Monte-Carlo-Type Techniques for Processing Interval Uncertainty, and Their Engineering Applications

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Uncertainty is Important

- In engineering, decisions are made under *uncertainty.*
- *Main source of uncertainty:* measurement errors.
- *Additional source of uncertainty:* we do not know how exactly the devices will be used.
- *Example:*
  - we have limits L_i on the loads l_i in different rooms i;
  - we do not know how exactly these loads will be distributed; and
  - we want to make sure that our design is safe for all possible $l_i \le L_i$.

Traditional Statistical Approach

- Traditionally, *statistical* methods are used.
- Usually, we can safely *linearize* the dependence of the desired quantities y (e.g., stress at different structural points) on the uncertain parameters x_i.
- Thus, we enable *sensitivity analysis.*
- *Problem:* for n parameters, we need n calls to the model.
- Often, the number n of uncertain parameters is huge.
- *Example:* in ultrasonic testing, we record (= measure) signal values at thousands moments of time.
- *Solution:* Monte-Carlo simulations.
- *Advantage:* the number of calls to a model depends only on the desired accuracy ε and not on n.
- So, for large n, these methods are much faster.

Formulation of the Problem

- *Problem:* in real life, we often do not know the exact probability distribution of measurement errors and/or of user loads.
- *Interval uncertainty:* often, all we know is the *intervals* of possible values of the corresponding parameters.
- *Example:* we know that the load l_i is in $[0, L_i]$.
- *Sensitivity analysis:* we can use sensitivity analysis, we can use interval techniques.
- *Problem:* for large n, this takes too long.
- *What we are planning to do:* describe Monte-Carlo type techniques for interval uncertainty.
- *Advantage:* these techniques lead to faster computations.

Formulation of the Problem (cont-d)

- *We know:*
  - the algorithm $f(x_1, \ldots, x_n)$;
  - the measured values $\widetilde{x}_1, \ldots, \widetilde{x}_n$; and
  - the information about the uncertainty

    $$\Delta x_i \stackrel{\text{def}}{=} \widetilde{x}_i - x_i$$

    of each direct measurement.
- *We must estimate:* the uncertainty $\Delta y = \widetilde{y} - y$ of the algorithm's output.

Types of Uncertainty

- *Idea.* We must know:
  - what are the possible values of Δx, and
  - how often can different possible values occur.
- *Ideal case – full info:* we know the cdf $F_i(t)$ for each variable x_i (and we know that x_i are independent).
- *Interval case – no info about probabilities:* we know the interval $[\underline{x}_i, \overline{x}_i]$ of possible values of each x_i.
- *General case: partial into:* we know the intervals $[\underline{F}_i(t), \overline{F}_i(t)]$ that contain $F_i(t)$ (*p-boxes*).
- *Important case – DS:* we know that $x_i \in [\underline{x}_i^{(k)}(t), \overline{x}_i^{(k)}(t)]$ with probability $p_i^{(k)}$.
- *Comment:* we may have different info for different x_i.
- *Comment:* we may have dependent x_i.

Black Box

- *Traditional approach:* apply f step-by-step to the corresponding "uncertain numbers".
- E.g.: probability distributions, intervals, p-boxes.
- *Problem:* in several practical situations, f is given as a *black box*:
  - we do not know the sequence of steps forming f;
  - we can only plug in different values into f and see the results.
- *Examples:*
  - *commercial software*: safeguard vs. competitors;
  - *classified security-related software*: safeguard vs. adversary.
- *Additional problem:* sometimes, f takes so much time that it is only possible to run it a few times.

Sensitivity Analysis: Reminder

- *When applicable:* $f(x_1, \ldots, x_n)$ is monotonic (increasing or decreasing) with respect of each of its variables.
- *Example:* linearizable f.
- *Algorithm:*
  - Compute $\widetilde{y} = f(\widetilde{x}_1, \ldots, \widetilde{x}_n)$.
  - For each i, compute

    $y'_i = f(\widetilde{x}_1, \ldots, \widetilde{x}_i, \widetilde{x}_i + h, \widetilde{x}_{i+1}, \ldots, \widetilde{x}_n)$ $(h > 0)$.
  - Compute $\underline{y} = f(x_1^-, \ldots, x_n^-)$ and $\overline{y} = f(x_1^+, \ldots, x_n^+)$, where:
    * if $y'_i \geq \widetilde{y}$, then $x_i^- = \underline{x}_i$ and $x_i^+ = \overline{x}_i$;
    * if $y'_i < \widetilde{y}$, then $x_i^- = \overline{x}_i$ and $x_i^+ = \underline{x}_i$.
- *Problem:* we need $n + 3$ calls to f.
- For large n and for complex f, this is too slow.

Cauchy Deviates Method

- *When applicable:* linearizable f.
- *In this case:* $[\underline{y}, \overline{y}] = [\widetilde{y} - \Delta, \widetilde{y} + \Delta]$, where $\Delta = \sum\limits_{i=1}^{n} |c_i| \cdot \Delta_i$, and $c_i = \dfrac{\partial f}{\partial x_i}$.
- *What is Cauchy:* $\rho(x) = \dfrac{\Delta}{\pi \cdot (x^2 + \Delta^2)}$.
- *Why Cauchy:* if $\xi_1, \ldots, \xi_n$ are independent Cauchy w/Δ_i, then $\sum\limits_{i=1}^{n} c_i \cdot \xi_i$ is Cauchy with desired Δ.
- *Algorithm:*
  - compute $\delta x_i^{(k)} = \Delta_i \cdot \tan(\pi \cdot (r_i - 0.5))$, $r_i = U[0,1]$.
  - compute $\delta y^{(k)} \stackrel{\text{def}}{=} f(\widetilde{x}_1 + \delta x_1^{(k)}, \ldots, \widetilde{x}_n + \delta x_n^{(k)}) - \widetilde{y}$;
  - find Δ from the MLM:

$$\frac{1}{1 + \left(\dfrac{\delta y^{(1)}}{\Delta}\right)^2} + \ldots + \frac{1}{1 + \left(\dfrac{\delta y^{(N)}}{\Delta}\right)^2} = \frac{N}{2}.$$

- *Advantage:* after $N = 200$ runs, we get 20% accuracy $0.2 \cdot \Delta$ with 95% certainty (corr. to $2\sigma_e$).

Applications: Brief Overview

- *Environmental and power engineering:* safety analysis of complex systems.
- *Civil engineering:* building safety (f is FEM).
- *Petroleum and geotechnical engineering:* f solves inverse problem (x_i – traveltimes).
- *Results:*
  - In the environmental and civil engineering, same results as sensitivity analysis, but faster.
  - In geotechnical engineering, the dependence of the accuracy on the location and depth fits much better with the geophysicists' understanding than statistical estimates.

Limitations of Cauchy Deviate Techniques

- Cauchy deviate technique is based on the following *assumptions:*
  - that the measurement errors are *small*, so we can safely linearize the problem;
  - that we only have *interval* information about the uncertainty, and
  - that we can actually call the program f 200 times.
- *Problem:* in real-life engineering problems, these assumptions are often not satisfied.
- *What we plan to do:* we describe how we can modify the Cauchy techniques to overcome these limitations.

What If We Cannot Perform Many Iterations

- *Problem:* in many real-life engineering problems, we cannot run f 200 times.
- *Idea:* use Cauchy estimates with the available amount of $N \ll 200$ iterations, but use new formulas for Δ.
- *Fact:* for reasonable large N, $\widetilde{\Delta} - \Delta$ is $\approx$ Gaussian.
- *Solution:* $\Delta \le \widetilde{\Delta} \cdot \left(1 + k_0 \cdot \sqrt{\frac{2}{N}}\right)$ (where $k_0 = 2$) w/certainty 95%.
- *Comment:* to get 99.9% certainty, take $k_0 = 3$.
- *Example:* for $N = 50$, $\Delta \le 1.4 \cdot \widetilde{\Delta}$ – not bad.
- *Problem:* for smaller N, $\widetilde{\Delta} - \Delta$ is not Gaussian.
- *Solution:* we empirically find the factor.

Dempster-Shafer Knowledge Bases: An Idea

- *Problem:* for each i, instead of a single interval $\mathbf{x}_i$, we have several intervals $\mathbf{x}_i^{(k)}$ with probabilities $p_i^{(k)}$.
- *Difficulty:* even if we have 2 intervals for $n = 50$ inputs, we have an astronomical number of $2^{50} \approx 10^{15}$ output intervals.
- *Fact:* when $\mathbf{x}_i = [x_i^{\text{mid}} - \Delta_i, x_i^{\text{mid}} + \Delta x_i]$, then $\mathbf{y} = [y^{\text{mid}} - \Delta, y^{\text{mid}} + \Delta]$, where:
$$y^{\text{mid}} = \widetilde{y} + \sum_{i=1}^n c_i \cdot (y_i^{\text{mid}} - \widetilde{y}_i); \quad \Delta = \sum_{i=1}^n |c_i| \cdot \Delta_i.$$
- *DS case:* we have different pairs $(y_i^{\text{mid}}, \Delta_i)$ with different probabilities.
- *Idea:* due to Central Limit Theorem, (y^{mid}, Δ) is approximately normally distributed.
- *Comment:* not *exactly* normal since $\Delta \geq 0$.

Analysis of the Problem

- *Previously:* Cauchy distribution with given Δ.
- *Characteristic function:*

$$E[\exp(\mathrm{i}\cdot\omega\cdot\xi)] = \exp(-|\omega|\cdot\Delta).$$

- *Now:* Gaussian mixture of several Cauchy distributions, with given Δ.
- *Characteristic function:*

$$E[\exp(\mathrm{i}\cdot\omega\cdot\xi)] =$$

$$\int \frac{1}{\sqrt{2\cdot\pi}\cdot\sigma}\cdot\exp\left(-\frac{\Delta-\mu}{2\sigma^2}\right)\cdot\exp(-|\omega|\cdot\Delta)\,\mathrm{d}\Delta.$$

- *Simplified expression:*

$$E[\exp(\mathrm{i}\cdot\omega\cdot\xi)] = \exp\left(\frac{1}{2}\cdot\sigma^2\cdot\omega^2-\mu\cdot|\omega|\right).$$

Algorithm

- For different real values $\omega_1, \ldots, \omega_k > 0$, compute $l(\omega_k) \stackrel{\text{def}}{=} -\ln(c(\omega_k))$, where

$$c(\omega_k) \stackrel{\text{def}}{=} \frac{1}{N} \cdot \sum_{k=1}^{N} \cos(\omega \cdot y^{(k)}).$$

- Use the Least Squares Method to find the values μ and σ for which

$$\mu \cdot \omega_k - \frac{1}{2}\sigma^2 \cdot \omega_k^2 \approx l(\omega_k).$$

- The resulting value μ is the average Δ.
- We repeat the above algorithm twice:
  - for samples for which $y^{\text{mid}} \le E[y^{\text{mid}}]$, and
  - for samples for which $y^{\text{mid}} > E[y^{\text{mid}}]$.
- Based on two μ's, we compute $E[\Delta]$ and $\sigma[\Delta]$.

What About p-Boxes?

- *Known fact:* a p-box can be described as a DS knowledge base.
- *Specifics:* a p-box $[\underline{F}(t), \overline{F}(t)]$ can be described by listing, for each p, the interval $[\underline{f}(p), \overline{f}(p)]$ of the possible quantile values:
  - the function $\underline{f}(p)$ is an inverse function to $\overline{F}(t)$, and
  - the function $\overline{f}(p)$ is an inverse function to $\underline{F}(t)$.
- *Conclusion:* whatever method we have for DS knowledge bases, we can apply it to p-boxes as well.
- *Handling different types of uncertainty for different x_i:* just translate them into p-boxes.

Cauchy Method for Quadratic f

- *Linear case:* quadratic and higher order terms can be ignored.

- *Next case:* linear terms are still prevailing, but quadratic terms can no longer be ignored:

$$\delta y \stackrel{\text{def}}{=} \sum_{\imath=1}^{n} c_i \cdot \delta x_i + \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} \cdot \delta x_i \cdot \delta x_j.$$

- *Analysis:* since linear terms are prevailing, max and min are attained when $\delta x_i = \pm \Delta_i$ (depending on $\varepsilon_i \stackrel{\text{def}}{=} \text{sign}(x_i)$):

$$\Delta^+ = \sum_{i=1}^{n} |c_i| \cdot \Delta_i + \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} \cdot \varepsilon_i \cdot \varepsilon_j \cdot \Delta_i \cdot \Delta_j;$$

$$\Delta^- = \sum_{i=1}^{n} |c_i| \cdot \Delta_i - \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} \cdot \varepsilon_i \cdot \varepsilon_j \cdot \Delta_i \cdot \Delta_j.$$

- *Problem:* for large n, literal computation takes too long.

- *Objective:* design a Cauchy-type method for this case.

Algorithm

- *Auxiliary algorithm:* $z = (z_1, \ldots, x_n) \to g(z)$: apply the linear Cauchy deviate method to the auxiliary function $t \to \frac{1}{2} \cdot (f(x^{\text{mid}} + z + t) - f(x^{\text{mid}} + z - t))$ and the values $t_i \in [-\Delta_i, \Delta_i]$.

- *Main algorithm:*

  - We apply the algorithm $g(z)$ to the vector $0 = (0, \ldots, 0)$, thus computing the value $g(0)$.

  - We apply the linear Cauchy deviate method to the auxiliary function
  $$h(z) = \frac{1}{2} \cdot (g(z) - g(0) + f(x^{\text{mid}} + z) - f(x^{\text{mid}} - z));$$
  the result is the desired value Δ^+.

  - Finally, we compute Δ^- as $2g(0) - \Delta^+$.

- *Computational complexity:* $2N^2$ calls to f.

- *Conclusion:* this method is better if $n \gg 8 \cdot 10^4$.

Acknowledgments

This work was supported in part:

- by NASA under cooperative agreement NCC5-209;
- by NSF grants EAR-0112968, EAR-0225670, and EIA-0321328;
- by Army Research Laboratories grant DATM-05-02-C-0046;
- by NIH grant 3T34GM008048-20S1;
- by Applied Biomathematics;
- by a research grant from Sandia National Laboratories as part of the Department of Energy Accelerated Strategic Computing Initiative (ASCI).