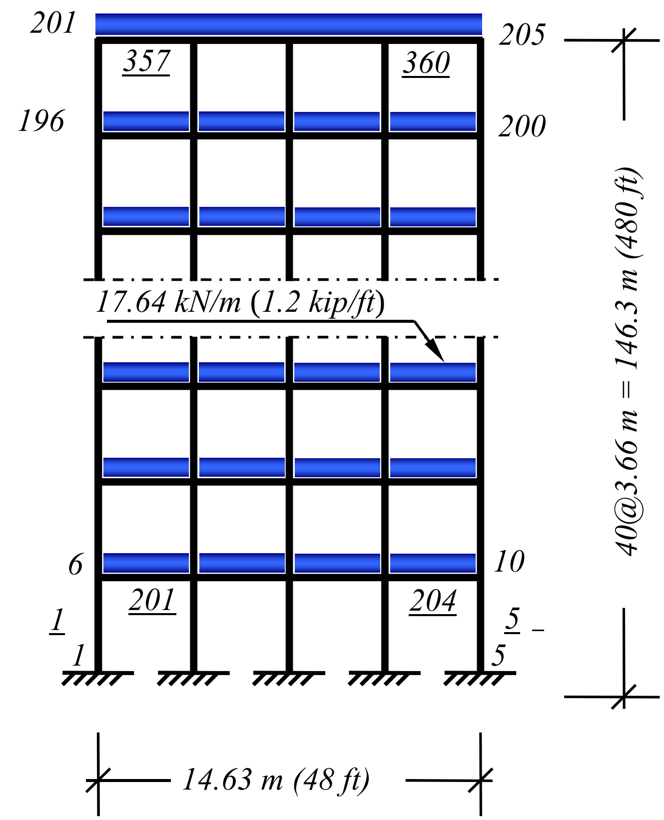
$2^{160} = 1.46 \times 10^{48}$

If one were able to calculate

10,000 *patterns / s*

there has not been sufficient time since the creation of the universe **(4-8)** billion years ? to solve all load patterns for this simple structure

Material *A36*, Beams *W24* x *55*, Columns *W14* x *398*



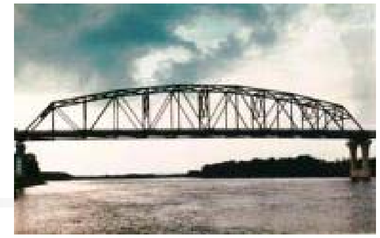
Examples – Load Uncertainty

- Four-bay forty-story frame

Four bay forty floor frame - Interval solutions for shear force and bending moment of first floor columns

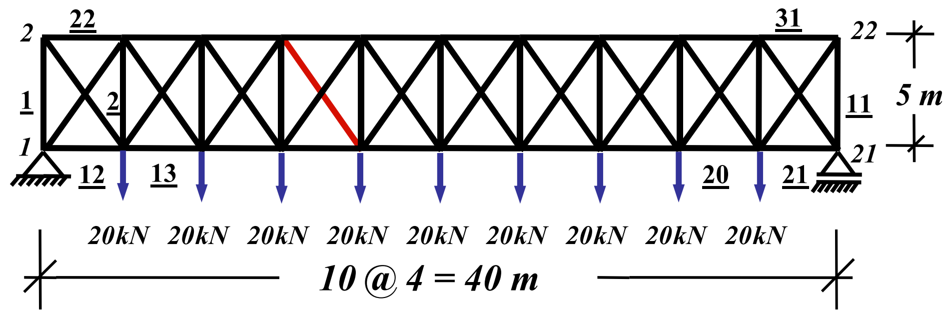
Elements		1		2		3	
Nodes		1	6	2	7	3	8
Combination solution		Total number of required combinations = $1.461501637 \times 10^{48}$					
Interval	Axial force (kN)	[-2034.5, 185.7]		[-2161.7, 0.0]		[-2226.7, 0.0]	
solution	Shear force (kN)	[-5.1, 0.9]		[-5.8, 5.0]		[-5.0, 5.0]	
	Moment (kN m)	[-10.3, 4.5]	[-15.3, 5.4]	[-10.6, 9.3]	[-17, 15.2]	[-8.9, 8.9]	[-16, 16]

Examples – Load Uncertainty

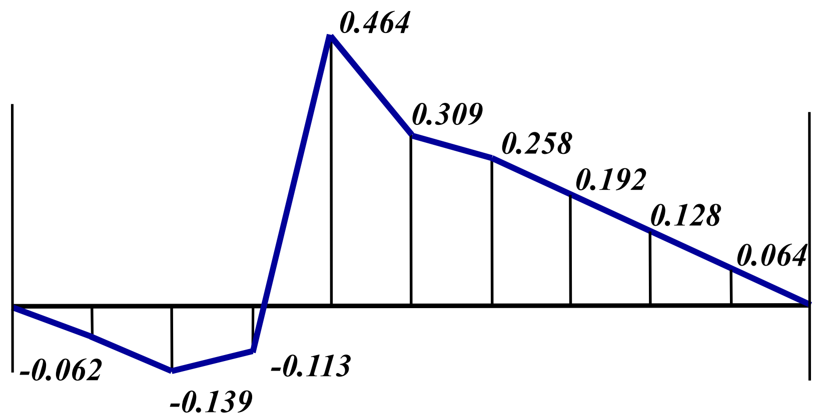


- **Ten-bay truss**

$A = 0.006\ m^2$
$E = 2.0 \times 10^8\ kPa$



$F = [-4.28, 28.3]\ kN$

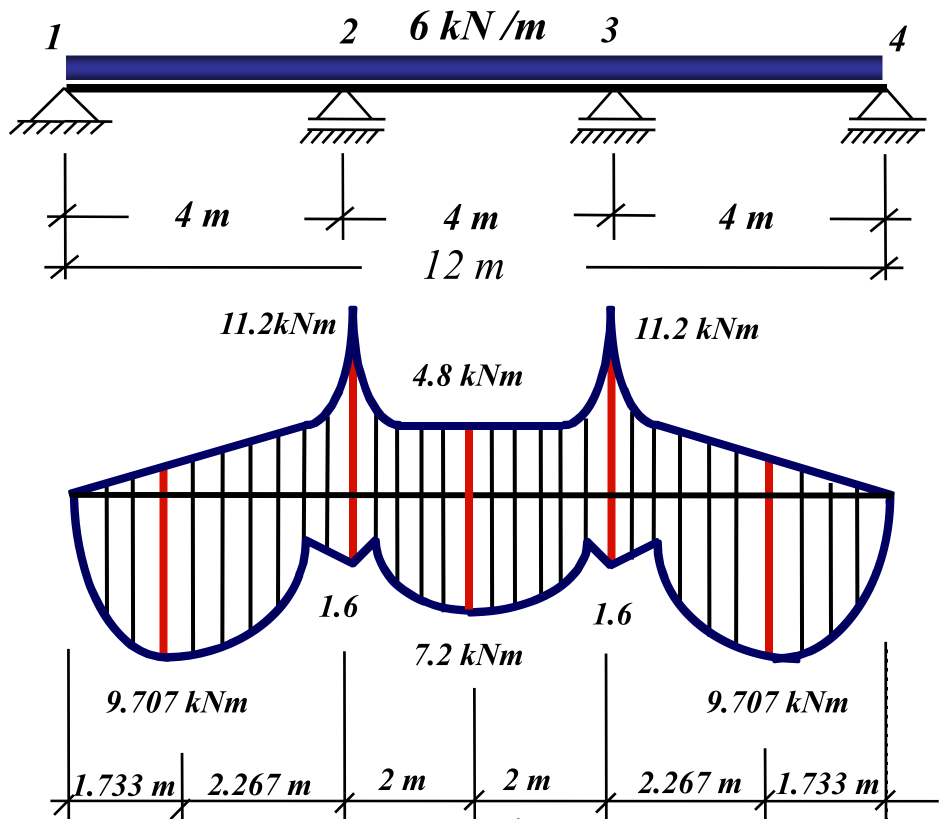


$F_{min} = -(0.062+0.139+0.113)\ 20 = \mathbf{-4.28\ kN}$

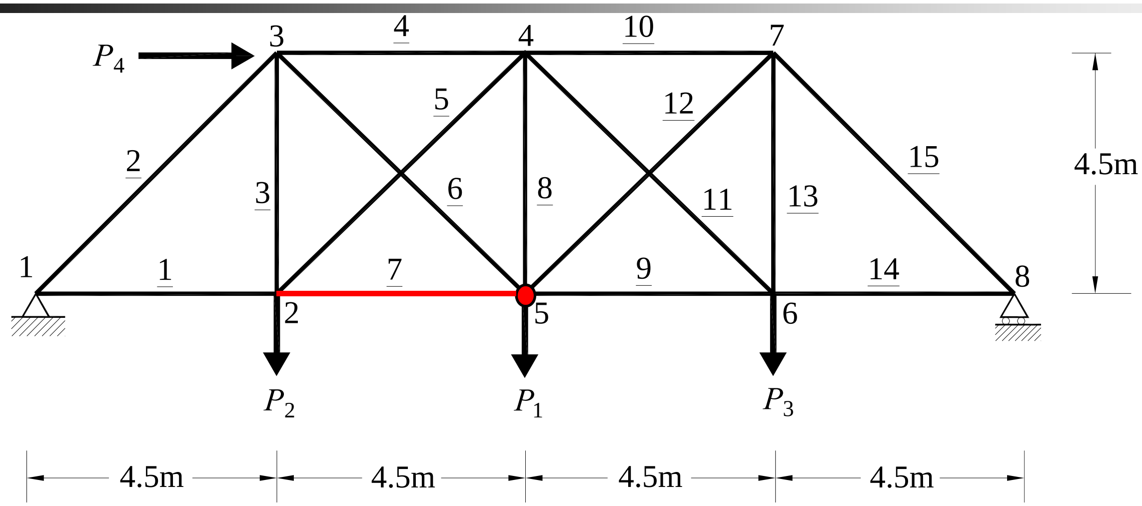
$F_{max} = (0.464+0.309+0.258+0.192+0.128+0.064)\ 20 = \mathbf{28.3\ kN}$

Examples – Load Uncertainty

- **Three-Span Beam**



Truss structure



$A_1, A_2, A_3, A_{13}, A_{14}, A_{15}$: $[9.95,\ 10.05]\ \text{cm}^2$ (1% uncertainty)

cross-sectional area

of all other elements: $[5.97,\ 6.03]\ \text{cm}^2$ (1% uncertainty)

modulus of elasticity of all elements: 200,000 MPa

$\boldsymbol{p}_1 = [190, 210]$ kN, $\boldsymbol{p}_2 = [95, 105]$ kN

$\boldsymbol{p}_3 = [95, 105]$ kN, $\boldsymbol{p}_4 = [85.5, 94.5]$ kN (10% uncertainty)

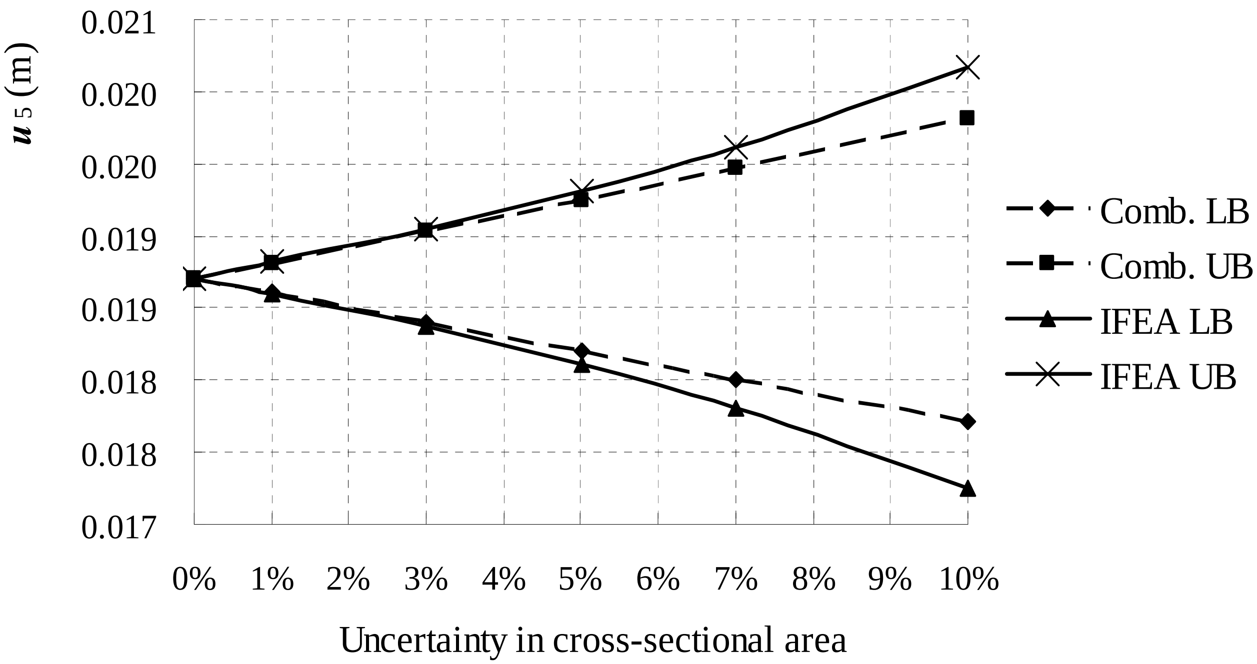
Truss structure - results

Table: results of selected responses

Method	u_5(LB)	u_5(UB)	N_7(LB)	N_7(UB)
Combinatorial	0.017676	0.019756	273.562	303.584
Naïve IFEA	– 0.011216	0.048636	– 717.152	1297.124
δ	163.45%	146.18%	362%	327%
Present IFEA	0.017642	0.019778	273.049	304.037
δ	0.19%	0.11%	0.19%	0.15%

unit: u_5 (m), N_7 (kN). LB: lower bound; UB: upper bound.

Truss structure – results



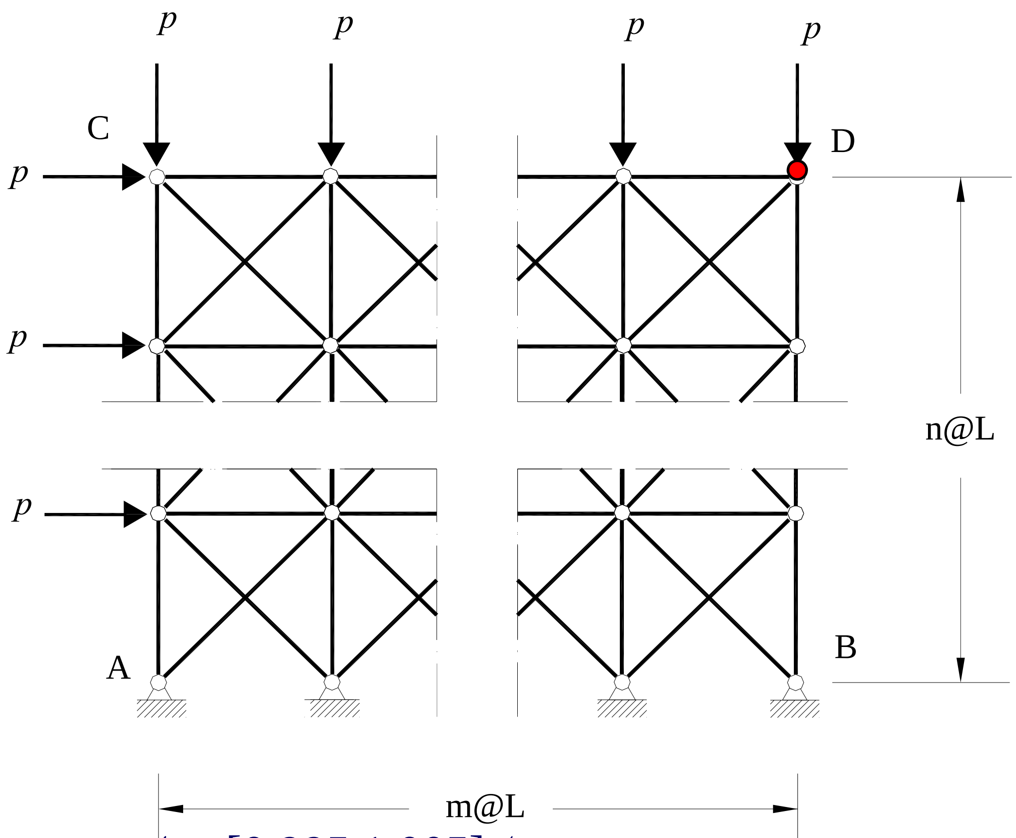
- for moderate uncertainty (≤ 5%), very sharp bounds are obtained
- for relatively large uncertainty, reasonable bounds are obtained

in the case of 10% uncertainty:

Comb.: $\boldsymbol{u}_5$ = [0.017711,0.019811], IFEM: $\boldsymbol{u}_5$ = [0.017252,0.020168]

(relative difference: 2.59%, 1.80% for LB, UB, respectively)

Truss with a large number of interval variables



story × bay	N_e	N_v
3×10	123	246
4×12	196	392
4×20	324	648
5×22	445	890
5×30	605	1210
6×30	726	1452
6×35	846	1692
6×40	966	1932
7×40	1127	2254
8×40	1288	2576

$\boldsymbol{A}_i = [0.995, 1.005] A_0,$

$\boldsymbol{E}_i = [0.995, 1.005] E_0 \quad \text{for } i = 1, \ldots, N_e$

Scalability study

vertical displacement at right upper corner (node D): $\boldsymbol{v}_D = \boldsymbol{a}\dfrac{PL}{E_0 A_0}$

Table: displacement at node D

Story×bay	Sensitivity Analysis		Present IFEA				
	LB^*	UB^*	LB	UB	δ_{LB}	δ_{UB}	wid/d_0
3×10	2.5143	2.5756	2.5112	2.5782	0.12%	0.10%	2.64%
4×20	3.2592	3.3418	3.2532	3.3471	0.18%	0.16%	2.84%
5×30	4.0486	4.1532	4.0386	4.1624	0.25%	0.22%	3.02%
6×35	4.8482	4.9751	4.8326	4.9895	0.32%	0.29%	3.19%
7×40	5.6461	5.7954	5.6236	5.8166	0.40%	0.37%	3.37%
8×40	6.4570	6.6289	6.4259	6.6586	0.48%	0.45%	3.56%

$\delta_{LB} = |LB - LB^*| / LB^*$, $\delta_{LB} = |UB - UB^*| / UB^*$, $\delta_{LB} = (LB - LB^*) / LB^*$

Efficiency study

Table: CPU time for the analyses with the present method (unit: seconds)

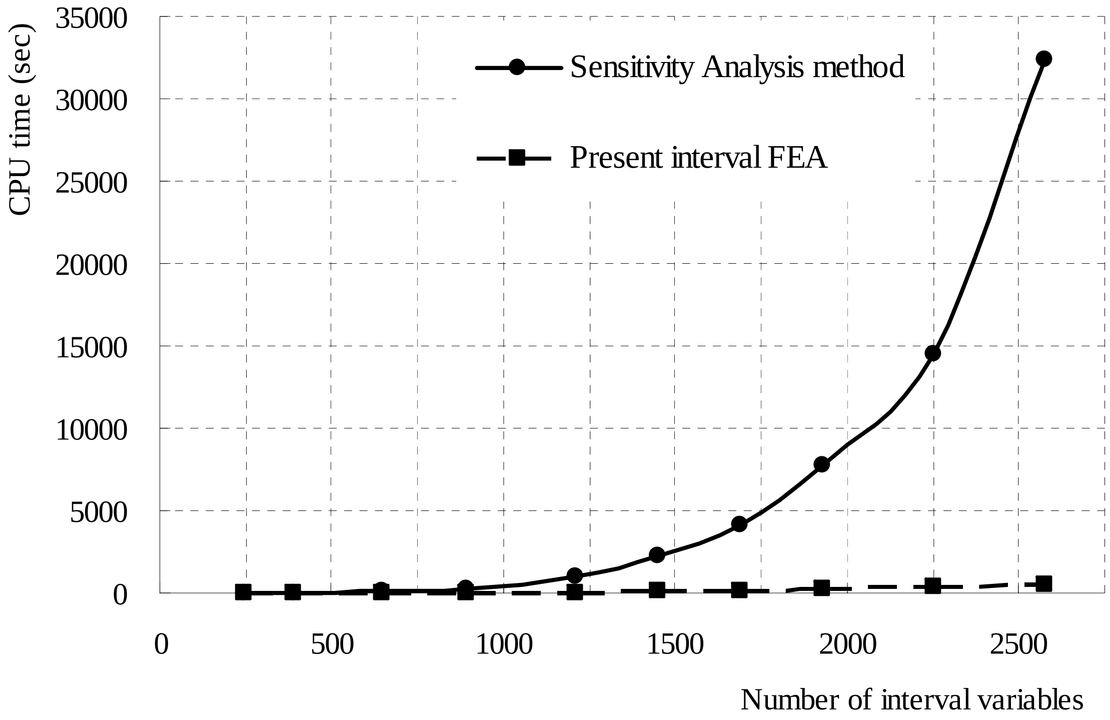
Story×bay	N_v	Iteration	t_i	t_r	t	t_i/t	t_r/t
3×10	246	4	0.14	0.56	0.72	19.5%	78.4%
4×20	648	5	1.27	8.80	10.17	12.4%	80.5%
5×30	1210	6	6.09	53.17	59.70	10.2%	89.1%
6×35	1692	6	15.11	140.23	156.27	9.7%	89.7%
7×40	2254	6	32.53	323.14	358.76	9.1%	90.1%
8×40	2576	7	48.454	475.72	528.45	9.2%	90.0%

t_i : iteration time, t_r : CPU time for matrix inversion, t : total comp. CPU time

- majority of time is spent on matrix inversion

Efficiency study

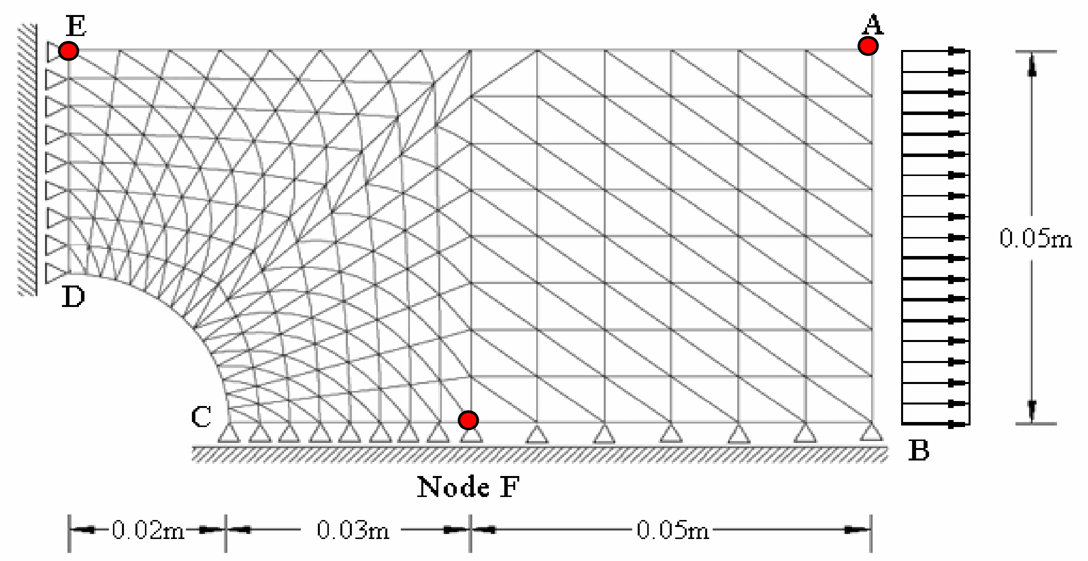
Computational time: a comparison of the senstitivity analysis method and the present method



Computational time (seconds)

N_v	Sens.	Present
246	1.06	0.72
648	64.05	10.17
1210	965.86	59.7
1692	4100	156.3
2254	14450	358.8
2576	32402	528.45
	9 hr	9 min

Plate with quarter-circle cutout



thickness: 0.005m

Possion ratio: 0.3

load: 100kN/m

modulus of elasticity:

$\boldsymbol{E}$ = [199, 201]GPa

number of element: 352

element type: six-node isoparametric quadratic triangle

results presented: $\boldsymbol{u}_A$, $\boldsymbol{v}_E$, $\boldsymbol{\sigma}_{xx}$ and $\boldsymbol{\sigma}_{yy}$ at node F

Plate with quarter-circle cutout

Case 1: the modulus of elasticity for each element varies independently in the interval [199, 201] GPa.

Table: results of selected responses

Response	Monte Carlo sampling*		Present IFEA	
	LB	UB	LB	UB
$\boldsymbol{u}_A$ (10^{-5} m)	1.19094	1.20081	1.18768	1.20387
$\boldsymbol{v}_E$ (10^{-5} m)	−0.42638	−0.42238	−0.42894	−0.41940
$\boldsymbol{\sigma}_{xx}$ (MPa)	13.164	13.223	12.699	13.690
$\boldsymbol{\sigma}_{yy}$ (MPa)	1.803	1.882	1.592	2.090

*10^6 samples are made.

Outline

- Introduction
- Interval Arithmetic
- Interval Finite Elements
- Element-By-Element
- Examples
- Conclusions

Conclusions

- Development and implementation of IFEM
  - uncertain material, geometry and load parameters are described by interval variables
  - interval arithmetic is used to guarantee an enclosure of response
- Enhanced dependence problem control
  - use Element-By-Element technique
  - use the penalty method or Lagrange multiplier method to impose constraints
  - modify and enhance fixed point iteration to take into account the dependence problem
  - develop special algorithms to calculate stress and element nodal force

Conclusions

- The method is generally applicable to linear static FEM, regardless of element type
- Evaluation of the present method
  - Rigorousness: in all the examples, the results obtained by the present method enclose those from the alternative methods
  - Accuracy: sharp results are obtained for moderate parameter uncertainty (no more than 5%); reasonable results are obtained for relatively large parameter uncertainty (5%~10%)

Conclusions

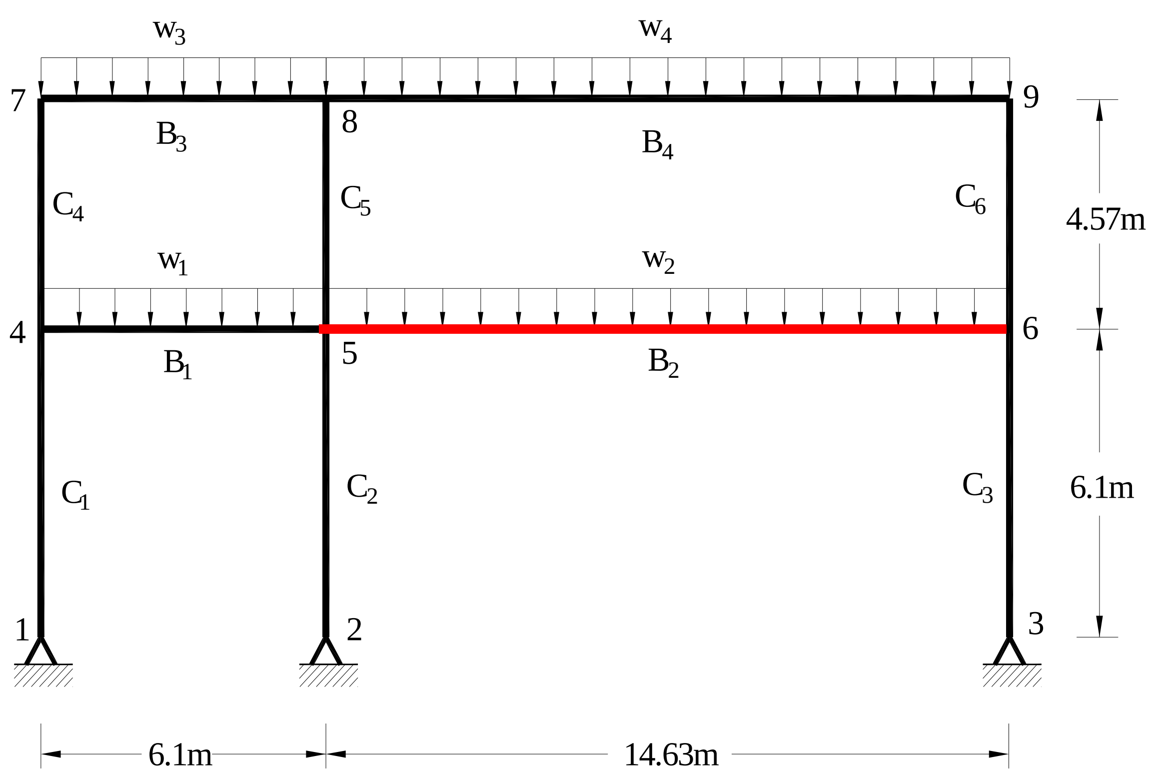
- Scalability: the accuracy of the method remains at the same level with increase of the problem scale
- Efficiency: the present method is significantly superior to the conventional methods such as the combinatorial, Monte Carlo sampling, and sensitivity analysis method

- The present IFEM represents an efficient method to handle uncertainty in engineering applications

Center for Reliable Engineering Computing (REC)

We handle computations with care

Frame structure



Member	Shape
C_1	W12×19
C_2	W14×132
C_3	W14×109
C_4	W10×12
C_5	W14×109
C_6	W14×109
B_1	W27×84
B_2	W36×135
B_3	W18×40
B_4	W27×94

results listed: nodal forces at the left node of member B_2

Frame structure – case 1

Case 1: load uncertainty

$\mathbf{w}_1 = [105.8, 113.1]$ kN/m, $\mathbf{w}_2 = [105.8, 113.1]$ kN/m,

$\mathbf{w}_3 = [49.255, 52.905]$ kN/m, $\mathbf{w}_4 = [49.255, 52.905]$ kN/m,

Table: Nodal forces at the left node of member B_2

Nodal force	Combinatorial		Present IFEA	
	LB	UB	LB	UB
Axial (kN)	219.60	239.37	219.60	239.37
Shear (kN)	833.61	891.90	833.61	891.90
Moment (kN·m)	1847.21	1974.95	1847.21	1974.95

- exact solution is obtained in the case of load uncertainty

Frame structure – case 2

Case 2: stiffness uncertainty and load uncertainty

1% uncertainty introduced to $\boldsymbol{A}$, $\boldsymbol{I}$, and $\boldsymbol{E}$ of each element.

Number of interval variables: 34.

Table: Nodal forces at the left node of member B_2

Nodal force	Monte Carlo sampling*		Present IFEA	
	LB	UB	LB	UB
Axial (kN)	218.23	240.98	219.35	242.67
Shear (kN)	833.34	892.24	832.96	892.47
Moment (kN.m)	1842.86	1979.32	1839.01	1982.63

*10^6 samples are made.