Structural Design under Fuzzy Randomness

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Abstract. In this paper a procedure for designing structures under uncertainty is presented. The uncertainty model of fuzzy randomness is employed to take into account the uncertainty of structural parameters in a realistic and comprehensive manner. This uncertainty model includes real valued random variables and fuzzy variables as special cases. Objective uncertainty and subjective uncertainty are processed simultaneously.

Algorithms of fuzzy structural analysis (processing of fuzzy variables in structural analysis) and fuzzy probabilistic safety assessment (processing of fuzzy random variables, real random variables, and fuzzy variables in safety assessment) are used to compute fuzzy structural responses and fuzzy safety prognoses, which are the backbone of the new design concept. Comparing fuzzy structural responses and the fuzzy safety level with permissible values, discrete permissible and nonpermissible parameter vectors are identified. These are introduced into a fuzzy cluster analysis to obtain permissible and nonpermissible clusters (continuous sets of real parameter vectors with similar properties), which represent the basis for generating uncertain structural design parameters.

This concept is referred to as fuzzy cluster design. It can be combined with arbitrary fundamental solutions for deterministic and probabilistic structural safety analyses. For instance, well developed algorithms of Monte Carlo simulation and codes of nonlinear structural analysis can be incorporated in the procedure.

The algorithm of fuzzy cluster design is presented in detail and demonstrated by way of a numerical example.

Keywords: Fuzzy variables; Fuzzy random variables; Fuzzy structural analysis; Fuzzy probabilistic safety assessment; Fuzzy clustering; Uncertain structural design; Nonlinear structural design.

1. Introduction

Structural engineering focuses on computing structural responses, assessing structural safety, and determining parameters for structural design that meet all relevant requirements. For these purposes, the structural engineer has to apply appropriate structural models, suitably-matched computational models and reliable structural parameters close to reality. Computational models must be capable of numerically simulating the system behavior of the chosen structural model. Such models have already been developed up to a high quality level and are available as nonlinear numerical procedures for solving many problems. Structural models and structural parameters, however, have to be established in the particular case on the basis of plans, drawings, measurements, observations, experiences, expert knowledge, codes, and standards. As a rule certain information regarding structural models and precise values of structural parameters do not exist. Structural models and structural parameters are characterized by uncertainty. Human errors in the manufacture, the use and maintenance of constructions, expert evaluations, and insufficient information sources represent only some examples of uncertain influences. In order to perform realistic structural analysis and safety assessment this uncertainty must be appropriately taken into consideration.

Various concepts are available for mathematically describing and quantifying uncertainty such as, e.g., probability theory [17], including subjective probability approach [31] and Bayes methods [6], interval mathematics [1], convex modeling [5], theory of rough sets [23], fuzzy set theory [2], theory of fuzzy random variables [15] and chaos theory [13]. In the scientific literature the new uncertainty models are not only controversially discussed [9] but also increasingly implemented for the solution of practice-relevant problems [4, 8, 12, 16, 22, 24, 29] These different developments of uncertainty models do not directly contradict each other but rather constitute an entirety.

The procedure presented in this paper takes account of uncertainty that may be quantified using the concept of fuzzy randomness. Structural parameters are modeled as fuzzy random variables, real random variables, or fuzzy variables. These are simultaneously processed in special procedures of fuzzy structural analysis and fuzzy probabilistic safety assessment to yield the associated uncertain structural responses and uncertain safety levels. The uncertainty of the structural parameters is apparent in the results. Generally, arbitrary computational models may be employed as deterministic and probabilistic fundamental solutions in these special procedures. If necessary, sophisticated nonlinear codes for structural analysis such as described in [26] and well developed algorithms of computational stochastics such as Monte Carlo simulation [27] may be applied to obtain results close to reality.

The uncertain results from fuzzy structural analysis and fuzzy probabilistic safety assessment provide a suitable basis for deriving an uncertain structural design. With the aid of a fuzzy cluster analysis algorithm permissible and nonpermissible clusters are detected in the space of the design parameters. These are taken as the basis to generate modified uncertain structural parameters, which represent alternative design variants. By comparing requirements regarding structural responses and safety levels with the fuzzy results associated with these design variants, their permissibility and quality is finally assessed to find the optimum uncertain structural design.

This fuzzy cluster design concept does not make any demands on the underlying deterministic and probabilistic fundamental solutions for fuzzy structural analysis and fuzzy probabilistic safety assessment. As it solves the inverse problem of structural analysis and safety assessment numerically, it is generally applicable and may also be employed to solve deterministic design problems in arbitrary nonlinear cases.

The basis and algorithmic development of fuzzy cluster design are discussed in detail in the sequel.

2. Processing Uncertainty in Structural Safety Assessment

For introducing uncertain structural parameters into structural analysis and safety assessment these must be quantified. Depending on the characteristic of the uncertainty appropriate mathematical models are applied. The uncertain parameters are classified according to the cause of their uncertainty. If exclusively informal or lexical uncertainty appears, the uncertainty characteristic fuzziness may be stated. If the uncertain parameter considered is partly influenced by stochastic uncertainty, but cannot be described clearly using random variables, then the characteristic fuzzy randomness may be assigned. Thereby, real valued random variables may be treated as a special case of fuzzy random variables. These represent uncertain variables whose uncertainty exclusively arises from stochastic causes.

Uncertain parameters whose uncertainty characteristic has been identified as fuzziness are treated on the basis of fuzzy set theory. They are described as fuzzy variables $\tilde{x}$ quantified by membership functions $\mu(x)$ [2, 32], see Figure 1. The specification of the membership function is referred to as fuzzification. General algorithms for fuzzification cannot be provided as it basically represents a subjective assessment [20, 30].

Uncertainty with the characteristic fuzzy randomness is described, quantified, and processed on the basis of the theory of fuzzy random variables. For quantifying fuzzy random variables $\tilde{X}$ fuzzy probability distributions $\tilde{F}(x)$ are introduced [18, 30], see Figure 1. These may be understood as being a bunch of real valued probability distributions assessed by membership values μ indicating their degree of plausibility. The fuzzy probability distribution functions are analytically described by introducing fuzzy functional parameters $\tilde{p}$ into the common equations for distribution functions. Additionally, fuzzy functional types of distributions may be defined. The fuzzy parameters $\tilde{p}$ of the fuzzy probability distributions represent fuzzy variables characterized by membership functions $\mu(p)$. These fuzzy parameters may be determined on the basis of common statistical methods in combination with procedures from fuzzy set theory[18, 20, 30].

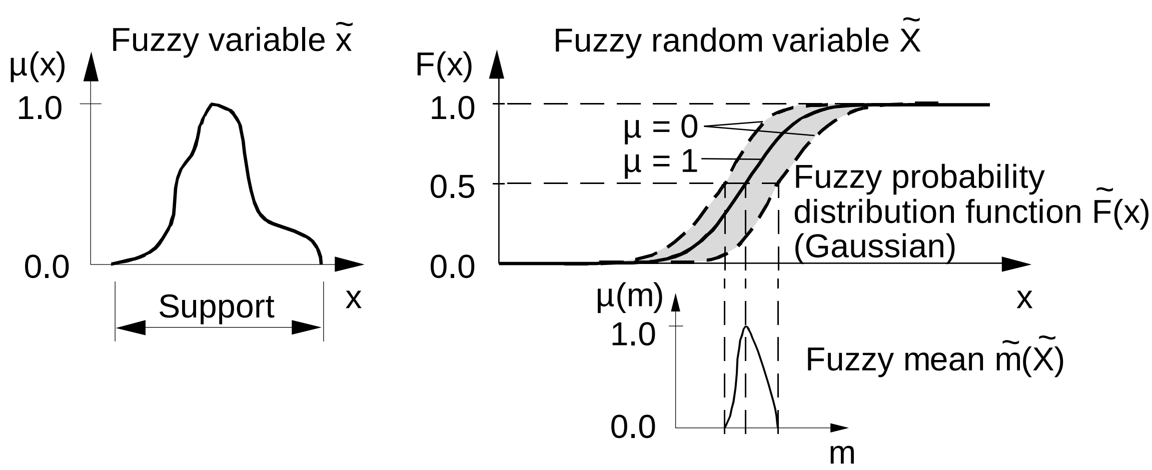

Figure 1. Fuzzy variable $\tilde{x}$ and fuzzy random variable $\tilde{X}$

For the purpose of fuzzy cluster design, the uncertain structural parameters are mapped to fuzzy structural responses and fuzzy safety levels, which are compared with required or permissible values as design constraints.

Fuzzy structural responses are obtained from fuzzy structural analysis [18, 20]. Fuzzy structural analysis implies the analysis of a structure with the aid of a crisp (or uncertain) algorithm applied to fuzzy variables for structural parameters. It may formally be described as the mapping of the fuzzy input vector $\underline{\tilde{x}}$ consisting of the fuzzy structural parameters $\tilde{x}_k$ to the vector $\underline{\tilde{z}}$ containing the fuzzy structural responses $\tilde{z}_j$,

$$\underline{\tilde{x}} \rightarrow \underline{\tilde{z}} \, , \qquad (1)$$

see Figure 2. The mapping 1 is realized with the aid of the mapping model f,

$$\underline{z} = (z_1, ..., z_j, ..., z_m) = f(x_1, ..., x_k, ..., x_r) \, , \qquad (2)$$

which is represented here by a deterministic algorithm for statical or dynamic structural analysis as a deterministic fundamental solution.

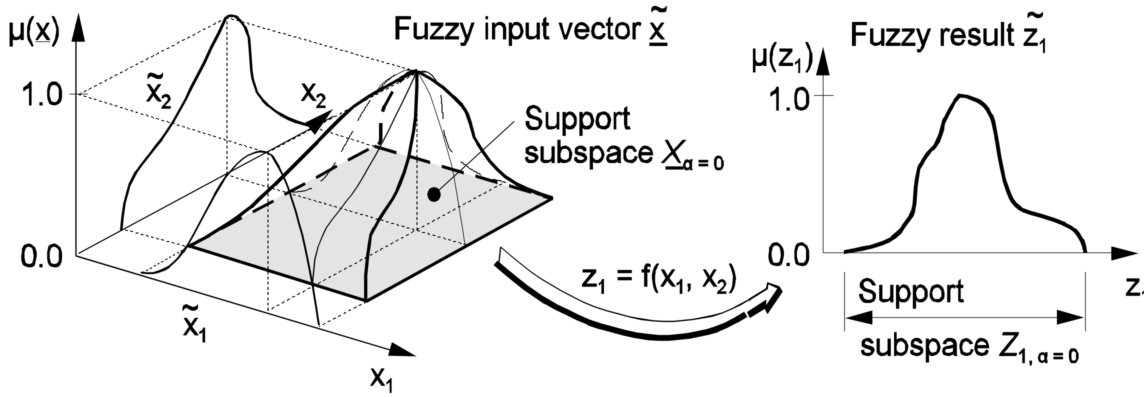

Figure 2. Mapping of fuzzy input variables to a fuzzy result variable

Fuzzy safety levels are computed by applying the concept of fuzzy probabilistic safety assessment. The developed Fuzzy First Order Reliability Method (FFORM) [18, 21] is selected here. Fuzzy random variables together with fuzzy structural parameters are introduced into the algorithms of FFORM to compute a fuzzy reliability index $\tilde{\beta}$. In terms of fuzzy analysis this may be formulated as the mapping

$$\underline{\tilde{x}}^e = (\tilde{x}_1, ..., \tilde{x}_k, ..., \tilde{x}_r, \tilde{p}_1, ..., \tilde{p}_t, ..., \tilde{p}_q) \rightarrow \tilde{\beta} \, , \qquad (3)$$

of an extended fuzzy input vector $\underline{\tilde{x}}^e$ to the fuzzy reliability index $\tilde{\beta}$ representing a special fuzzy result variable. The extended fuzzy input vector $\underline{\tilde{x}}^e$ not only contains fuzzy structural parameters $\tilde{x}_k$ but also comprises the fuzzy parameters $\tilde{p}_t$ of the fuzzy random variables $\tilde{X}_s$.

For purposes of notation convenience, all fuzzy input variables are subsequently denoted by $\tilde{x}_k$, whereas all fuzzy result variables are designated as $\tilde{z}_j$.

For processing the fuzziness of the fuzzy input vector a generally applicable as well as efficient numerical algorithm has been developed and formulated in terms of α-level optimization [18, 20]. This concept permits to implement an arbitrary nonlinear deterministic fundamental solution without any special properties. The membership scale of the fuzzy input vector is discretized (α-discretization), which leads to a certain number of α-level sets

$$\underline{X}_\alpha = \{\underline{x} \in \tilde{\underline{x}} \mid \mu(\underline{x}) \geq \alpha\} \; . \tag{4}$$

The mapping 1 may then be described by

$$\underline{X}_\alpha \rightarrow \underline{Z}_\alpha \, \forall \, \alpha \in (0, 1] \; . \tag{5}$$

The α-level sets $\underline{X}_\alpha$ are mapped to the associated α-level sets $\underline{Z}_\alpha$ of the fuzzy results. For $\alpha = 0$ (as an abbreviation for the limit $\alpha \rightarrow +0$) this mapping concerns the support subspaces $\underline{X}_{\alpha=0}$ and $\underline{Z}_{\alpha=0}$, which contain all elements of the fuzzy input vector and the fuzzy result, respectively, see Figure 2. The resulting α-level sets $\underline{Z}_\alpha$ are described with the aid of their boundings. For their determination a modified evolution strategy has been developed and combined with a repeated solution of the associated optimization problem.

As a consequence, the fuzzy results are obtained with numerically determined membership functions. This means that fuzzy results are not available as connected sets but represent discrete sets of randomly distributed points in the space of the fuzzy result variables. Additionally, the parameter coordinates in the space of the fuzzy input variables which belong to these result points are known.

3. Fuzzy Cluster Design

3.1. Concept

The basic idea of fuzzy cluster design is to derive an appropriate structural design accounting for the uncertainty of design parameters. It is thus reasonable to make use of the numerical results from fuzzy structural analysis and fuzzy probabilistic safety assessment. In contrast to deterministic computations, these results not only provide a single input-output dependency but offer some systematic insight into relationships between structural parameters and structural responses and safety levels over a certain range of parameter values. If the uncertain structural parameters are initially defined in such a way that they cover a proper design domain, then the information gained from fuzzy structural analysis and fuzzy probabilistic safety assessment may be used to find continuous sets of suitable design parameter vectors within this defined domain.

As a result of the mapping 5, a set of input points $\underline{x}_i$ from the support subspace $\underline{X}_{\alpha=0}$ of $\tilde{\underline{x}}$ and a set of result points $\underline{z}_i$ from the support subspace $\underline{Z}_{\alpha=0}$ of $\tilde{\underline{z}}$ are obtained. The result points $\underline{z}_i$ are assigned to the input points $\underline{x}_i$ by means of the mapping model according to 2 on a point-by-point basis,

$$\underline{z}_i = f(\underline{x}_i) \ . \tag{6}$$

From α-level optimization, the inverse assignment is also available point-by-point,

$$\underline{x}_i = f^{-1}(\underline{z}_i) \ . \tag{7}$$

The fact that these dependencies are only known in a discrete form excludes a closed solution of the design problem; continuous sets of permissible design parameter vectors cannot be determined by simply applying 7. The pointwise information from 6 and 7, however, permits designing structures virtually directly with the aid of cluster analysis methods.

The points $\underline{x}_i$, i = 1, ..., n, are lumped together in the discrete point set M_x, whereas the points $\underline{z}_i$, i = 1, ..., n, are combined to form the discrete point set M_z. According to the design constraints CT_h with h = 1, ..., q, the point set M_z is subdivided into two disjoint subsets M_{z+} and M_{z-} comprising exclusively permissible points $\underline{z}_i^+ = (z_{i,1}^+, ..., z_{i,m}^+)$ or only nonpermissible points $\underline{z}_i^- = (z_{i,1}^-, ..., z_{i,m}^-)$, respectively. As a consequence the point set M_x also becomes decomposed due to the dependencies in 6 and 7. The resulting disjoint subsets M_{x+} and M_{x-} of M_x contain only permissible points $\underline{x}_i^+ = (x_{i,1}^+, ..., x_{i,r}^+)$ or only nonpermissible points $\underline{x}_i^- = (x_{i,1}^-, ..., x_{i,r}^-)$, respectively. A cluster analysis is then separately applied to both the permissible points $\underline{x}_i^+$ and the nonpermissible points $\underline{x}_i^-$. The obtained clusters comprised of permissible points represent alternative structural design variants [3, 18, 19], see Figure 3.

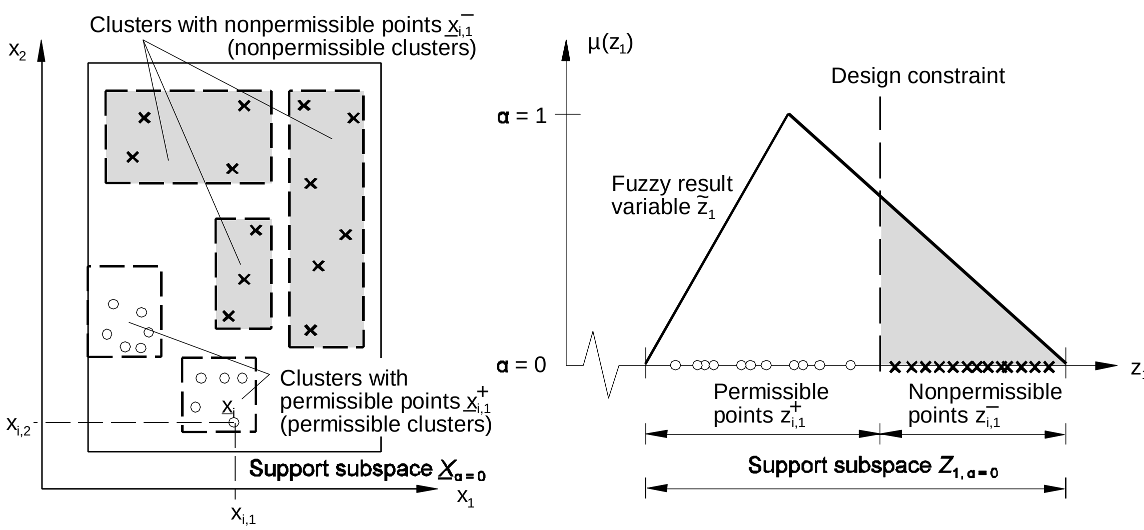

Figure 3. Clustering permissible and nonpermissible input points

The concept presented may be applied to designing structures based on fuzzy structural analysis as well as on fuzzy probabilistic safety assessment. Depending on the particular design problem fuzzy structural parameters or fuzzy parameters of fuzzy probability distributions may be selected as design parameters. Design constraints may be formulated for fuzzy structural responses or for the fuzzy safety level. Furthermore, the concept of fuzzy structural design for the first time provides a tool for designing structures directly and independently of the computational model. That is, every arbitrary nonlinear structural analysis algorithm may be taken as a basis.

3.2. Fuzzy Cluster Analysis

For generating permissible and nonpermissible clusters arbitrary established cluster analysis algorithms may be taken as a basis. The aim of these algorithms is to generate clusters from a given set of data, which are referred to as objects. Objects that are similar to each other are lumped together to form one cluster, whereas objects that are dissimilar are assigned to different clusters. Each cluster should only contain objects of high similarity. That is, the distances d(i, j) between its objects $\square_i$ and $\square_j$ should be as small as possible. Objects from different clusters are characterized by dissimilarity, the distances between these objects should be as large as possible. In structural design with clusters the input points $\underline{x}_i^+$ and $\underline{x}_i^-$ in the support subspace $\underline{X}_{\alpha=0}$ are the objects.

For determining clusters from point sets a variety of methods are available [10, 14, 28]. Fuzzy cluster methods are particularly suitable for designing structures. These are characterized by the fact that the assignment of objects to clusters is fuzzy, which yields fuzzy clusters. The degree with which an object $\square_i$ belongs to a particular cluster C_v is expressed by the membership value $\mu_{iv} \in [0, 1]$. An object may simultaneously be assigned to different clusters with different membership values.

Herein, the fuzzy cluster method presented in [14] is selected as a basis for fuzzy cluster design. In this method the determination of the cluster configuration is formulated as the nonlinear optimization problem

$$\sum_{v=1}^{k} \frac{\sum_{i=1}^{n} \sum_{j=1}^{n} \mu_{iv}^{r} \cdot \mu_{jv}^{r} \cdot d\,(\square_i,\ \square_j)}{2 \cdot \sum_{j=1}^{n} \mu_{jv}^{r}} \ \Rightarrow\ \mathrm{Min} \tag{8}$$

with the equality and inequality constraints

$$\mu_{iv}\ \geq\ 0;\ i = 1,\ ...,\ n;\ v = 1,\ ...,\ k \tag{9}$$

and

$$\sum_{v=1}^{k} \mu_{iv}\ =\ 1;\ i = 1,\ ...,\ n\ . \tag{10}$$

The number k of clusters must be predefined. Then, the objective function is minimized by iteratively improving the membership values μ_{iv} and μ_{jv}. The exponent $r \in [1, \infty)$ controls the influence of the membership values sought. For large values r the result of the cluster analysis exhibits a strong fuzziness characterized by equal membership values $\mu_{iv} = k^{-1}$ for all objects. In contrast to this, a value r close to unity results in an almost crisp clustering. For applications the exponent r = 2 is frequently chosen.

In order to determine a suitable number k of clusters the cluster analysis is usually repeated with a varying number of clusters. The results are then compared based on quality measures to find the optimum solution. The absolute value of the quality measure, however, should not be taken into consideration as the sole criterion. Additionally, the relative improvement in the quality measure when sucessively increasing the number of clusters should be accounted for. In the present study the partition coefficient and the separation degree are applied as major quality measures [10].

The partition coefficient evaluates the "clearness" with which the objects are assigned to clusters. The more uncertain an assignment is obtained, the worse the cluster configuration is assessed to be. It is defined as

$$\mathrm{PC} = \frac{1}{n} \sum_{v=1}^{k} \sum_{i=1}^{n} \mu_{iv}^{2} \,. \tag{11}$$

As an enhancement, the normalized partition coefficient

$$\mathrm{PCN} = 1 - \frac{k}{k-1} \cdot (1 - \mathrm{PC}) \tag{12}$$

is defined [28], which takes values from the interval [0, 1]. An appropriate clustering is characterized by a PCN close to unity.

The separation degree measures the separation of the clusters. That is, the average distance between the clusters and their assigned objects is evaluated with respect to the squared minimum distance between different clusters,

$$\mathrm{SD} = \frac{\sum_{i=1}^{n} \sum_{v=1}^{k} \mu_{iv}^{2} \cdot d\left(\Box_{i}, \Box_{p_v}\right)}{n \cdot \min\left[d^{2}\left(\Box_{p_v}, \Box_{p_w}\right) \;\middle|\; \Box_{p_v}, \Box_{p_w} \in C_1, \ldots, C_k;\ \Box_{p_v} \neq \Box_{p_w}\right]} \,. \tag{13}$$

The objects $\Box_{p_v}$ and $\Box_{p_w}$ denote the prototypes of the clusters C_v and C_w, respectively, which may be interpreted as "artificial, representative objects" of the clusters. In the desired case of a high similarity between the objects of the particular clusters and, simultaneously, of a distinctive dissimilarity between objects from different clusters, the separation degree 13 takes on small values.

3.3. Algorithmic Procedure

The concept from Section 3.1 together with the method of fuzzy cluster analysis from Section 3.2 is now taken as the basis to formulate an algorithm for fuzzy cluster design. This is comprised of the following six steps:

Step I. Fuzzy structural parameters and fuzzy parameters of fuzzy probability distributions that are chosen as fuzzy design parameters are initially defined in such a way that they cover a proper design domain. The parameter values taken into account by this means must be technically reasonable and practically realizable. Design constraints CT_h, $h = 1, \ldots, q$, are formulated according to the particular problem. Fuzzy structural analysis and fuzzy probabilistic safety assessment are performed on the basis of α-level optimization. The final results as well as the intermediate results, which are accumulated during the included repeated optimization, are summarized in the point sets $M_x = \{\underline{x}_i;\ i = 1, ..., n\}$ and $M_z = \{\underline{z}_i;\ i = 1, ..., n\}$ and plotted in the support subspaces $\underline{X}_{\alpha=0}$ and $\underline{Z}_{\alpha=0}$, respectively.

Step II. The result points $\underline{z}_i$in $\underline{Z}_{\alpha=0}$ are evaluated by checking the design constraints CT_h. This permits the separation of the point set M_z into the subset M_{z^+} comprising all points $\underline{z}_i^+$ proven to be permissible and the subset M_{z^-} containing all nonpermissible points $\underline{z}_i^-$,

$$M_z = \{M_{z^+},\ M_{z^-}\}\ . \tag{14}$$

With the aid of the inverse assignment 7, which is known from α-level optimization, the assigned permissible points $\underline{x}_i^+$ and nonpermissible points $\underline{x}_i^-$ in $\underline{X}_{\alpha=0}$ are determined. Accordingly, the point set M_x is subdivided into $M_{x^+} = \left\{\underline{x}_i^+\right\}$ and $M_{x^-} = \left\{\underline{x}_i^-\right\}$,

$$M_x = \{M_{x^+},\ M_{x^-}\}\ . \tag{15}$$

Step III. Fuzzy cluster analysis is applied separately to the point sets M_{x^+} and M_{x^-}. The set M_{x^+} is decomposed into k_1 permissible clusters. From the set M_{x^-} k_2 nonpermissible clusters are determined. The search for an appropriate cluster configuration is performed in two stages. First, clustering is carried out for a varying number of clusters k_1 and k_2. Second, the obtained cluster configurations are assessed based on the numerical quality measures 12 and 13 to select the most suitable clustering. In addition to the values of these quality measures, their relative variation when changing the number of clusters is taken into consideration. Demanding a minimum cluster membership μ_{iv} of all points $\underline{x}_i^+$ and $\underline{x}_i^-$the size of the clusters and hence the intersections between permissible and nonpermissible clusters may be reduced. Permissible clusters may be merged to obtain superclusters covering a bigger domain of permissible points $\underline{x}_i^+$, which may be advantageous when finally defining the structural design.

Step IV. The k_1 permissible clusters C_v are taken as the basis for constructing modified uncertain structural parameters of the first generation. Intersections with nonpermissible clusters are removed from the permissible ones. The remaining reduced clusters $C_{v,red}$ then comprise exclusively permissible parameter combinations to a high probability. The boundaries of the reduced permissible clusters are used to form the α-level sets $X_{1,\alpha=0}^{[v]}, ..., X_{r,\alpha=0}^{[v]}$ for constructing the modified fuzzy design parameters $\tilde{x}_1^{[v]}, ..., \tilde{x}_r^{[v]}$ with $v = 1, \ldots, k_1$. Each reduced permissible cluster yields one sequence of α-level sets $X_{1,\alpha=0}^{[v]}, ..., X_{r,\alpha=0}^{[v]}$. That is, a total of k_1 sequences are generated. The associated support subspaces $\underline{X}_{\alpha=0}^{[v]}$ may intersect each other. With the sets $X_{1,\alpha=0}^{[v]}, ..., X_{r,\alpha=0}^{[v]}$ the supports of the modified fuzzy design parameters are already determined. The membership functions of the $\tilde{x}_1^{[v]}, ..., \tilde{x}_r^{[v]}$ may generally be constructed with arbitrary shapes. As simple shapes are usually preferred, e.g., fuzzy triangular numbers may be used. Their mean values may be determined, e.g., by considering the cluster centers, the prototypes, or other specifically selected points. Moreover, the curves of the membership functions may be oriented to, e.g., the membership values of the points resulting from fuzzy cluster analysis, the local density of points in the cluster, or other characteristics and properties of the cluster. Design parameter values that are preferred due to constructional convenience may be furnished with higher membership values. Generally speaking, all concepts of fuzzification may be applied [18]. The obtained sequences of modified fuzzy design parameters $\tilde{x}_1^{[v]}$, ..., $\tilde{x}_r^{[v]}$ represent alternative structural design variants [v] of the first generation. These design variants may be partly included in each other.

Step V. Clustering permissible points and generating reduced permissible clusters do not guaranty that all design variants include exclusively permissible design parameter values. For the purpose of verification fuzzy structural analysis and fuzzy probabilistic safety assessment are carried out for each design variant. That is, the sequences of modified fuzzy design parameters $\tilde{x}_1^{[v]}$, ..., $\tilde{x}_r^{[v]}$, v = 1, ..., k_1, are introduced into α-level optimization one after the other. This leads to k_1 sequences of modified fuzzy results $\tilde{z}_1^{[v]}$, ..., $\tilde{z}_m^{[v]}$, v = 1, ..., k_1, of the first generation with associated point sets $M_x^{[v]} = \left\{ \underline{x}_i^{[v]};\ i = 1,\ ...,\ n^{[v]} \right\}$ and $M_z^{[v]} = \left\{ \underline{z}_i^{[v]};\ i = 1,\ ...,\ n^{[v]} \right\}$.

Step VI. For each of the k_1 structural design variants of the first generation all points $\underline{z}_i^{[v]}$; i = 1, ..., $n^{[v]}$ obtained from α-level optimization are evaluated by means of the design constraints CT_h. A design variant is considered permissible only if all points $\underline{z}_i^{[v]}$; i = 1, ..., $n^{[v]}$ comply with the requirements according to the CT_h. As soon as one point $\underline{z}_i^{[v]}$ does not meet all CT_h, the associated design variant [v] is considered as not permissible. If the remaining k_1^+ permissible design variants do not represent satisfying results for some reason or if no permissible design variants have been found, $k_1^+ = 0$, fuzzy cluster analysis according to Step III may be repeated to obtain modified design variants of the second generation. Taking account of the additional points $\underline{x}_i^{[v]}$ and $\underline{z}_i^{[v]}$ from the verification analyses and selecting an other cluster configuration may so lead to more satisfying results. The k_1^+ alternative permissible design variants are compared to find out the most suitable one. For this comparison the numerical assessment criteria considered in the subsequent section may be used. Finally, the selected design variant is taken as the basis for defining the structural design. The final structural design parameters must lie within the uncertainty of the permissible design parameters. They may be crisp or uncertain depending on the conditions for their specification.

3.4. Assessment of Alternative Design Variants

For assessing the alternative permissible structural design variants obtained in Step VI of the design procedure (Section 3.3) arbitrary criteria may be formulated and combined with each other. The following two criteria, however, may represent the most important ones being of major interest in almost all design problems.

Criterion I - Constraint Distance. This criterion assesses the degree of exploitation of the design constraints for a design variant. The fuzzy result variables $\tilde{z}_1^{[v]}$, ..., $\tilde{z}_m^{[v]}$ are defuzzified with the aid of suitably selected methods, see [18, 25]. For example, the centroid method and the defuzzification algorithms after Chen [7] and Jain [11] may be applied. The Chen/Jain algorithms permit a "biased" defuzzification. That is, the defuzzification can be focused on small values (e.g., when defuzzifying a fuzzy reliability index) or on large values (e.g., when evaluating an internal force). Defuzzification leads to the crisp values $z_{10}^{[v]}$, ..., $z_{j0}^{[v]}$, ..., $z_{m0}^{[v]}$ for the results. Assessment Criterion I is then defined as the sum of the weighted distances $d^{[v]}(.)$ between the crisp values $z_{j0}^{[v]}$ and the design constraints CT_h,

$$A^{[v]} = \sum_{h=1}^{q} w_h \cdot d\left(CT_h,\ \left\{z_{j0}^{[v]}\right\}\right) \tag{16}$$

with w_h representing real weighting factors. In the simple but common case that a particular design constraint CT_h is given for each fuzzy result $\tilde{z}_j^{[v]}$ alone (that is, $q = m$ and $h = j$) as a permissible value $perm\_z_j$, the sum 16 may be rewritten as

$$A^{[v]} = \sum_{j=1}^{m} w_j \cdot \left|\, perm\_z_j - z_{j0}^{[v]} \,\right| \ . \tag{17}$$

The weighting factors w_h are introduced for taking account of the importance of the particular constraints. A small value $A^{[v]}$ characterizes a structural design that exploits the constraints to a high degree. On the other hand, a large $A^{[v]}$ indicates a structural design providing some reserves, e.g., in load-bearing capacity, which may be considered as advantageous.

Criterion II - Robustness. The designed structure is considered as being robust if the sensitivity of the result variables (structural responses or safety measures) is low with regard to fluctuations of the design parameters. That is, a robust structure is characterized by a low uncertainty of the fuzzy results $\tilde{z}_j^{[v]}$ in relation to the uncertainty of the fuzzy design parameters $\tilde{x}_k^{[v]}$. The uncertainty of fuzzy values is computed on the basis of an analog to Shannon's entropy [2, 32]. This yields the absolute uncertainty of $\tilde{z}_j^{[v]}$ as [18]

$$H(\tilde{z}_j^{[v]}) = -k \cdot \int_{z_j^{[v]} \in \tilde{z}_j^{[v]}} g\left(\mu(z_j^{[v]})\right) dz_j^{[v]} \ ;\ k > 0 \tag{18}$$

with

$$g\left(\mu(z_j^{[v]})\right) = \mu(z_j^{[v]}) \cdot \ln\left(\mu(z_j^{[v]})\right) + \left(1-\mu(z_j^{[v]})\right) \cdot \ln\left(1-\mu(z_j^{[v]})\right) \ . \tag{19}$$

The sensitivity of a structure is then defined as

$$B^{[v]} = \sum_{j=1}^{m} u_m \sum_{k=1}^{r} \frac{H(\tilde{z}_j^{[v]})}{H(\tilde{x}_k^{[v]})} \tag{20}$$

with the weighting factors u_m for emphasizing particular result variables $\tilde{z}_j^{[v]}$ depending on their importance. A low value $B^{[v]}$ indicates a low sensitivity (high robustness) of the structure according to design variant [v].

4. Examples

4.1. Problem Definition

The presented concept of fuzzy structural design is demonstrated for the plane reinforced concrete frame shown in Figure 4. The geometrical and physical nonlinear behavior of the structure is numerically simulated on the basis of the analysis algorithms in [26]. Physical nonlinearities are accounted for by using material laws for reinforcement steel and concrete after OETES, which are also provided in [26]. Tension stiffening and the effects of stirrup reinforcement are accounted for in the concrete material law. As for geometrical nonlinearities, the quadratic terms in the deformation-displacement dependencies are taken into account in addition to considering equilibrium for the deformed system. This enables the algorithm to simulate large displacements and moderate rotations. The stiffness of the system is determined by numerical integration incorporated into an incremental iterative approach. The selected computational model is thus capable of considering all essential nonlinearities. The load bearing behavior of the structure is numerically simulated in a sufficiently realistic manner.

The system is modeled using three bars. Fifty integration increments are chosen for each bar and each cross section is subdivided into 60 layers. The loading process is comprised of dead weight, horizontal load P_H, vertical nodal loads $\nu \cdot P_{V0}$, and the line load $\nu \cdot p_0$. After applying dead weight the horizontal load P_H is introduced. Finally, P_{V0} and p_0 are increased incrementally using the load factor ν.

Uncertainty is present in the load factor ν and in the rotational spring stiffness k_φ. For different uncertainty models applied to these structural parameters an appropriate structural design is determined with the aid of fuzzy structural analysis and fuzzy probabilistic safety assessment.

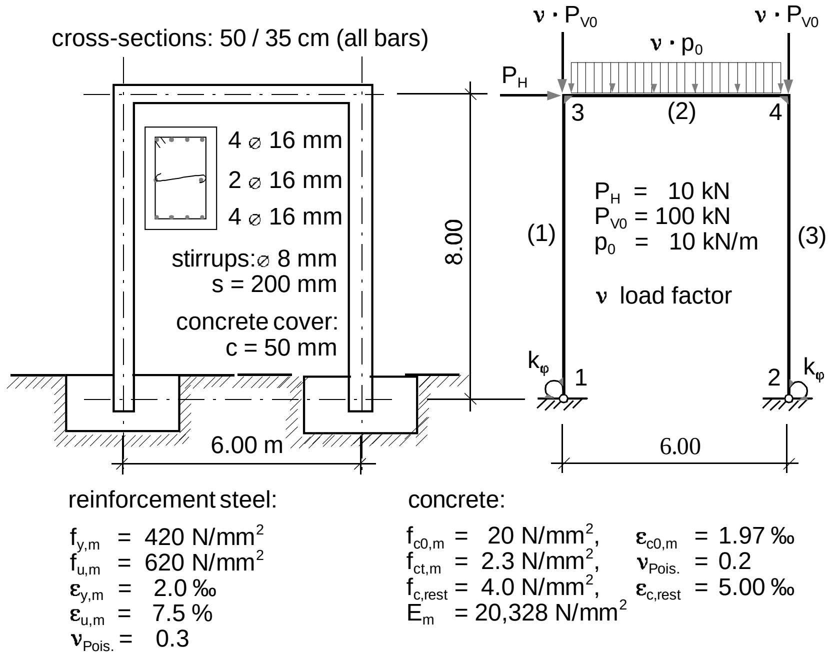

Figure 4. Reinforced concrete frame

4.2. Design with the Aid of Fuzzy Structural Analysis

The uncertainty of the structural parameters ν and k_{φ} is modeled as fuzziness. Load factor and rotational spring stiffness represent fuzzy design parameters. These are described with the aid of fuzzy triangular numbers,

$$\tilde{\nu} \;=\; < 5.5, 5.9, 6.7 > \,, \tag{21}$$

$$\tilde{\mathrm{k}}_{\varphi} \;=\; < 5.0, 9.0, 13.0 > \left[\frac{\mathrm{MNm}}{\mathrm{rad}}\right] . \tag{22}$$

Design constraint CT_1 is a serviceability requirement for the maximum horizontal displacement $\mathrm{v_H}(3)$ of node 3. The permissible displacement is defined as $perm\_\mathrm{v_H}(3) = 4.0$ cm.

Fuzzy structural analysis [18, 20] yields a nonlinear fuzzy load-displacement dependency and, finally, the fuzzy result variable $\tilde{\mathrm{v}}_{\mathrm{H}}(3)$ for the horizontal displacement of node 3, see Figure 5.

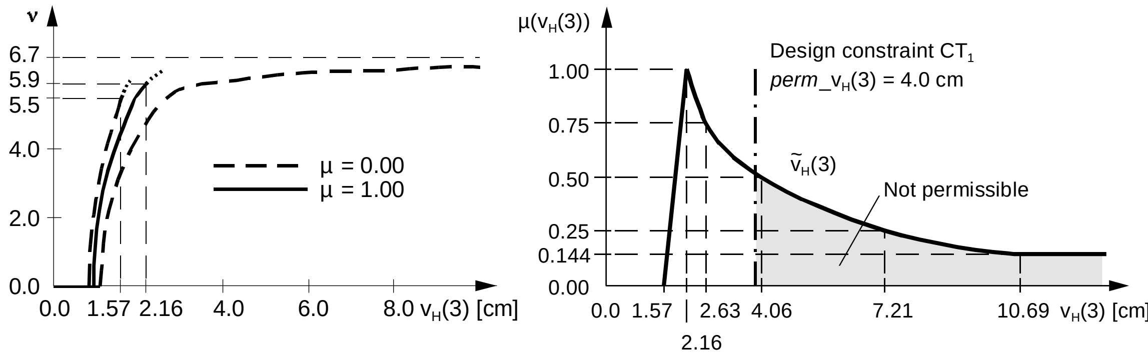

Figure 5. Fuzzy load-displacement dependency and fuzzy result $\tilde{v}_H(3)$

Moreover, from fuzzy structural analysis 75 points $(\nu,\ k_\varphi)_i$ from the space of the fuzzy design parameters and the associated result values $v_{H,i}(3)$ are known. The fuzzy result $\tilde{v}_H(3)$ contains both permissible and nonpermissible points, see Figure 5. Accordingly, 62 permissible and 13 non-permissible points are identified in the space of the fuzzy design parameters.

Fuzzy cluster analysis is separately applied to these permissible and nonpermissible points. A suitable cluster configuration is obtained with three permissible clusters and one nonpermissible cluster, see Figure 6.

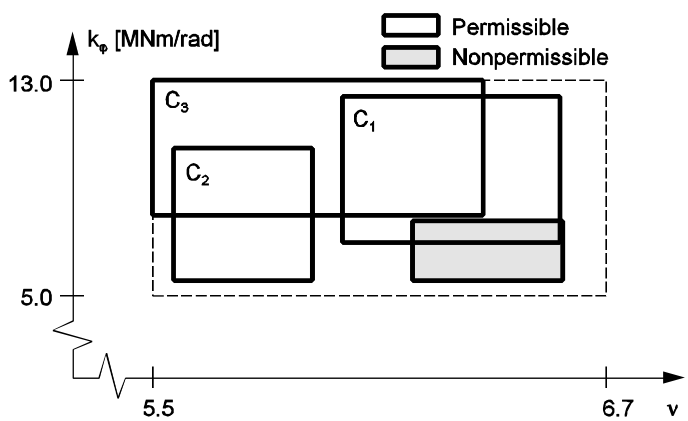

Figure 6. Cluster configuration

A minimum cluster membership of $\mu_{iv} = 0.30$ is hereby prescribed for the assignment of objects to the clusters. Cluster C_1 possesses some intersection with the nonpermissible cluster and is thus excluded from the further design procedure. Clusters C_2 and C_3 cover only permissible points $(\nu,\ k_\varphi)_i^+$ and are considered as capable for generating modified fuzzy design parameters. This yields two resulting alternative design variants,

$$\tilde{\nu}^{[2]} = < 5.56, 5.74, 5.92 > , \tag{23}$$

$$\tilde{\mathrm{k}}_{\varphi}^{[2]} = < 5.60, 8.00, 10.40 > \left[\frac{\mathrm{MNm}}{\mathrm{rad}}\right] , \tag{24}$$

and

$$\tilde{\nu}^{[3]} = < 5.50, 5.94, 6.38 > , \tag{25}$$

$$\tilde{\mathrm{k}}_{\varphi}^{[3]} = < 8.00, 10.50, 13.00 > \left[\frac{\mathrm{MNm}}{\mathrm{rad}}\right] . \tag{26}$$

For verifying the permissibility of the design variants fuzzy structural analysis is carried out for both pairs of modified design parameters. The associated fuzzy results meet the design constraint regarding the permissible displacement in either case, see Figure 7. Defuzzifying the fuzzy displacements $\tilde{\mathrm{v}}_{\mathrm{H}}^{[2]}(3)$ and $\tilde{\mathrm{v}}_{\mathrm{H}}^{[3]}(3)$ after JAIN and computing the sensitivity of the structure lead to

- cluster C_2: $\mathrm{v}_{\mathrm{H}\,0}^{[2]}(3) = 2.49$ cm, $\mathrm{B}^{[2]} = 3.81$
- cluster C_3: $\mathrm{v}_{\mathrm{H}\,0}^{[3]}(3) = 2.54$ cm, $\mathrm{B}^{[3]} = 2.03$

The results show no significant difference in the constraint distance, whereas the sensitivity of design variant [2] is almost twice as high as the sensitivity of design variant [3]. The fuzzy design parameters according to 25 and 26 corresponding to cluster C_3 are thus selected as being the most suitable ones. These fuzzy structural design parameters have to be ensured by an appropriate construction of the system, whereby the remaining uncertainty initialized by noncontrollable parameters, such as soil stiffness, must lie within the design uncertainty.

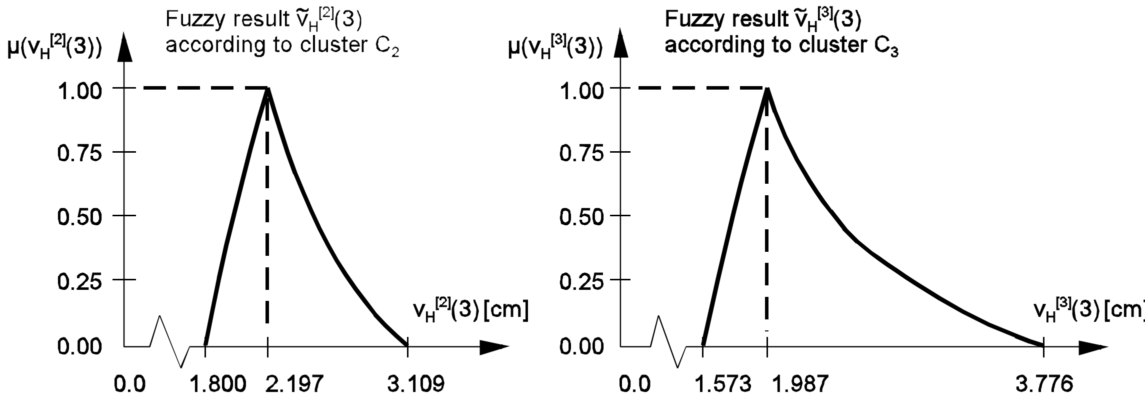

Figure 7. Fuzzy results according to the alternative design variants

4.3. Design with the Aid of Fuzzy Probabilistic Safety Assessment

The reinforced concrete frame shown in Figure 4 is designed with an alternative modeling of the structural parameters. Both the load factor ν and the rotational spring stiffness k_φ are described with fuzzy random variables,

$$\nu \;\rightarrow\; \tilde{X}_1 \;, \tag{27}$$

$$k_\varphi \;\rightarrow\; \tilde{X}_2 \;. \tag{28}$$

The load factor is assumed to follow an extreme value distribution of Ex-Max Type I. The spring stiffness is described by a logarithmic normal distribution. The expected values m and standard deviations σ of both distributions are modeled as fuzzy triangular numbers,

$$\tilde{m}_{X_1} \;=\; < 5.70, 5.90, 6.30 > \;, \tag{29}$$

$$\tilde{\sigma}_{X_1} \;=\; < 0.08, 0.11, 0.15 > \;, \tag{30}$$

and

$$\tilde{m}_{X_2} \;=\; < 8.50, 9.00, 10.00 > \left[\frac{\text{MNm}}{\text{rad}}\right] , \tag{31}$$

$$\tilde{\sigma}_{X_2} \;=\; < 1.00, 1.35, 1.50 > \left[\frac{\text{MNm}}{\text{rad}}\right] . \tag{32}$$

The minimum value of the logarithmic normal distribution of k_φ is assumed to be crisp with $x_{0,2} = 0\,\text{MNm/rad}$. The fuzzy parameters $\tilde{m}_{X_1}$, $\tilde{\sigma}_{X_1}$, $\tilde{m}_{X_2}$, and $\tilde{\sigma}_{X_2}$ are chosen as fuzzy design parameters. Design constraint CT_1 is the required reliability index $req\_\beta = 3.8$ for global system failure.

The actual safety level is computed on the basis of the Fuzzy First Order Reliability Method (FFORM)[18, 21]. This yields the fuzzy reliability index $\tilde{\beta}$ shown in Figure 8. From

α-level optimization a total of 609 result points are known in $\tilde{\beta}$, which are subdivided into 414 permissible points and 195 nonpermissible points by means of the design constraint CT_1. Also, the associated permissible and nonpermissible points in the four-dimensional space of the fuzzy design parameters are known from α-level optimization.

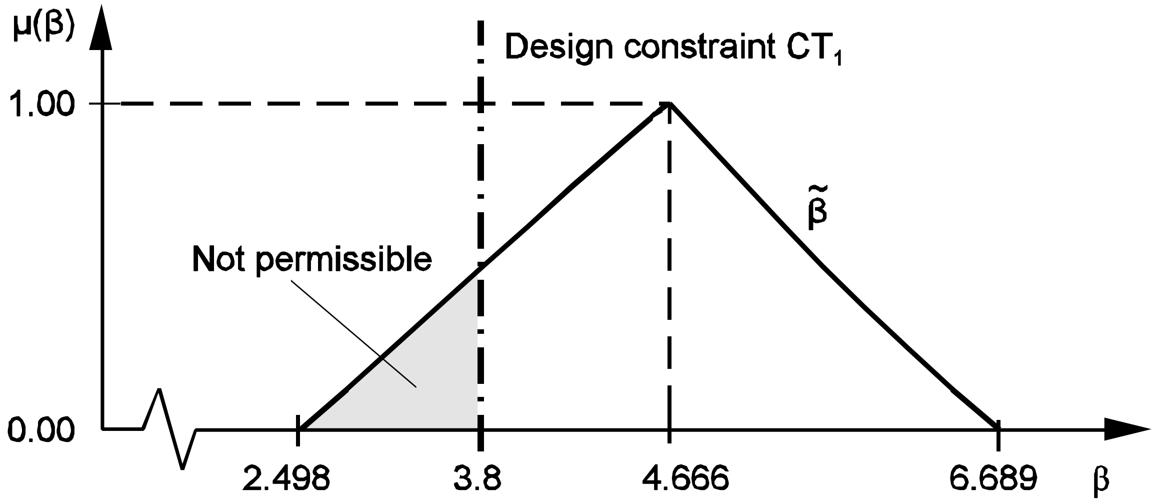

Figure 8. Fuzzy result $\tilde{\beta}$ and design constraint CT_1

The fuzzy cluster method is separately applied to permissible and nonpermissible points in the space of the fuzzy design parameters. The number of clusters is varied within the interval [1, 10]. The obtained cluster configurations are assessed on the basis of the numerical quality measures from Section 3.2. As a compromise a clustering with $k_1 = 6$ permissible clusters is considered as being suitable, see Figure 9.

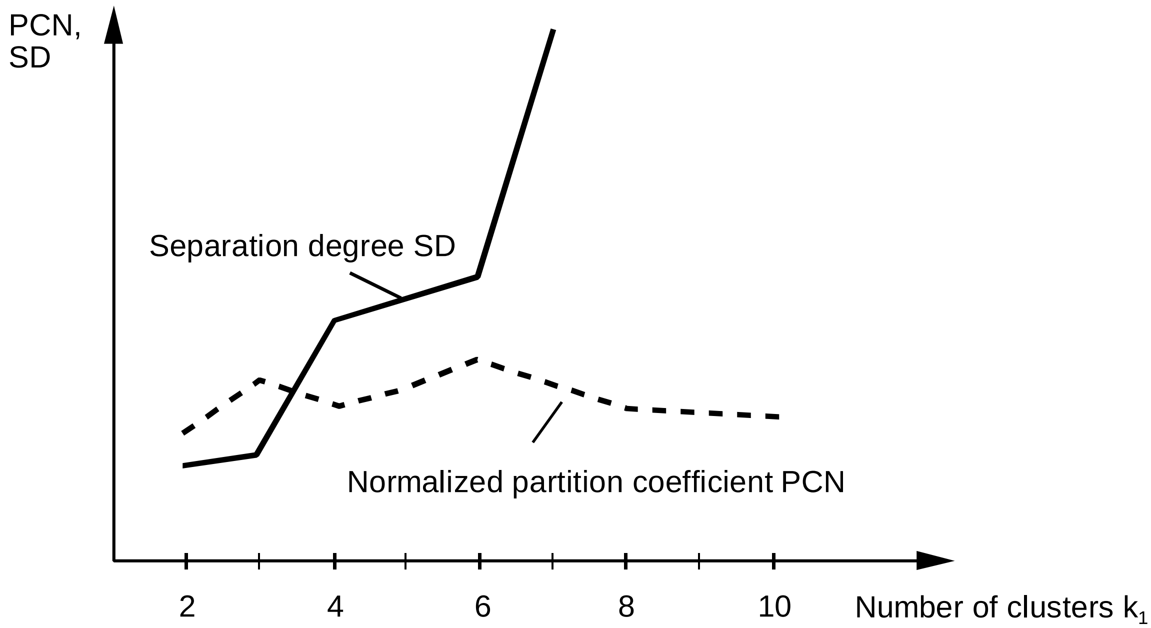

Figure 9. Assessing cluster configurations for permissible points

For the number of nonpermissible clusters $k_1 = 7$ is selected. With a minimum cluster membership of $\mu_{iv} = 0.25$ a cluster configuration without intersections between permissible and nonpermissible clusters is obtained. The permissible clusters C_1, C_2, C_5, and C_6 are very small in size. They cover only a very limited value range of permissible design parameters and are thus not taken as a basis for generating alternative design variants. Clusters C_3 and C_4 are approximately congruent and are merged to constitute the supercluster $C_{3,4}$. Hence, only one set of modified fuzzy design parameters is obtained,

$$\tilde{m}_{X_1}^{[3,4]} = < 5.85, 5.90, 6.00 > , \tag{33}$$

$$\tilde{\sigma}_{X_1}^{[3,4]} = < 0.08, 0.10, 0.11 > , \tag{34}$$

and

$$\tilde{m}_{X_2}^{[3,4]} = < 8.50, 9.50, 10.00 > , \tag{35}$$

$$\tilde{\sigma}_{X_2}^{[3,4]} = < 1.30, 1.40, 1.50 > \ . \tag{36}$$

For verifying this design variant FFORM is applied again with the modified fuzzy design parameters. This yields the modified fuzzy reliability index $\widetilde{\beta}^{[3,4]}$, see Figure 10. All elements of $\widetilde{\beta}^{[3,4]}$ comply with the design constraint CT_1. The modified fuzzy design parameters define modified fuzzy probability distribution functions for the modified fuzzy random variables $\widetilde{X}_1$ and $\widetilde{X}_2$.

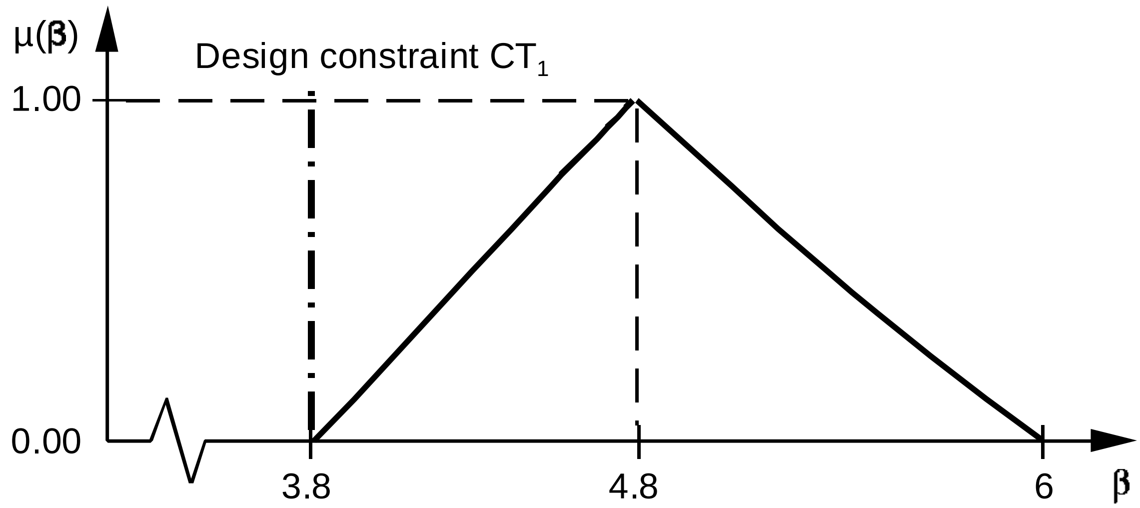

Figure 10. Modified fuzzy result variable $\tilde{\beta}^{[3,4]}$

To ensure the permissibility of the structural design, the final design parameters must be elements of the modified fuzzy design parameters. This may be realized by acquiring more information about ν and k_φ. For example, additional prior information or additional sample elements drawn subsequently may be accounted for with the aid of a Bayesian approach. Also, a larger sample may be taken as the basis for an interval estimation of the parameters. The extent to which additional information is needed can be determined iteratively or inferred by inverting the parameter estimation problem. As a result, a minimum sample size may so be determined.

5. Conclusions

In this paper a concept for designing structures based on nonlinear fuzzy structural analysis (processing of fuzzy variables in nonlinear structural analysis) and fuzzy probabilistic safety assessment (processing of fuzzy random variables, real random variables, and fuzzy variables in safety assessment) has been provided. Thereby, uncertainty is accounted for as fuzziness, randomness, and fuzzy randomness of structural parameters. This is exploited to analyze an initially defined proper range of design parameter values with regard to the associated structural responses and safety levels. With the aid of fuzzy cluster analysis (to determine clusters of design parameter vectors as continuous sets of real vectors with similar properties) alternative uncertain structural design variants are generated from the analysis results. These are compared on the basis of numerical criteria to select the optimum design variant. For instance, a robust structural design may be found in this manner. The final design parameters may then be determined within the fuzziness of the optimum uncertain design variant. They are not necessarily crisp but may contain uncertainty that fits into the uncertainty of the design variant. This is particularly suitable if the information about these parameters is limited, e.g., due to a small sample size.

This concept permits designing structures directly in combination with arbitrary nonlinear computational models and under consideration of nonstochastic uncertainty.

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