Optimal Multilevel System Design under Uncertainty

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Abstract. In this paper we consider hierarchically decomposed multilevel systems, and extend previous deterministic methodologies for optimal and consistent design of such systems to account for the presence of uncertainties. Specifically, we use the probabilistic formulation of the analytical target cascading process to solve the multilevel problem, and use an advanced mean value-based technique to estimate uncertainty propagation. The proposed methodology is demonstrated by means of a simple yet illustrative optimal bi-level system design example.

Keywords: design optimization, hierarchical multilevel systems, analytical target cascading, design under uncertainty, propagation of uncertainties

1. Introduction

Optimal design of complex engineering systems can be accomplished only by decomposition. The system is partitioned into subsystems, the subsystems are partitioned into components, the components into parts, and so on. This decomposition process results in a multilevel hierarchy of elements that comprise the system.

Deterministic optimization approaches assume that complete information of the problem is available, and that design decisions can be implemented. These assumptions imply that optimization results are as good (and therefore useful) as the design and simulation/analysis models used to obtain them, and that they are meaningful only if they can be realized exactly.

In reality, these assumptions do not hold. We are rarely in a position to represent a physical system without using approximations, have complete knowledge on all of its parameters, or control the design variables with high accuracy. It is therefore necessary to treat all quantities associated with uncertainty as stochastic.

In this paper, we consider hierarchically decomposed multilevel systems, and we extend deterministic methodologies for optimal and consistent design of such systems to account for the presence of uncertainties. Our objective is to introduce the concept of uncertainty, model its propagation through the multilevel hierarchy, set the ground for the application of "single-element" optimization under uncertainty methods in multilevel systems, and identify needs for future research.

To the best of our knowledge, no research work on addressing the presence of uncertainties in hierarchically decomposed multilevel systems has been reported in the literature.

However, there is ongoing work to take uncertainties into consideration in the multidisciplinary optimization (MDO) framework [1–10]. Most of these references utilize a simple first-order Taylor expansion to calculate the mean and variance of the response in robust multidisciplinary design or use "worst case" concepts based on first-order sensitivity to evaluate the performance range of a multidisciplinary system.

Although the calculation of the response mean and variance using first-order sensitivity may be adequate for robustness calculations, it does not provide enough statistical information to consider design feasibility under uncertainty. As will be illustrated in this paper, probabilistic representation of the constraints requires complete probabilistic distributions of the system output.

Reliability analysis using probabilistic distributions has been used in MDO [11–13]. Reliability analysis introduces an additional iteration loop resulting in coupled optimization problems that are computationally expensive. Response surfaces have been used to reduce the computational effort [1]. Decoupled reliability and optimization procedures in an MDO framework have been also proposed using approximate probabilistic constraint representations [12]. In general, a double-loop optimization process exists in reliability-based MDO analysis, which repeatedly calls expensive system-level multidisciplinary analyses. A single-loop collaborative reliability analysis method has been recently proposed in [11]. A Most Probable Point (MPP) reliability analysis method is combined with the collaborative disciplinary analyses to automatically satisfy the interdisciplinary consistency in reliability analyses. A single reliability optimization loop uses equality constraints to enforce disciplinary compatibility. Despite the use of a single optimization loop, it is a computationally expensive, "all-at-once" procedure due to the presence of the equality discipline constraints.

It is important to differentiate our research work from that related to multidisciplinary design optimization (MDO). MDO approaches are non-hierarchical in the sense that the optimal design problems are not decomposed according to disciplines into multilevel hierarchies. Discipline outputs are inputs to other disciplines and *vice versa*. This is the significant difference between MDO and our work. In hierarchically decomposed multilevel systems outputs of lower-level elements are inputs to higher-level elements, but not *vice versa*.

The paper is organized as follows. In the next section we present a methodology for optimal design of hierarchical multilevel systems, and extend its formulation to account for uncertainties. In Section 3 we address the issue of modeling uncertainty propagation in multilevel hierarchies and present some analytical examples. A simple yet illustrative simulation-based example is used in Section 4 to demonstrate our methodology for hierarchical multilevel system design. Finally, concluding remarks are summarized in Section 5.

2. Optimal Design of Hierarchically Decomposed Multilevel Systems

Our framework for hierarchical multilevel system optimization under uncertainty is based on analytical target cascading (ATC). In this section we first review the deterministic formulation of ATC, and then we present its extension to account for uncertainties.

2.1. Deterministic Formulation

ATC is a mathematical methodology for translating ("cascading") overall system design targets to element specifications based on a hierarchical multilevel decomposition [14–16]. The objective is to assess relations and identify possible trade-offs among elements early in the design development process, and to determine specifications that yield consistent system design with minimized deviation from design targets.

The ATC process is proven to be convergent when using a specific class of coordination strategies [17], and has been successfully applied to a variety of optimal design problems, e.g., [18–21].

We refer the reader to the above references for a detailed description of ATC. Here, we will briefly present the concept and the general mathematical formulation. In ATC a minimum deviation optimization problem is formulated and solved for each element in the multilevel hierarchy that reflects the decomposed optimal system design problem, *cf.* Figure 1. Therefore, responses of lower-level elements are inputs into higher-level elements.

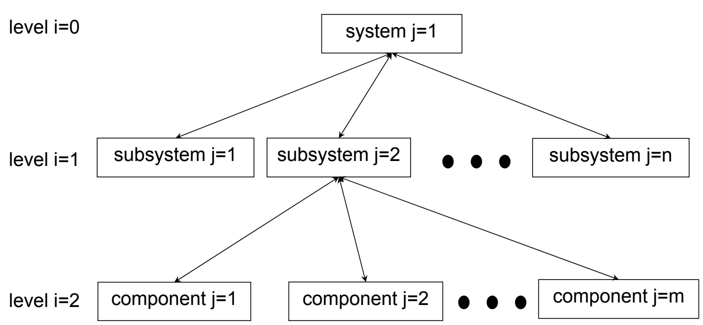

Figure 1. Example of hierarchically decomposed multilevel system

The ATC process aims at minimizing the gap between what higher-level elements "want" and what lower-level elements "can". If design variables are shared among some elements at the same level, their consistency is coordinated by their parent element at the level above.

The mathematical formulation of problem p_{ij}, where i and j denote level and element, respectively, is

$$\begin{aligned} \min_{\tilde{\mathbf{x}}_{ij},\epsilon^r_{ij},\epsilon^y_{ij}} \quad & \|\mathbf{r}_{ij} - \mathbf{r}^u_{ij}\|^2_2 + \|\mathbf{y}_{ij} - \mathbf{y}^u_{ij}\|^2_2 + \epsilon^r_{ij} + \epsilon^y_{ij} \\ \text{subject to} \quad & \textstyle\sum_{k=1}^{n_{ij}} \|\mathbf{r}_{(i+1)k} - \mathbf{r}^l_{(i+1)k}\|^2_2 \leq \epsilon^r_{ij} \end{aligned} \tag{1}$$

$$\begin{aligned} & \textstyle\sum_{k=1}^{n_{ij}} \|\mathbf{y}_{(i+1)k} - \mathbf{y}^l_{(i+1)k}\|_2^2 \leq \epsilon^y_{ij} \\ & \mathbf{g}_{ij}(\mathbf{r}_{(i+1)1}, \ldots, \mathbf{r}_{(i+1)n_{ij}}, \mathbf{x}_{ij}, \mathbf{y}_{ij}) \leq \mathbf{0} \\ & \mathbf{h}_{ij}(\mathbf{r}_{(i+1)1}, \ldots, \mathbf{r}_{(i+1)n_{ij}}, \mathbf{x}_{ij}, \mathbf{y}_{ij}) = \mathbf{0} \\ \text{with} \quad & \mathbf{r}_{ij} = \mathbf{f}_{ij}(\mathbf{r}_{(i+1)1}, \ldots, \mathbf{r}_{(i+1)n_{ij}}, \mathbf{x}_{ij}, \mathbf{y}_{ij}), \end{aligned}$$

where the vector of optimization variables $\tilde{\mathbf{x}}_{ij}$ consists of (n_{ij}) children response design variables $\mathbf{r}_{(i+1)1}, \ldots, \mathbf{r}_{(i+1)n_{ij}}$, local design variables $\mathbf{x}_{ij}$, local shared design variables $\mathbf{y}_{ij}$ (i.e., design variables that this element shares with other elements at the same level), and coordinating variables for the shared design variables of the children $\mathbf{y}_{(i+1)1}, \ldots, \mathbf{y}_{(i+1)n_{ij}}$, and where $\mathbf{g}_{ij}$ and $\mathbf{h}_{ij}$ denote local design inequality and equality constraints, respectively. Tolerance optimization variables ϵ^r and ϵ^y are introduced to coordinate responses and shared variables, respectively. Superscripts u (l) are used to denote response and shared variable values that have been obtained at the parent (children) problem(s), and have been cascaded down (passed up) as design targets (consistency parameters), *cf.* Figure 2.

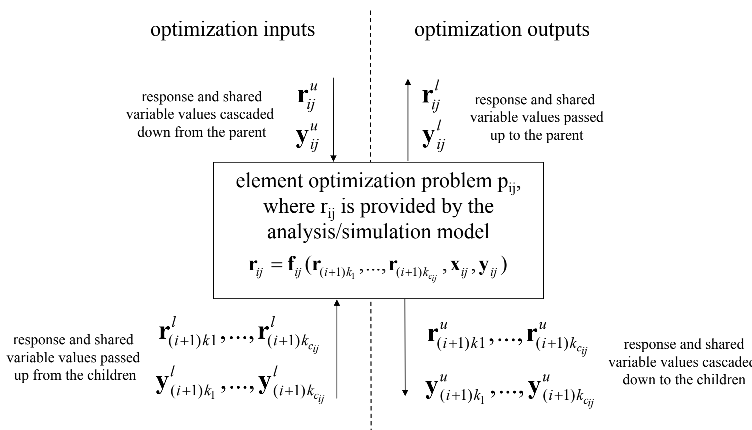

Figure 2. ATC information flow at element j of level i

Assuming that all the parameters have been updated using the solutions obtained at the parent- and children-problems, Problem (1) is solved to update the parameters of the parent- and children-problems. This process is repeated until the tolerance optimization variables in all problems cannot be reduced any further.

2.2. Non-deterministic Formulations

In this section, the ATC formulation is modified to account for uncertainties. Stochastic quantities are represented by random variables and parameters (denoted by upper case latin symbols). For the sake of simplicity, in the following formulations we will assume that all design variables are random and that there exist no random parameters.

2.2.1. *Stochastic Formulation*

In the stochastic formulation, each random variable is represented by a parameter that describes its probabilistic characteristics. Typically, this parameter is the first moment, or mean, of the random variable. Responses and other functions of random variables are expressed as expected values. Thus, Problem (1) becomes

$$\begin{aligned} \min_{\mu_{\tilde{\mathbf{X}}_{ij}},\epsilon_{ij}^R,\epsilon_{ij}^Y} \quad & \|E[\mathbf{R}_{ij}] - \mu^u_{\mathbf{R}_{ij}}\|_2^2 + \|\mu_{\mathbf{Y}_{ij}} - \mu^u_{\mathbf{Y}_{ij}}\|_2^2 + \epsilon_{ij}^R + \epsilon_{ij}^Y \\ \text{subject to} \quad & \textstyle\sum_{k=1}^{n_{ij}} \|\mu_{\mathbf{R}_{(i+1)k}} - E[\mathbf{R}_{ij}]^l\|_2^2 \leq \epsilon_{ij}^R \\ & \textstyle\sum_{k=1}^{n_{ij}} \|\mu_{\mathbf{Y}_{(i+1)k}} - \mu^l_{\mathbf{Y}_{(i+1)k}}\|_2^2 \leq \epsilon_{ij}^Y \\ & E[\mathbf{g}_{ij}(\mathbf{R}_{(i+1)1}, \ldots, \mathbf{R}_{(i+1)n_{ij}}, \mathbf{X}_{ij}, \mathbf{Y}_{ij})] \leq \mathbf{0} \\ & E[\mathbf{h}_{ij}(\mathbf{R}_{(i+1)1}, \ldots, \mathbf{R}_{(i+1)n_{ij}}, \mathbf{X}_{ij}, \mathbf{Y}_{ij})] = \mathbf{0} \\ \text{with} \quad & \mathbf{R}_{ij} = \mathbf{f}_{ij}(\mathbf{R}_{(i+1)1}, \ldots, \mathbf{R}_{(i+1)n_{ij}}, \mathbf{X}_{ij}, \mathbf{Y}_{ij}), \end{aligned} \tag{2}$$

where $E[\cdot]$ denotes the expectation operator.

In words, this formulation attempts to

1. Match the expected values of the local responses with the targets cascaded from the higher level; these targets are the optimal values of the random design variables, i.e., the means, of the higher-level problem.
2. Match the optimal values of the random response design variables, i.e., the means, with the expected values of the children responses.
3. Match the optimal values of the local and children random shared variables, i.e., the means, with the target values cascaded from the higher and lower levels, respectively.

The challenge in solving stochastic optimization problems such as Problem (2) is that evaluating expectations requires knowledge of the probability density functions of the random variables and evaluation of multidimensional integrals.

The solution of Problem (2) satisfies the design inequality and equality constraints in an average sense, but does not provide any information on the percentage of constraint violations due to uncertainty. In practical applications, however, there is a need to satisfy the constraints at a specified target reliability level.

2.2.2. *Probabilistic Formulation*

The constraints are thus reformulated. We now require that the probability of satisfying a constraint under the presence of uncertainties greater than some appropriately selected threshold, or, alternatively, that the probability of violating a constraint is less than some pre-specified probability of failure. The formulation of Problem (2) becomes

$$\begin{aligned} \min_{\mu_{\tilde{\mathbf{X}}_{ij}},\epsilon_{ij}^R,\epsilon_{ij}^Y} \quad & \|E[\mathbf{R}_{ij}] - \mu^u_{\mathbf{R}_{ij}}\|_2^2 + \|\mu_{\mathbf{Y}_{ij}} - \mu^u_{\mathbf{Y}_{ij}}\|_2^2 + \epsilon_{ij}^R + \epsilon_{ij}^Y \\ \text{subject to} \quad & \textstyle\sum_{k=1}^{n_{ij}} \|\mu_{\mathbf{R}_{(i+1)k}} - E[\mathbf{R}_{ij}]^l\|_2^2 \leq \epsilon_{ij}^R \end{aligned} \tag{3}$$

$$\begin{aligned} & \textstyle\sum_{k=1}^{n_{ij}} \|\mu_{\mathbf{Y}_{(i+1)k}} - \mu^l_{\mathbf{Y}_{(i+1)k}}\|_2^2 \leq \epsilon_{ij}^Y \\ & P[\tilde{\mathbf{g}}_{ij}(\mathbf{R}_{(i+1)1}, \ldots, \mathbf{R}_{(i+1)n_{ij}}, \mathbf{X}_{ij}, \mathbf{Y}_{ij}) > 0] \leq \mathbf{P}_f, \\ \text{with} \quad & \mathbf{R}_{ij} = \mathbf{f}_{ij}(\mathbf{R}_{(i+1)1}, \ldots, \mathbf{R}_{(i+1)n_{ij}}, \mathbf{X}_{ij}, \mathbf{Y}_{ij}), \end{aligned}$$

where $P[\cdot]$ denotes probability measure and $\mathbf{P}_f$ is a vector of prespecified probability of failure thresholds.

Note that the mathematical formulation of Problem (3) does not contain equality constraints. Equality constraints do not make sense in a probabilistic framework (it is meaningless to require that a function takes exactly a specific value under the presence of uncertainty, since the probability of a continuous random variable taking an exact value is zero), one has to introduce some slack and treat equality constraints as inequality constraints. For example, if in a deterministic framework it is required that $h(\mathbf{x}) = 0$, in a probabilistic framework it is required that $\mid h(\mathbf{X}) \mid \leq \delta$, where δ is a small positive constant, so that the constraint is formulated as $P[\mid h(\mathbf{X}) \mid -\delta > 0] \leq P_f$. Therefore, we rewrite equality constraints as inequality constraints and unite the two constraint function vectors into one, denoted by $\tilde{\mathbf{g}}$.

Problem (3) can be solved with any of the available commercial software packages or the methods reported recently in the literatures, e.g., the hybrid mean value (HMV) method or the sequential optimization and reliability assessment (SORA) method [22, 23]. We adopt a recently developed single-loop method that is as accurate as the HMV and the SORA methods, but much more efficient [24].

3. Propagation of Uncertainties

The responses of the elements in the multilevel hierarchy are typically nonlinear functions of the elements' inputs, which include random variables and parameters. Thus, responses are themselves random variables, whose expected value must be computed to evaluate objective and constraints when solving probabilistic optimization problems. Moreover, estimated variance of responses is required if robustness considerations are included.

In a multilevel hierarchy, responses of lower-level subsystems are inputs to higher-level subsystems. Therefore, it is necessary to obtain probability distribution information required for the solution of the higher-level problems. This is an issue of outmost importance in design optimization of hierarchically decomposed multilevel systems. An efficient and accurate mechanism is required for propagating probabilistic information in the form of cumulative distribution and probability density functions throughout the hierarchy.

3.1. Estimating Moments Using the Mean-Value First-Order Second-Moment Method

In an initial effort, a mean-value first-order second-moment (MVFOSM) approach was adopted to estimate the mean and standard deviation of a nonlinear function of random variables [25]. Specifically, a first-order Taylor expansion about the current design, represented by the mean vector $\mu_{\mathbf{X}}$ of the random variables $\mathbf{X}$, was used to linearize a nonlinear random response R:

$$R = f(\mathbf{X}) \approx f(\mu_{\mathbf{X}}) + \sum_{i=1}^{n} \frac{\partial f(\mu_{\mathbf{X}})}{\partial X_i}(X_i - \mu_{X_i}), \tag{4}$$

where n is the dimension of the vector $\mathbf{X}$. Assuming that all the random variables are statistically independent (uncorrelated), the first-order approximations of the mean and the variance of R were given by

$$E[R] = \mu_R \approx f(\mu_{\mathbf{X}}) \tag{5}$$

and

$$Var[R] = \sigma_R^2 \approx \sum_{i=1}^{n} \left(\frac{\partial f(\mu_{\mathbf{X}})}{\partial X_i} \right)^2 \sigma_{X_i}^2, \tag{6}$$

respectively.

The advantage of this approach, besides efficiency, is that it allowed us to assume that the responses are normally distributed if all input random variables and parameters were normal. Therefore, propagation of uncertainty in ATC was modeled as a linear process. With the distribution information known, all that was necessary was the estimation of the first two moments, which characterize a normal distribution completely. The validity of the successive linearizations during the ATC process was ensured by virtue of the ATC consistency constraints that do not allow large deviations from current designs.

To our knowledge, this linearization approach is currently embedded in all state-of-the-art software packages for optimization under uncertainty. As will be demonstrated shortly, the linearization approach does a fairly good job in estimating the expected value of nonlinear functions of random variables. However, it can be quite inaccurate in estimating higher moments, e.g., the standard deviation. Moreover, it is limiting in that it does not provide us with the correct probability distribution information of the random nonlinear responses.

It is also important to note that if the linearization approach is used to compute expectations in the stochastic formulation, Problems (1) and (2) generate identical solutions. There is no value in solving the stochastic ATC formulation if expectations are not computed exactly, which requires accurate probability distribution information and multidimensional integrations. This is an additional reason that may explain why the probabilistic constraint formulation is used universally today to solve non-deterministic problems.

3.2. Generating Distributions Using the Advanced Mean Value Method

In this paper, we utilize the advanced mean value (AMV) method to generate the cumulative distribution function (CDF) of a nonlinear response. The AMV method [26] is a computationally efficient method for generating the CDF of nonlinear functions of random variables. It improves the Mean Value (MV) prediction (Section 3.1) by using a simple correction to compensate for errors introduced from the Taylor series truncation. A response performance function $R = f(\mathbf{X})$ is linearized as shown in Eq. (4) and its first and second order moments μ_R and σ_R are calculated using Eqs. (5) and (6), respectively.

A limit state function is then defined as

$$g(\mathbf{X}) = f(\mathbf{X}) - f_0, \tag{7}$$

where f_0 is a particular value of the performance function. The reliability index β is then given by

$$\beta = \frac{\mu_g}{\sigma_g}, \tag{8}$$

where $\mu_g = \mu_R - f_0$ and $\sigma_g = \sigma_R$. The CDF value of f at f_0 is calculated from the first-order relation

$$P[f \leq f_0] = P[g \leq 0] = \Phi(-\beta), \tag{9}$$

where Φ is the standard normal cumulative distribution function. It is emphasized that Eq. (8) is equivalent to calculating the most probable point (MPP) using the linear approximation of Eq. (4). The MPP in the standard normal space is given by

$$\mathbf{U}^* = -\beta \frac{\nabla g(\mathbf{X})}{|\nabla g(\mathbf{X})|}. \tag{10}$$

In the original X space, the MPP coordinates vector is

$$\mathbf{X}^* = \mathbf{U}^* \sigma_{\mathbf{x}} + \mu_{\mathbf{x}}, \tag{11}$$

where $\mu_{\mathbf{x}}$ and $\sigma_{\mathbf{x}}$ are the mean and standard deviation vectors, respectively, of the vector of random variables $\mathbf{X}$.

In the AMV method, the following relation is used instead of Eq. (9):

$$P[f \leq f(\mathbf{X}^*)] = \Phi(-\beta), \tag{12}$$

i.e., the f_0 value at which the reliability index β is calculated is replaced by $f(\mathbf{X}^*)$.

To generate the CDF of $R = f(\mathbf{X})$, the Most Probable Point is first approximated using the simple MV method, which has minimal computational requirements relative to existing MPP-based reliability analysis methods. Once all MPP's $\mathbf{X}_i^*$ for an appropriately discretized range of the performance function at points f_i are obtained, the so-called MPP locus (MPPL) is identified, and is equivalent with the CDF of $R = f(\mathbf{X})$. Subsequently, a single function evaluation $f(\mathbf{X}_i^*)$ is used at each CDF level i to correct the CDF value obtained with the MV method. This so-called AMV-based method is computationally efficient since it requires only a single linearization of the performance function at the mean value and an additional function evaluation at each CDF level (discretized f range at values f_i). It is also very accurate as repeatedly demonstrated in the literature [27–29]. Note that the MPPL-based CFD generation concept has been reported before, but is was based on a less efficient MPP determining procedure [30].

With the CDF available, one can differentiate numerically to obtain the probability density function (PDF). We use central differences to obtain second-order accurate approximations. Finally, to compute moments, we integrate numerically, using spline interpolation to estimate response values that lie between the available PDF values. As will be shown by means of several analytical examples, this method is quite accurate.

3.3. Examples

The MVFOSM-based and AMV-based methods were used to estimate the first two moments of several nonlinear analytical expressions. All random variables were assumed to be normal. Test functions and input statistics are presented in Table I and results are summarized in Table II. One million samples were used for the Monte Carlo simulations.

Table I. Test functions and input statistics

#	Expression	Input Statistics
1	$X_1^2 + X_2^2$	$X_1 \sim N(10, 2)$, $X_2 \sim N(10, 1)$
2	$-\exp(X_1 - 7) - X_2 + 10$	$X_{1,2} \sim N(6, 0.8)$
3	$1 - \frac{X_1^2 X_2}{20}$	$X_{1,2} \sim N(5, 0.3)$
4	$1 - \frac{(X_1+X_2-5)^2}{30} - \frac{(X_1-X_2-12)^2}{30}$	$X_{1,2} \sim N(5, 0.3)$
5	$1 - \frac{80}{X_1^2+8X_2+5}$	$X_{1,2} \sim N(5, 0.3)$

Table II. Estimated moments and errors relative to Monte Carlo simulation (MCS) results

#	1	2	3	4	5
μ_{lin}	200.0	3.6321	−5.25	−1.0333	−0.1428
μ_{AMV}	203.4	3.6029	−5.3495	−1.0380	−0.1454
μ_{MCS}	205.0	3.4921	−5.3114	−1.0404	−0.1448
ϵ_{lin} [%]	−2.44	4.00	−1.15	−0.68	−1.30
ϵ_{AMV} [%]	−0.78	3.17	0.71	−0.23	0.41
σ_{lin}	44.72	1.9386	0.8385	0.1166	0.00627
σ_{AMV}	45.20	0.9013	0.8423	0.1653	0.00631
σ_{MCS}	45.10	0.9327	0.8407	0.1653	0.00630
ϵ_{lin} [%]	−0.84	107.85	−0.26	29.46	−0.47
ϵ_{AMV} [%]	0.22	−3.36	0.19	0	0.15

By inspecting Table II, it can be seen that while the mean-related errors of the linearization approach are within acceptable limits, standard deviation errors can be quite large. The AMV-based moment estimation method performs always better, and never exhibits unacceptable errors. Moreover, the AMV-method provides accurate probability distribution information of nonlinear responses. For example, Figure 3 depicts the CDF and PDF, respectively, of function # 1, obtained using both the MVFOSM-based and the AMV-based method. It can be seen that, using the linearization approach, the nonlinear response would be incorrectly assumed as normally distributed.

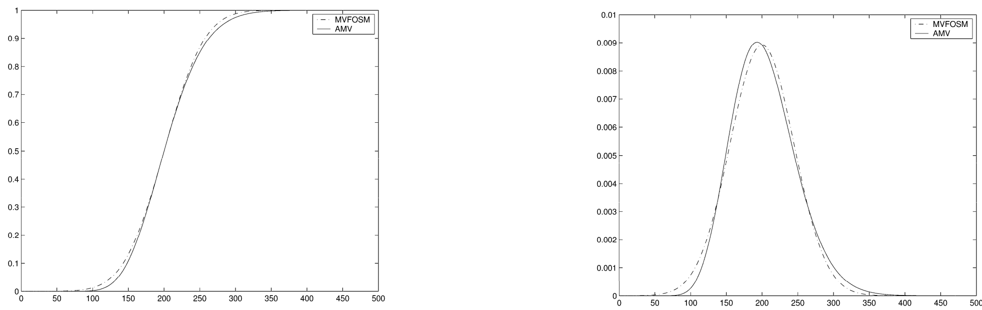

Figure 3. Cumulative distribution and probability density functions for analytical example #1

3.4. Propagating Uncertainty in ATC

Our methodology for propagating uncertainty information during the ATC process can be summarized in the following steps:

1. Start at the bottom level of the hierarchy, where probability distribution on the input random variables and parameters is assumed as known. If such information is not available at the bottom level, start at the lowest level possible where such information is available.

2. Solve the probabilistic design optimization problems for the level specified in step 1.

3. Use the approach described in Section 3.2 to obtain distribution information for the response variables that are inputs to higher-level ("parent") problems.

4. Using the information obtained at step 3, solve the parent problems. Note that the CDFs and PDFs of lower-level ("children") responses that constitute optimization variables in the parent problems are required for solving these problems correctly. Second moment (variance) information alone is inadequate to guarantee proper solution process and uncertainty propagation throughout the hierarchy (as opposed, e.g., to "single"-element robust design optimization).

5. Move your way to the top of the hierarchy.

6. Once you have reached the top-level problem start moving towards the bottom using previous solutions to update parameters as shown in Figure 2.

7. Keep iterating until all ϵ values in all problems in the hierarchy have been reduced as much as possible, i.e., have converged to a steady state value. Note that the ϵ variables

are deterministic, as are the constraints they appear in. While uncertainties are taken into account in the probabilistic design constraints, the non-deterministic ATC process aims at coordinating values of shared variables and responses in an average sense.

Since the linearization approach is sufficiently accurate for estimating expected values, it can be used to reduce computational cost. However, the AMV-based method is so efficient, that it is suggested for use in estimating expected values to improve accuracy and thus, possibly, the convergence rate of the ATC process.

4. Example

The probabilistic formulation of the ATC process (Problem (3)) is used to solve a simple yet illustrative simulation example. We consider a V6 gasoline engine as the system, which is "decomposed" into a subsystem that represents the piston-ring/cylinder-liner subassembly of a single cylinder. The system simulation predicts engine performance in terms of brake-specific fuel consumption. Although the engine has six cylinders, they are all designed to be identical. For this reason, we only consider one subsystem.

The associated bi-level hierarchy, shown in Figure 4, includes the engine as a system at the top level and the piston-ring/cylinder-liner subbassembly as a subsystem at the bottom level. The ring/line subassembly simulation takes as inputs the surface roughness of the

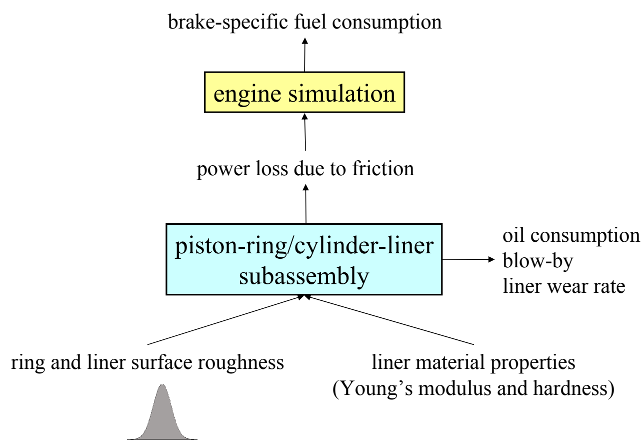

Figure 4. Hierarchical bi-level system

ring and the liner and the Young's modulus and hardness and computes power loss due to friction, oil consumption, blow-by, and liner wear rate. The root mean square (RMS) of asperity height is used to represent asperity roughness, which is assumed to be normally distributed. The engine simulation takes then as input the power loss and computes brake-specific fuel consumption of the engine. Commercial software packages were used to perform the simulations. A detailed description of the problem can be found in [25].

4.1. Problem Formulation

Due to the simplicity of the given problem structure, we will use here a modified version of the notation introduced earlier. Since there are only two levels with only one element in each, we skip element indices and denote the upper-level element with subscript 0 and the lower-level element with subscript 1. We use second indices to denote entries in the design variable vector of the lower-level element optimization problem. The design problem is to find optimal mean values $\mu_{X_{11}}$ and $\mu_{X_{12}}$ for the piston-ring and cylinder-liner surface roughness random variables X_{11} and X_{12}, respectively, and optimal values for the deterministic design variables representing the material properties (Young's modulus x_{13} and hardness x_{14}) of the liner that yield minimized expected value of brake-specific fuel consumption R_0. The optimal design is subject to constraints on liner wear rate, oil consumption, and blow-by. The power loss due to friction R_1 links the two levels.

The top- and bottom-level ATC problems are formulated as

$$\begin{aligned} \min_{\mu_{R_1}, \epsilon^R} \quad & (E[R_0] - T)^2 + \epsilon^R \\ \text{subject to} \quad & (\mu_{R_1} - E[R_1]^l)^2 \leq \epsilon^R \\ \text{with} \quad & R_0 = f_0(R_1) \end{aligned} \tag{13}$$

and

$$\begin{aligned} \min_{\mu_{X_{11}}, \mu_{X_{12}}, x_{13}, x_{14}} \quad & (E[R_1] - \mu_{R_1}^u)^2 \\ \text{subject to} \quad & P[\text{liner wear rate} > 2.4 \times 10^{-12}\ m^3/s] \leq P_f \\ & P[\text{blow-by} > 4.25 \times 10^{-5}\ kg/s] \leq P_f \\ & P[\text{oil consumption} > 15.3 \times 10^{-3}\ kg/hr] \leq P_f \\ & P[X_{11} < 1\mu m] \leq P_f \\ & P[X_{11} > 10\mu m] \leq P_f \\ & P[X_{12} < 1\mu m] \leq P_f \\ & P[X_{12} > 10\mu m] \leq P_f \\ & 340\ GPa \geq x_{13} \geq 80\ GPa \\ & 240\ BHV \geq x_{14} \geq 150\ BHV \\ \text{with} \quad & R_1 = f_1(X_{11}, X_{12}, x_{13}, x_{14}), \end{aligned} \tag{14}$$

respectively. The standard deviation of the surface roughnesses was assumed to be 1.0 μm, and remained constant throughout the ATC process. The assigned probability of failure P_f was 0.13%, which corresponds to the target reliability index $\beta = 3$. The fuel consumption target T was simply set to zero to achieve the best fuel economy possible.

Note that since the random variables are normally distributed, the associated linear probabilistic bound constraints can be reformulated as deterministic. For example,

$$P[X_{11} < 1\mu m] \leq P_f \Leftrightarrow P[X_{11} - 1\mu m < 0] \leq P_f \Leftrightarrow$$

$$\Phi(0-\frac{\mu_{X_{11}}-1\mu m}{\sigma_{X_{11}}})\leq\Phi(-\beta)\Rightarrow-\frac{\mu_{X_{11}}-1\mu m}{\sigma_{X_{11}}}\leq-\beta\Leftrightarrow$$
$$\frac{\mu_{X_{11}}-1\mu m}{\sigma_{X_{11}}}\geq\beta\Leftrightarrow\mu_{X_{11}}-1\mu m\geq\beta\sigma_{X_{11}}\Leftrightarrow$$
$$\mu_{X_{11}}\geq 1\mu m+\beta\sigma_{X_{11}}\Leftrightarrow\mu_{X_{11}}\geq 4\mu m$$

Similarly, the other three probabilistic bound constraints in Problem (14) can be reformulated as

$$\mu_{X_{11}}\leq 7\mu m;\quad \mu_{X_{12}}\geq 4\mu m;\quad \mu_{X_{12}}\leq 7\mu m.$$

4.2. Results

It is desired to minimize power loss due to friction in order to optimize engine operation and thus maximize fuel economy. Therefore, it was anticipated that the bottom-level optimization problem would yield a design with as smooth surfaces (low surface roughnesses) as possible.

The probabilistic ATC process of solving Problems (14) and (13) iteratively converged after two iterations. The obtained optimal ring/liner subassembly design is shown in Table III. The ring surface roughness optimal value is at its probabilistic lower minimum,

Table III. Optimal ring/liner subassembly design

Variable	Description	Value
X_{11}	Ring surface roughness, $[\mu m]$	4.00
X_{12}	Liner surface roughness, $[\mu m]$	6.15
x_{13}	Liner Young's modulus, $[GPa]$	80
x_{14}	Liner hardness, $[BHV]$	240

while the liner's Young's modulus and hardness optimal values are at their deterministic lower and upper bounds, respectively.

The liner surface roughness is not, however, at its lower bound because the problem is bounded by the oil consumption constraint. A certain degree of surface roughness is required to maintain an optimal oil film thickness in order to avoid excessive oil consumption. For this reason, the associated constraint is active, and the surface roughness of the liner is an interior optimizing argument.

An interesting theoretical issue arises. How do we define activity for probabilistic constraints? The definition of constraint activity in deterministic optimization is the following: A constraint is active if removing it or moving its boundary affects the location of the optimum. In probabilistic design, a constraint is active if the reliability index associated with the constraint's MPP is equal to the target reliability index. In other words, the constraint's MPP lies on the target reliability circle.

A Monte Carlo simulation was performed to assess the accuracy of the reliability analyses of the probabilistic constraints. One million samples were generated using the mean and standard deviation values of the design variables, and the constraints were evaluated using these samples to calculate the probability of failure. Results are summarized in Table IV.

Table IV. Reliability analysis results

Constraint	Active	P_f	MCS P_f
Liner wear rate	No	≤ 0.13 %	0 %
Blow-by	No	≤ 0.13 %	0 %
Oil consumption	Yes	0.13 %	0.16 %

The obtained design is actually 0.03% less reliable than found. This error is due to the first-order reliability approximation used in the probabilistic optimization problem.

Propagation of uncertainty was modeled using the approach described in Section 3.2. Table V summarizes the estimated moments for the two responses of the bi-level hierarchy.

Table V. Estimated moments and errors relative to Monte Carlo simulation (MCS) results for the simulation example

Response	Power loss	Fuel consumption
μ_{lin}	0.3950	0.5341
μ_{AMV}	0.3922	0.5431
μ_{MCS}	0.3932	0.5432
ϵ_{lin} [%]	0.45	−0.01
ϵ_{AMV} [%]	−0.25	−0.01
σ_{lin}	0.0481	0.00757
σ_{AMV}	0.0309	0.00760
σ_{MCS}	0.0311	0.00759
ϵ_{lin} [%]	54.6	−0.25
ϵ_{AMV} [%]	−0.64	0.13

The linearization approach results are included to illustrate the large error that this approach introduces to the top-level problem. This happens because the power loss function is highly nonlinear. In fact, its PDF is multi-modal, as illustrated in Figure 5. Figure 5 also depicts the histogram obtained by Monte Carlo simulation using one million samples; note that the perpendicular axis of the histogram must be divided by 1,000,000 to obtain the probability density relative to the sample size. The agreement is quite satisfactory and illustrates the usefulness of the AMV-based approach to propagate uncertainty for highly nonlinear functions. The fuel consumption is almost a linear function of the power loss.

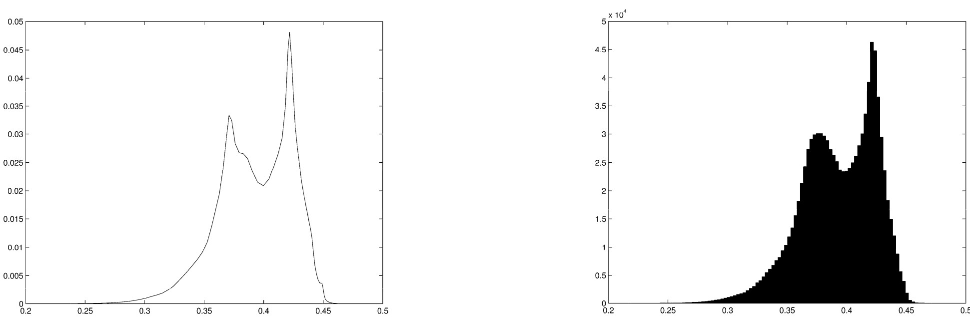

Figure 5. Power loss PDF (left) and histogram obtained using Monte Carlo simulation (right)

5. Summary and Conclusions

We have presented a methodology for design optimization of hierarchically decomposed multilevel systems under uncertainty. We extended the deterministic formulation of analytical target cascading (ATC) to account for uncertainties. We modeled the propagation of uncertainty in the ATC process by using the advanced mean value (AMV) method to generate accurate probability distributions of nonlinear responses. We demonstrated the presented methodology by means of a simple yet illustrative engine design example. The proposed methodology for simulation-based optimal system design by decomposition is not related to multidisciplinary design optimization (MDO) methods in either its deterministic or its probabilistic formulation. Stochastic formulations are meaningful only if expectations of nonlinear responses are computed exactly, which requires probability distribution information of the input random variables and parameters and accurate multidimensional integrations. Probabilistic formulations are suggested for practical applications. The linearization approach for propagating uncertainties yields inaccurate second moment estimations and is inadequate for multilevel optimization under uncertainty since it does not provide probability distribution information that is necessary for solving higher-level problems.

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