ON RANGE COMPUTATIONS USING EXTRAPOLATION AND NIE

P.S.V. Nataraj and **Shanta Sondur**

Department of Systems and Control Engineering
Indian Institute of Technology, Bombay
Powai, Mumbai 400 076
nataraj@ee.iitb.ac.in
shanta@ee.iitb.ac.in

Extended Abstract

The Natural Interval Extension (NIE) used widely in interval analysis is known to have the first order convergence property, i.e., the excess width of the range enclosures obtained with the NIE goes down at least linearly with the domain width [3]. In this paper, we show how range enclosures of higher convergence orders can be obtained from the sequence of range enclosures generated with the NIE and uniform subdivision. We combine the well-known Richardson Extrapolation Process [4] with Brezenski's error control method [1] to generate nonvalidated range enclosures containing the true range (upto machine precision accuracy). We demonstrate our proposed method for accelerating the convergence orders on several multidimensional examples, varying from one to six dimensions.

Illustrative Example: Consider the 3-dimensional example of Berz and Makino [2].

$$f(x,y,z) = \frac{4\tan(3y)}{3x + x\sqrt{\frac{6x}{-7(x-8)}}} - 120 - 2x - 7z(1+2y) - \sinh(0.5 + \frac{6y}{8y+7}) + \frac{(3y+13)^2}{3z} -20z(2z-5) + \frac{5x\tanh(0.9z)}{\sqrt{5y}} - 20y\sin(3z)$$

The domain is $([1.75, 2.25],\ [0.75, 1.25]^2)$.

The true range is $[-10.39014529023959, 17.296315827104715]$.

Table 1 shows the range overestimations computed using the proposed extrapolation method. Column 1 of the table gives the uniform subdivision factor denoted as N, column 2 gives the range overestimation obtained with the NIE, and columns 3 to 8 give the range overestimations for the k^{th} extrapolated column (obtained with Richardson Extrapolation Process and Brezenski's error control method).

Table 1: Range overestimation for the function $f(x)$

N	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
8	16.5	16.5					
16	8.16	8.16					
32	4.06	4.06					
64	2.03	2.03	$8.71e-3$	$7.54e-4$			
128	1.01	1.01	$2.43e-3$	$9.32e-5$	$5.68e-7$	$1.06e-7$	
256	0.51	0.51	$6.39e-4$	$1.16e-5$	$3.63e-8$	$3.32e-9$	$2.07e-11$
512	0.25	0.25	$1.64e-4$	$1.44e-6$	$2.29e-9$	$1.04e-10$	$4.67e-13$

Remark: In the above table 1, for the uniform subdivision factor $N = 512$, the second extrapolated column ($k = 2$) gives a reduction in the overestimation by 1524.4 times (from 0.25 to1.64$e-4$), whereas in the 6^{th} extrapolated column ($k = 6$) the reduction is $5.35e + 11$ times (from 0.25 to $4.67e-13$).

Table 2: Quotients of the entries of Table 1

N	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
8	2.035						
16	2.024						
32	2.017	2.017					
64	2.009	2.009					
128	2.005	2.005	3.580	8.092			
256	2.003	2.003	3.802	8.061	15.647	31.9162	
512	2.0007	2.0007	3.903	8.034	15.851	31.9161	44.202
	$O\left(\frac{1}{N}\right)$	$O\left(\frac{1}{N}\right)$	$O\left(\frac{1}{N^2}\right)$	$O\left(\frac{1}{N^3}\right)$	$O\left(\frac{1}{N^4}\right)$	$O\left(\frac{1}{N^5}\right)$	$O\left(\frac{1}{N^6}\right)$

Remark: In the above Table 2, we see that the excess width obtained with the NIE (given in column 2) goes down linearly with $O\left(\frac{1}{N}\right)$. The rate of convergence is accelerated in the subsequent extrapolated columns from $O\left(\frac{1}{N}\right)$in column 3 to $O\left(\frac{1}{N^6}\right)$in column 8. Therefore, we conclude that the approach developed in this paper provides efficient new method for computing the range enclosures with the NIE and uniform subdivision.

References

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