An Efficient Unified Approach for Reliability and Robustness in Engineering Design

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Abstract. Mathematical optimization plays an important role in engineering design, leading to greatly improved performance. Deterministic optimization however, can lead to undesired choices because it neglects input and model uncertainty. Reliability-based design optimization (RBDO) and robust design improve optimization by considering uncertainty. A design is called reliable if it meets all performance targets in the presence of variation/uncertainty and robust if it is insensitive to variation/uncertainty. Ultimately, a design should be optimal, reliable, and robust. Usually, some of the deterministic optimality is traded-off in order for the design to be reliable and/or robust. This paper describes the state-of-the-art in assessing reliability and robustness in engineering design and proposes a new unifying formulation. The principles of deterministic optimality, reliability and robustness are first defined. Subsequently, the design compromises for simultaneously achieving optimality, reliability and robustness are illustrated. Emphasis is given to a unifying probabilistic optimization formulation for both reliability-based and robust design, including variation of all performance measures. The robust engineering problem is investigated as a part of a "generalized" RBDO problem. Because conventional RBDO optimizes the mean performance, its objective is only a function of deterministic design variables and the means of the random design variables. The conventional RBDO formulation is expanded to include performance variation as a design criterion. This results in a multi-objective optimization problem even with a single performance criterion. A preference aggregation method is used to compute the entire Pareto frontier efficiently. Examples illustrate the concepts and demonstrate their applicability.

1. Introduction

Deterministic mathematical optimization has led to greatly improved performance in all areas of engineering design. It can however, lead to undesired choices, if uncertainty/variation is ignored. In deterministic design we assume that there is no uncertainty in the design variables and/or modeling parameters. Therefore, there is no variability in the simulation outputs. However, there exists inherent input and parameter variation that results in output variation. Deterministic optimization typically yields optimal designs that are pushed to the limits of design constraint boundaries, leaving little or no room for tolerances (uncertainty) in manufacturing imperfections, modeling and design variables. Therefore, deterministic optimal designs that are obtained without taking into account uncertainty are usually unreliable. Input variation is fully accounted for in Reliability-Based Design Optimization (RBDO) and robust design.

In RBDO, probability distributions describe the stochastic nature of the design variables and model parameters. Variations are represented by standard deviations (typically assumed to be constant) and a *mean* performance measure is optimized subject to *probabilistic* constraints. RBDO can be a powerful tool which can assist in design under uncertainty, since it provides optimum designs in the presence of uncertainty in design variables/parameters and simulation models. For this reason, it has been extensively studied [1-8].

Robust designs methods are also widely used because they can improve the quality of products and processes [9]. Robust design minimizes performance variation without eliminating the sources of variation [10]. The product quality is commonly defined using a quality loss function [10,11]. Various methods have been proposed for estimating the product quality loss [12-16] using the mean and standard deviation of the response (performance measure). A review of existing robust optimization methods can be found in [17,18].

It is common in product design to have multiple performance measures. A robust design therefore, must simultaneously minimize the variation of all performance measures using a multi-objective optimization approach. It is however, common to use a single-objective robust design formulation by either minimizing a heuristic quality loss function [18,19] or form a single objective utilizing weighting factors in a weighted-sum approach [20]. We will show in this work that the weighted-sum approach may lead to inaccurate results. A detailed examination of the weighted-sum approach drawbacks is provided in [21]. There are only a few multi-objective approaches to robust design [17,22-24].

Reliability and robustness are attributes of design under uncertainty. It makes sense therefore, to combine them in a *unified,* multi-objective approach where the mean and variation of multiple performance measures are simultaneously minimized, subject to probabilistic constraints for design feasibility. Such an approach is proposed in this paper. The concept of a unified methodology for reliability and robustness is not new, as references [17,18,25] for example, indicate. However, major simplifications are usually made. In general, researchers use one or both of 1) the weighted-sum simplification for the general multi-objective problem and 2) a simplified representation of the probabilistic design feasibility using the worst-case scenario [23,17] or the moment matching formulation [26,16]. Furthermore, a first-order Taylor expansion is usually performed for estimating the performance variance. This linearization approach does a fairly good job in estimating the expected value of the nonlinear objective function. However, it can be quite inaccurate in estimating its higher moments as is the standard deviation [27]. Moreover, it is limiting in that it does not provide us with the correct probability distribution information of the objective function.

In this paper, a computationally efficient unified approach to reliability and robustness is proposed which alleviates the described shortcomings of the available methods. A preference aggregation method [28-30] is used to choose the "best" solution of a multi-objective optimization problem based on designer preferences. The performance variation is assessed by a percentile difference method originally proposed in [25]. The percentiles are efficiently calculated using a variation of the Advanced Mean Value method [31]. The "best" design is calculated using an efficient single-loop probabilistic optimization method [32]. Examples illustrate the methodology.

2. Definition of Optimality, Reliability and Robustness

In deterministic design optimization an objective function is usually minimized subject to certain constraints which define a feasible region. A conventional deterministic optimization problem with inequality constraints only, is stated as

$$\min_{\mathbf{d}} f(\mathbf{d}) \tag{1}$$

$$\text{s.t.} \quad G_i(\mathbf{d}) \geq 0\,, \quad i = 1,\ldots,n$$

$$\mathbf{d}^L \leq \mathbf{d} \leq \mathbf{d}^U$$

where $\mathbf{d} \in R^k$ is the vector of deterministic design variables. A bold letter indicates a vector.

In optimization under uncertainty the task is to minimize (or maximize) an objective while 1) all constraints are satisfied and 2) the performance of the design is insensitive to the existing variation or uncertainty. *Variation* or *stochastic uncertainty* is defined as that "irreducible" uncertainty which, being inherent in the physical system, ought not to depend on the amount of available statistical data. It is usually modeled probabilistically. In this work, a design is called *reliable* if it meets all performance targets in the presence of variation/uncertainty and *robust* if it is insensitive to variation/uncertainty. Ultimately, *a design should be optimal, reliable, and robust*. Usually, some of the deterministic optimality is traded-off in order for the design to be reliable and/or robust.

A typical RBDO problem is formulated as

$$\min_{\mathbf{d},\boldsymbol{\mu}_\mathbf{X}} f(\mathbf{d},\boldsymbol{\mu}_\mathbf{X},\boldsymbol{\mu}_\mathbf{P}) \tag{2}$$

$$\text{s.t.} \quad P(G_i(\mathbf{d},\mathbf{X},\mathbf{P}) \geq 0) \geq R_i = 1 - p_{f_i}, \quad i = 1,\dots,n$$

$$\mathbf{d}^L \leq \mathbf{d} \leq \mathbf{d}^U$$

$$\boldsymbol{\mu}_\mathbf{X}^L \leq \boldsymbol{\mu}_\mathbf{X} \leq \boldsymbol{\mu}_\mathbf{X}^U$$

where $\mathbf{X} \in R^m$ is the vector of random design variables and $\mathbf{P} \in R^q$ is the vector of random design parameters. According to the used notation, an upper case letter indicates a random variable or a random parameter and a lower case letter indicates a realization of a random variable or parameter. If the target probability of failure p_f is approximated using the target reliability index β_t and the standard normal cumulative distribution function Φ, the actual reliability level for the i^{th} deterministic constraint $G_i(\mathbf{d},\mathbf{X},\mathbf{P}) \geq 0$ is $R_i = 1 - p_{f_i}$ where

$$p_{f_i} = P(G_i(\mathbf{d}, \mathbf{X}, \mathbf{P}) \leq 0) = F_{G_i}(0) \leq \Phi(-\beta_{t_i}) \tag{3}$$

and $F_{G_i}(\ )$ is the cumulative distribution function of G_i.

The principles of deterministic optimality and reliability are demonstrated graphically in Fig. 1 using a hypothetical design. The design compromises for achieving optimality and reliability are illustrated. For a hypothetical design with two constraints in two dimensions, the deterministic optimum is denoted by point **A** in Fig. 1. It is the constrained optimum, where the objective is minimized and both constraints are active. If the two design variables are random with their means specified by the deterministic optimum, all possible design realizations fall within a closed domain (indicated for simplicity by a circle centered at point **A**) due to the variation of the two design variables. In this case, a large percentage of design realizations violate at least one constraint or performance target, rendering design **A** unreliable. To achieve reliability, the circle around point **A** must be moved inside the feasible domain with its center at point **B**. As the circle moves within the feasible domain, the design simultaneously becomes more reliable and less optimal. The circle must be moved to accommodate the uncertainty indicated by its radius. At the reliable design **B**, the circle may be tangent to a number of performance targets which become "probabilistically" active. The process of moving the circle from the deterministic optimum to the reliability optimum is known as Reliability-Based Design Optimization. It can be implemented mathematically by solving Problem (2).

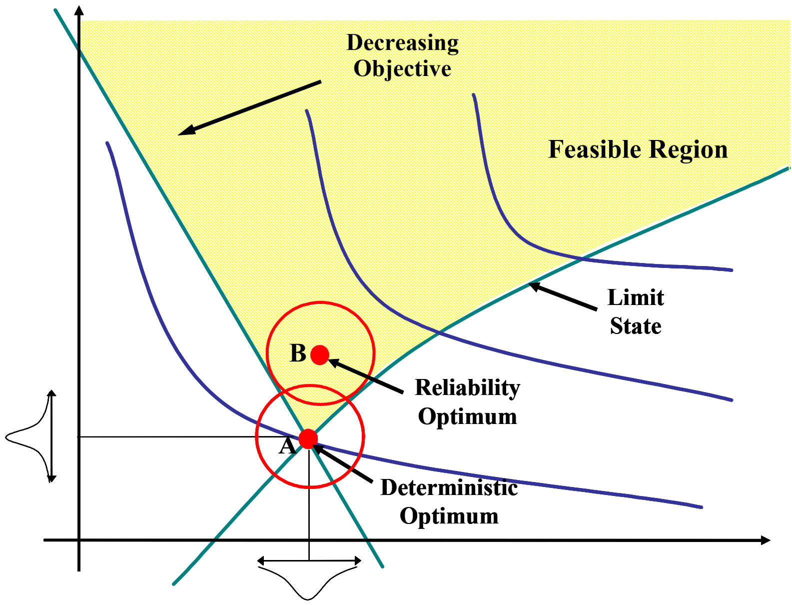

Figure 1. Geometric interpretation of deterministic and reliable designs

At point **B** of Fig. 1, the design is at its reliability optimum where the objective function is optimized given that the circle around **B** is within the bounds of the constraints. However, it may not be robust. It is robust if the performance of each design realization within the circle is as close to constant as possible, indicating insensitivity to variation. Robustness can therefore, be achieved by placing the final optimum at a region, where the response is "flat" or insensitive to the design variables. This is illustrated in Fig. 2 for a hypothetical one-dimensional design. Assuming the variation/uncertainty of the design variable **x** is constant, the variation of the response is much smaller if $\mathbf{x} = \mathbf{x_2}$. It should be noted that the reliable and robust design is usually (although not necessarily), suboptimal to the reliable design **B** which is in turn suboptimal to the deterministic design **A**. This is the design trade-off among optimality, reliability and robustness.

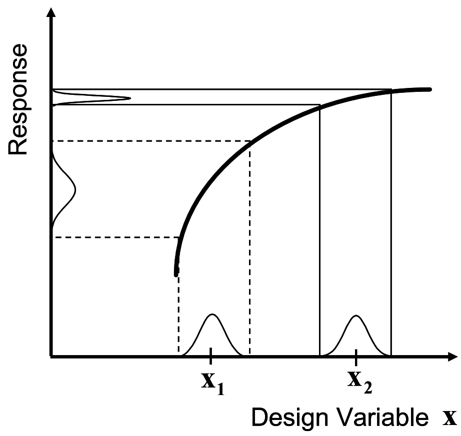

Figure 2. Geometric interpretation of robust design

The robust design problem is in general, expressed as

$$\min_{\mathbf{d},\boldsymbol{\mu}_\mathbf{X}} \mathbf{V_f} = [V_{f_1}, V_{f_2}, \ldots, V_{f_m}] \tag{4}$$

$$\text{s.t.} \quad P\left(G_i\left(\mathbf{d}, \mathbf{X}, \mathbf{P}\right) \geq 0\right) \geq R_i = 1 - p_{f_i}, \quad i = 1, \ldots, n$$

$$\mathbf{d}^L \le \mathbf{d} \le \mathbf{d}^U$$
$$\boldsymbol{\mu}_{\mathbf{X}}^L \le \boldsymbol{\mu}_{\mathbf{X}} \le \boldsymbol{\mu}_{\mathbf{X}}^U$$

where V indicates a variation measure. For example, V_{f_1} is a variation measure of the first objective f_1 of the multi-objective Problem (4). Section 4 discusses the commonly used variation measures and explains what we propose in this work.

A unifying formulation for reliability and robustness is described in Section 4. The solution methodology is based on a "generalized" RBDO formulation which includes robustness considerations. For this reason, an overview of the existing RBDO methods is given next.

3. Overview of Reliability-Based Design Optimization

Optimization is concerned with achieving the best outcome of a given objective while satisfying certain restrictions. It has been observed that the deterministic optimum design does not necessarily have high reliability. To ensure that the optimum design is also reliable, the optimization formulation must include reliability constraints. Such a formulation is commonly referred as Reliability-Based Design Optimization (RBDO). Problem (2) is a typical RBDO formulation.

The classical RBDO method is the so-called double-loop approach. It employs two nested optimization loops; the design optimization loop (outer) and the reliability assessment loop (inner). The latter is needed for the evaluation of each probabilistic constraint. There are two different methods for the reliability assessment; the Reliability Index Approach (RIA) [2] and the Performance Measure Approach (PMA) [6,7]. Although either approach can be used, PMA is in general more efficient, especially for high reliability problems [7]. Every time the design optimization loop calls for a constraint evaluation, a reliability assessment loop is executed which searches for the Most Probable Point (MPP) in the standard normal space, based on First-Order Reliability Methods (FORM).

The PMA-based RBDO problem, which is practically the inverse of the RIA-based RBDO problem, is stated as [7]

$$\min_{\mathbf{d},\boldsymbol{\mu}_{\mathbf{X}}} f(\mathbf{d},\boldsymbol{\mu}_{\mathbf{X}},\boldsymbol{\mu}_{\mathbf{P}}) \tag{5}$$

$$\text{s.t. } G_{p_i} = (F_{G_i}{}^{-1}(\Phi(-\beta_{t_i})) \ge 0, \; i = 1,...,n$$
$$\mathbf{d}^L \le \mathbf{d} \le \mathbf{d}^U$$
$$\boldsymbol{\mu}_{\mathbf{X}}^L \le \boldsymbol{\mu}_{\mathbf{X}} \le \boldsymbol{\mu}_{\mathbf{X}}^U$$

where Eq. (3) has been used to transform each probabilistic constraint to an equivalent non-negative constraint for a performance measure G_p. G_p is a function of the target reliability index β_t. It is calculated from the following reliability minimization problem

$$G_p = \min_{\mathbf{U}} G(\mathbf{U}) \tag{6}$$
$$\text{s.t } \|\mathbf{U}\| = \beta_t$$

where the vector $\boldsymbol{U}$ represents the random variables in the standard normal space.

Using a percentile formulation, the general RBDO formulation of Eq. (2), can be equivalently stated as [6,7]

$$\min_{\mathbf{d},\boldsymbol{\mu}_{\mathbf{X}}} f(\mathbf{d},\boldsymbol{\mu}_{\mathbf{X}},\boldsymbol{\mu}_{\mathbf{P}}) \tag{7}$$

$$\text{s.t. } G_i^R(\mathbf{d},\mathbf{X},\mathbf{P}) \ge 0, \quad i = 1,\cdots,n$$
$$\mathbf{d}^L \le \mathbf{d} \le \mathbf{d}^U \;, \; \boldsymbol{\mu}_{\mathbf{X}}^L \le \boldsymbol{\mu}_{\mathbf{X}} \le \boldsymbol{\mu}_{\mathbf{X}}^U$$

where G^R is the R-percentile of the constraint $G(\mathbf{d},\mathbf{X},\mathbf{P})$. It is defined as

$$P(G(\mathbf{d},\mathbf{X},\mathbf{P}) \geq G^R) = R \tag{8}$$

where R is the target reliability for the constraint. Note that $P(G(\mathbf{d},\mathbf{X},\mathbf{P}) \geq 0) \geq R$ if $G^R \geq 0$. Therefore, $G^R \geq 0$ provides an equivalent expression of the probabilistic constraints in Eq. (2). After the MPP is calculated, the R-percentile is given by

$$G^R = G(\mathbf{d}, \mathbf{X}_{MPP}, \mathbf{P}_{MPP}). \tag{9}$$

The RBDO Problems (5) or (7) involve nested optimization loops which may hinder on their computational efficiency. For this reason, two new classes of RBDO formulations have been recently proposed. The first class decouples the RBDO process into a sequence of a deterministic design optimization followed by a set of reliability assessment loops [33,34]. The series of deterministic and reliability loops is repeated until convergence. The second class of RBDO methods converts the problem into an equivalent, single-loop deterministic optimization [35,32], leading therefore, to significant efficiency improvements.

3.1. Decoupled RBDO

Among the decoupled RBDO methods, the Sequential Optimization and Reliability Assessment (SORA) [33] method is the most promising. It uses the reliability information from the previous cycle to shift the violated deterministic constraints in the feasible domain. This is done sequentially until convergence is achieved. SORA employs a sequence of decoupled deterministic optimization and reliability assessment loops which are performed in series. At the end of a deterministic design optimization, the reliability of each constraint is assessed. If the reliability of a particular constraint is less than the specified target, a “shifting” vector is calculated which is used to push the constraint boundary in the feasible domain. The “shifted” constraints are then used to perform a new deterministic design optimization. The series of deterministic and reliability assessment loops continues until convergence is achieved; i.e. the objective function is minimized and the target reliability of each constraint is met. At convergence the shifting distance is zero. For the reliability assessment in SORA, either of the RIA or PMA approaches can be used. Detailed information is provided in [33].

3.2. Single-Loop RBDO

Based on the percentile formulation of Eq. (7), a computationally efficient single-loop RBDO method has been recently developed [32]. The method relates the $\boldsymbol{\mu}_\mathbf{X}, \boldsymbol{\mu}_\mathbf{P}$ and **X**, **P** vectors using the KKT optimality conditions of the inner reliability loops. In that case, the constraint gradient and the β hyper-sphere gradient must be collinear and pointing in opposite directions at the MPP point [32]. This is expressed as $\mathbf{X} = \boldsymbol{\mu}_\mathbf{X} - \boldsymbol{\sigma}\beta_t\boldsymbol{\alpha}$ and $\mathbf{P} = \boldsymbol{\mu}_\mathbf{P} - \boldsymbol{\sigma}\beta_t\boldsymbol{\alpha}$. Problem (7) can be therefore, expressed as

$$\begin{aligned} &\min_{\mathbf{d},\boldsymbol{\mu}_\mathbf{X}} \; f(\mathbf{d}, \boldsymbol{\mu}_\mathbf{X}, \boldsymbol{\mu}_\mathbf{P}) \\ &\text{s.t.}\; G_i(\mathbf{d}, \mathbf{X}_i, \mathbf{P}_i) \geq 0 \qquad i = 1, \ldots\ldots, n \\ &\mathbf{X}_i = \boldsymbol{\mu}_\mathbf{X} - \boldsymbol{\sigma}\beta_{t_i}\boldsymbol{\alpha}_i, \quad \mathbf{P}_i = \boldsymbol{\mu}_\mathbf{p} - \boldsymbol{\sigma}\beta_{t_i}\boldsymbol{\alpha}_i \\ &\boldsymbol{\alpha}_i = \boldsymbol{\sigma} * \nabla G_{i\,\mathbf{X},\mathbf{P}}(\mathbf{d}, \mathbf{X}_i, \mathbf{P}_i) / \| \boldsymbol{\sigma} * \nabla G_{i\,\mathbf{X},\mathbf{P}}(\mathbf{d}, \mathbf{X}_i, \mathbf{P}_i) \| \\ &\mathbf{d}^L \leq \mathbf{d} \leq \mathbf{d}^U, \quad \boldsymbol{\mu}_\mathbf{X}^L \leq \boldsymbol{\mu}_\mathbf{X} \leq \boldsymbol{\mu}_\mathbf{X}^U \end{aligned} \tag{10}$$

where $\boldsymbol{\mu}_\mathbf{X}, \boldsymbol{\mu}_\mathbf{P}$ are the mean values of vectors **X** and **P**, β_{t_i} is the target reliability index for the ith constraint, $\boldsymbol{\alpha}_i$ is the normalized gradient of the ith constraint and $\boldsymbol{\sigma}$ is the standard deviation vector of random variables **X** and parameters **P**.

It should be noted that the single-loop RBDO method does not search for the MPP of each constraint. Instead, the MPP of each active constraint is correctly identified at the optimum. This dramatically improves the efficiency without compromising the accuracy. The main advantage of the method is the elimination of the repeated reliability loops and its excellent convergence properties since it is based on an equivalent deterministic optimization. Detail information on the single-loop method is provided in [32].

A modified version of the single-loop probabilistic optimization of Problem (10) is used in this work for the proposed unified reliability and robust design formulation.

4. A Unified Formulation for reliability and Robustness

A computationally efficient unified method to reliability and robustness, based on a multi-objective optimization problem, is presented in this section. The method addresses all shortcomings of the existing methods as described in the introduction section. The preference aggregation method of Section 4.1 is used to choose the "best" solution of the multi-objective optimization problem based on designer preferences. The performance variation is assessed by a percentile difference method originally proposed in [25]. The percentiles are efficiently calculated using a variation of the Advanced Mean Value method as described in Section 4.2. The "best" design is calculated using the single-loop probabilistic optimization method of Section 3.2.

A multi-objective formulation for reliability and robustness is proposed by combining the RBDO and robust design formulations of Problems (2) and (4), respectively. The variation measure V_{f_i} of objective (performance) f_i is expressed by the spread of its PDF, which is simply the percentile difference $\Delta R_{f_i} = f_i^{R_2} - f_i^{R_1}$ where $f_i^{R_1}$ and $f_i^{R_2}$ are a low and high percentile of f_i, respectively. For example, the 5th and 95th percentiles can be used. There are several advantages of using the percentile difference instead of the standard deviation in assessing variability in robust design. The percentile is related to the probability at the tail areas of the distribution and therefore, it provides more information than the standard deviation. It considers for example, the skewness of the distribution while the standard deviation only captures the dispersion around the mean value. Also, with the percentile formulation, we can easily know the confidence level of the design robustness, which is simply equal to $R_2 - R_1$.

Using the percentile difference as a variation measure, the unified reliability and robustness formulation is stated by the following multi-objective optimization problem

$$\begin{aligned} &\min_{\mathbf{d},\boldsymbol{\mu}_\mathbf{X}} f\left(\mathbf{d},\boldsymbol{\mu}_\mathbf{X},\boldsymbol{\mu}_\mathbf{P}\right) \\ &\min_{\mathbf{d},\boldsymbol{\mu}_\mathbf{X}} \Delta\mathbf{R}_\mathbf{f} = [\Delta R_{f_1}, \Delta R_{f_2}, \ldots, \Delta R_{f_m}] \\ \text{s.t. } \; & P\left(G_i\left(\mathbf{d},\mathbf{X},\mathbf{P}\right) \geq 0\right) \geq 1 - p_{f_i}, \quad i = 1,\ldots,n \\ &\mathbf{d}^L \leq \mathbf{d} \leq \mathbf{d}^U \\ &\boldsymbol{\mu}_\mathbf{X}^L \leq \boldsymbol{\mu}_\mathbf{X} \leq \boldsymbol{\mu}_\mathbf{X}^U \end{aligned} \tag{11}$$

where $\Delta R_f = f^{R_2} - f^{R_1}$. The first objective minimizes a mean performance and the remaining objectives minimize the distribution spread of all performance measures. The trade-off between all objectives can play an important role in the selection of the best design. It is common to perform this trade-off using a weighted-sum approach which usually leads to undesired results.

A multi-objective design problem generally has a set of possible "best" solutions, known as the Pareto set or Pareto frontier. The Pareto set contains all feasible points for which there is no other point which performs better on all objectives. To decide which of all the Pareto points is the best design, the objectives must be traded off against each other.

Some researchers have proposed to use compromise programming (CP) [17,22,23] to address the trade-off mentioned above. The basic idea in CP is to identify the entire Pareto frontier, and allow the user to choose among the Pareto points. CP commences with the identification of an ideal solution (*utopia point*), where each attribute under consideration simultaneously achieves its optimum value. As the ideal point is unachievable in general, the designer seeks a solution which is as close as possible to the ideal point. The closest feasible point to the utopia point for a given weight set is guaranteed to belong to the set of Pareto points. By varying the weighting factors, the full set of Pareto points can be obtained [17]. In contrast, a weighted-sum (WS) method used for the same purpose may fail to locate all Pareto points. It has been shown [36] that for every Pareto point of a convex multi-objective optimization problem there exists a (nonzero) weight vector **w** > 0 such that this Pareto point is an optimal solution of the Weighted-sum Problem (WSP). However, not every Pareto solution of a general (nonconvex) problem can be found by solving the corresponding WSP. Also, it has been concluded from numerical experiments [17] that even for convex multi-objective optimization problems, an evenly distributed set of weights fails to produce an even distribution of points from all parts of the Pareto set if a weighted-sum aggregation is used. Details on compromise programming in engineering design can be found in [17].

4.1. Preference Aggregation Method

An alternative to compromise programming is preference aggregation. A family of aggregation functions for modeling all possible trade-offs in engineering design has been presented in [28-30]. Methods using these aggregation functions to aggregate preferences of designers on performance measures are called *preference aggregation methods*. In this work, a preference aggregation method is used to address the robust design of Problem (11). It has been mentioned that the weighted-sum methods have serious drawbacks for optimization [17,21] because they are usually unable to reach some Pareto solutions. The recovery of the entire Pareto frontier may be computationally intractable and even if it is available, it may be beyond the capacity of the human designer to choose the best point from the Pareto set. The preference aggregation method surmounts these difficulties. This is the main reason we use preference aggregation methods instead of compromise programming in this work.

Preference aggregation is a formal approach for reconciling multiple conflicting criteria in design [28,29]. *Preference functions* or *preferences* are defined for each criterion and the various functions are *aggregated* into a single overall preference function by means of *aggregation operators*. Preference functions take values on [0,1], where a preference of 0 indicates a criterion is unacceptable, while a preference of 1 denotes complete satisfaction. A set of properties was offered in [28] that seem intuitively reasonable for combining preferences in engineering design, indicating that decisions can have different *trade-offs*. The set includes the annihilation, idempotency, monotonicity, commutativity and continuity properties which are mathematically described as

$$h[(0,w_1),(h_2,w_2)] = h[(h_1,w_1),(0,w_2)] = 0 \quad \text{(annihilation)} \quad (12)$$

$$h[(h_1,w_1),(h_1,w_2)] = h_1 \quad \text{(idempotency)} \quad (13)$$

$$h[(h_1,w_1),(h_2,w_2)] \le h[(h_1,w_1),(h_2^*,w_2)] \ \text{ if } \ h_2 \le h_2^* \quad \text{(monotonicity)} \quad (14)$$

$$h[(h_1,w_1),(h_2,w_2)] = h[(h_2,w_2),(h_1,w_1)] \quad \text{(commutativity)} \quad (15)$$

$$h[(h_1,w_1),(h_2,w_2)] = \lim_{h_1^* \to h_1} h[(h_1^*,w_1),(h_2,w_2)]. \quad \text{(continuity)} \quad (16)$$

where (h_1,w_1) and (h_2,w_2) are the individual preference functions to be aggregated and their corresponding importance weights and *h* is the aggregate preference function.

The aggregation properties distinguish between *compensating* trade-offs, where high performance on one criterion can make up for lower performance on another, and *non-*

compensating trade-offs, in which the lowest performances should be raised first. It was shown [29] that the level of compensation can vary continuously, and that there is a family of aggregation operators h^s that satisfies the original set of properties for design (Eqs 12-16) and can capture all possible trade-offs. For a two-attribute design problem, h^s is given by

$$h^s[(h_1, w_1),(h_2, w_2)] = \left(\frac{w_1 h_1^s + w_2 h_2^s}{w_1 + w_2} \right)^{1/s} . \tag{17}$$

The parameter s can be interpreted as a measure of the level of compensation, or trade-off. Higher values of s indicate a greater willingness to allow high preference for one criterion to compensate for lower values of another. It is shown [29] that if $s \to 0$, the aggregation of the two preferences provides maximum compensation. In this case, Eq. (17) reduces to the following geometric product of the two preferences h_1, h_2

$$h^s = h^s_{prod} = \left[h_1^{w_1} h_2^{w_2} \right]^{1/w_1 + w_2} . \tag{18}$$

To the contrary, if $s \to -\infty$ the aggregation of the two preferences provides no compensation at all and Eq. (17) reduces to

$$h^s = \min(h_1, h_2). \tag{19}$$

When the parameters s and w (or equivalently, weight ratio w_2 / w_1) are correctly chosen, the "best" design can be located by maximizing h^s.

The weighted sum is a special case with s =1. It has been shown [29] that for any Pareto optimal point in a given set, there is always a choice of a weight ratio and a level of compensation s that selects that point as the most preferred. It has been also shown that for any fixed s, there are Pareto sets in which some Pareto points can *never* be selected by any choice of weights. In particular, the weighted-sum approach (s =1) may not be able to select all Pareto points. In order to avoid this arbitrary and meaningless use of weights, a rigorous, provable procedure of "indifference points" has been developed for calculating the proper trade-off parameters [30].

4.2. Percentile Calculation using the Advanced Mean Value Method

The percentiles f^{R_1} and f^{R_2} of a performance measure f (see Eq. 11) can be in general calculated using two reliability calculations for estimating the two Most Probable Points corresponding to R_1 and R_2. In this work, a computationally more efficient method is used based on the Advanced Mean Value (AMV) method [31].

The AMV method has been originally proposed as a computationally efficient method for generating the cumulative distribution function (CDF) of performance f. It uses a simple correction to compensate for errors introduced from a Taylor series truncation. The performance $f(\mathbf{X})$ is first linearized around the mean design point. A limit state function is then defined as

$$g(\mathbf{X}) = f(\mathbf{X}) - f_0 \tag{20}$$

where f_0 is a particular value of the performance function. Based on the CDF definition, we have the following first-order relation

$$P(f \le f_0) = P(g \le 0) = \Phi(-\beta), \tag{21}$$

where Φ is the standard normal cumulative distribution function and β is the reliability index.

For the calculation of the R-percentile f^R, the reliability index β is calculated from $\Phi(-\beta) = R$ if $R \ge 50\%$ and $\Phi(-\beta) = 1 - R$ if $R \le 50\%$.

Using the linear approximation of $g(\mathbf{X})$ at the mean value point $\boldsymbol{\mu}_{\mathbf{X}}$, the MPP is given by

$$\mathbf{U}^* = -\beta \boldsymbol{\sigma}_{\mathbf{X}} \frac{\nabla g(\boldsymbol{\mu}_{\mathbf{X}})}{|\boldsymbol{\sigma}_{\mathbf{X}} \nabla g(\boldsymbol{\mu}_{\mathbf{X}})|} = -\beta \boldsymbol{\sigma}_{\mathbf{X}} \frac{\nabla f(\boldsymbol{\mu}_{\mathbf{X}})}{|\boldsymbol{\sigma}_{\mathbf{X}} \nabla f(\boldsymbol{\mu}_{\mathbf{X}})|}. \tag{22}$$

in the standard normal space, if the random variables $\mathbf{X}$ are normally distributed with $\mathbf{X} \sim N(\boldsymbol{\mu}_{\mathbf{X}}, \boldsymbol{\sigma}_{\mathbf{X}})$. In the original X space, the MPP coordinates are

$$\mathbf{X}^* = \mathbf{U}^* \boldsymbol{\sigma}_{\mathbf{X}} + \boldsymbol{\mu}_{\mathbf{X}}. \tag{23}$$

For non-normal random variables, a non-linear transformation is needed.

The AMV method "corrects" the relation of Eq. (21) as

$$P(f \leq f(\mathbf{X}^*)) = \Phi(-\beta) \tag{24}$$

by replacing the f_0 value at which the reliability index β is calculated by $f(\mathbf{X}^*)$. Based on Eq. (24), the R-percentile is equal to $f(\mathbf{X}^*)$.

The described R-percentile calculation using the AMV method requires only one extra function evaluation (i.e. $f(\mathbf{X}^*)$). The gradient of $f(\mathbf{X})$ at the mean design point $\boldsymbol{\mu}_{\mathbf{X}}$ (see Eq. 22) is usually known, if a gradient-based optimization method is used for solving the robust optimization problem.

5. Examples

5.1. A Mathematical Example

A simple mathematical example is first used to demonstrate the proposed methodology for reliability and robustness, using preference aggregation methods to handle the trade-off between reliability and robustness. The following mathematical problem, first appeared in [17], is used

$$\begin{aligned} &\min_{\mathbf{x}} f(\mathbf{x}) = (x_1 - 4)^3 + (x_1 - 3)^4 + (x_2 - 5)^2 + 10 \\ &\text{s.t. } G(\mathbf{x}) = -x_1 - x_2 + 6.45 \leq 0 \\ &\quad 1 \leq x_i \leq 10, \quad i = 1,2. \end{aligned}$$

Assuming that only the variation of the objective is important, the reliable/robust problem is formulated as,

$$\begin{aligned} &\min_{\boldsymbol{\mu}_{\mathbf{x}}} f \\ &\min_{\boldsymbol{\mu}_{\mathbf{x}}} \Delta R_f(\mathbf{X}) \\ &\text{s.t. } P(G(\mathbf{X}) \geq 0) \geq R \\ &\quad P(1 \leq \mathbf{X} \leq 10) \geq R. \end{aligned}$$

The two design variables are assumed normally distributed with $X_i \sim N(\mu_{x_i}, 0.4)$, $i = 1,2$. The percentile difference is calculated as $\Delta R_f = f^{R_2} - f^{R_1}$ with $R_2 = 95\%$ and $R_1 = 5\%$. Two separate single-objective optimization problems are first solved in order to establish the *utopia* point. Each problem is composed of one of the two objectives and all constraints. The first problem is the conventional RBDO problem. It minimizes f subject to the probabilistic constraints. Its solution yields an optimum objective of $\mu_f^* = 5.4745$ at the design vector $\boldsymbol{\mu}_{\mathbf{x}}^* = [2.2, 5.9471]$. The superscript * indicates optimal value. The second problem minimizes ΔR_f. It is a single-objective, purely robust optimization problem. Its solution yields an optimum objective value and design vector of $\Delta R_f^* = 2.8982$ and $\boldsymbol{\mu}_\mathbf{x}^* = [3.4668, 5.5332]$, respectively.

Because we have two objectives, all optimal solutions belong to a Pareto set. For the calculation of the Pareto set, the two objectives are aggregated using the preference aggregation method of Section 4.1. Two preference functions h_1 and h_2 are first defined for *f* and ΔR_f, respectively. Fig. 3 shows h_1, which has the following linear form

$$h_1(\mu_f) = \begin{cases} \dfrac{3\mu_f^* - \mu_f}{3\mu_f^* - \mu_f^*} & if \quad \mu_f \le 3\mu_f^* \\ 0 & if \quad 3\mu_f^* < \mu_f \end{cases}.$$

Note that for all feasible designs, the mean objective value μ_f is *always* greater or equal to μ_f^*. A "cut-off" value of $\mu_f = 3\mu_f^*$ is used, assuming that if $\mu_f > 3\mu_f^*$ the design is unacceptable. Therefore, $h_1 = 1$ if $\mu_f = \mu_f^*$ and $h_1 = 0$ if $\mu_f > 3\mu_f^*$. The preference function h_2 for the second objective ΔR_f is defined in a similar manner. The assumed "cut-off" value is equal to $\Delta R_f = 8\Delta R_f^*$.

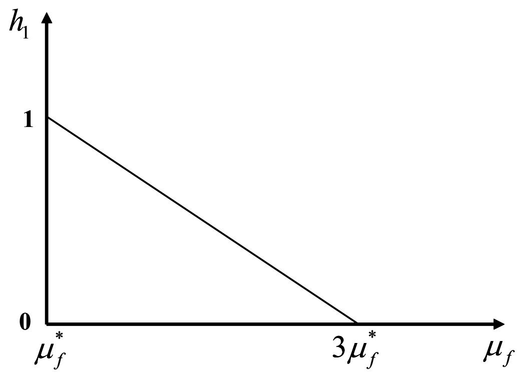

Figure 3. Preference function h_1 for the mathematical example

The two objectives are aggregated using

$$h = \left(\frac{w_1 h_1^s + w_2 h_2^s}{w_1 + w_2} \right)^{1/s}; \; w_1 + w_2 = 1, \; 0 \le w_1 \le 1.$$

The overall preference *h* is maximized by solving the following probabilistic optimization problem

$$\max_{\boldsymbol{\mu}_\mathbf{x}} h \qquad (25)$$

$$\text{s.t. } P(G(\mathbf{x}) \ge 0) \ge R$$

$$P(1 \le x_i \le 10) \ge R, \quad i = 1,2,$$

using the single-loop RBDO algorithm of Section 3.2.

The results are summarized in Fig. 4 and Table 1. For illustration purposes, we have assumed $s = -1$. Fig. 4 shows the trade-off between μ_f / μ_f^* and $\Delta R_f / \Delta R_f^*$. Note that the Pareto set does not cover the entire range between the reliable and robust optima. For $w_1 \geq 0.48$ (see Table 1), the overall design is dominated by the reliable design. Note that all designs are well spaced along the Pareto set.

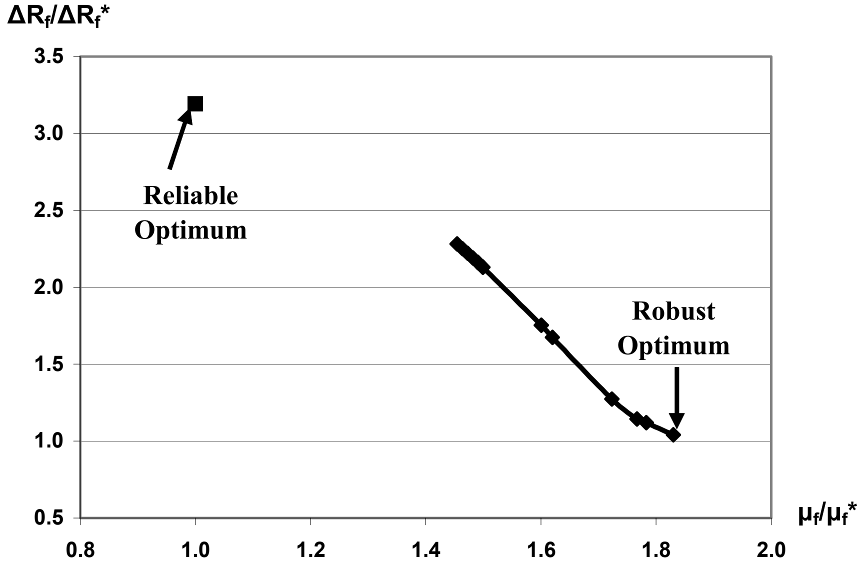

Figure 4. Pareto set for the mathematical example, using the preference aggregation method

Table 1 shows the exact values for some points on the Pareto set and the corresponding values of the optimal point $\mu_x = \lfloor \mu_{x_1} \quad \mu_{x_2} \rfloor$, overall preference function *h* and the value of constraint G for $0 \leq w_1 \leq 1$. As indicated by its zero value, constraint G is always active.

Table 1. Pareto set details for the mathematical example, using the preference aggregation method

w_1	w_2	μ_f / μ_f^*	$\Delta R_f / \Delta R_f^*$	μ_{x_1}	μ_{x_2}	h	G
0.00 [a]	***1.00***	***1.8300***	***1.0399***	***3.4775***	***4.6695***	***0.9943***	***0.0000***
0.10	0.90	1.7833	1.1190	3.3635	4.9443	0.9260	-0.1608
0.20	0.80	1.7673	1.1447	3.3036	4.9353	0.8761	-0.0918
0.30	0.70	1.7233	1.2747	3.1721	4.9785	0.8343	-0.0036
0.40	0.60	1.6199	1.6758	2.9455	5.2015	0.8040	0.0000
0.41	0.59	1.6006	1.7530	2.9104	5.2367	0.8019	0.0000
0.42	0.58	1.4993	2.1312	2.7498	5.3973	0.7990	0.0000
0.43	0.57	1.4907	2.1606	2.7377	5.4094	0.7980	0.0000
0.44	0.56	1.4819	2.1903	2.7254	5.4216	0.7972	0.0000
0.45	0.55	1.4730	2.2201	2.7131	5.4340	0.7965	0.0000
0.46	0.54	1.4638	2.2501	2.7007	5.4463	0.7960	0.0000
0.47	0.53	1.4545	2.2802	2.6882	5.4588	0.7956	0.0000
0.48	0.52	1.0000	3.1923	2.2000	5.9471	0.8083	0.0000
0.49	0.51	1.0000	3.1923	2.2000	5.9471	0.8113	0.0000
0.50	0.50	1.0000	3.1923	2.2000	5.9471	0.8143	0.0000
0.60	0.40	1.0000	3.1923	2.2000	5.9471	0.8457	0.0000
0.70	0.30	1.0000	3.1923	2.2000	5.9471	0.8797	0.0000
0.80	0.20	1.0000	3.1923	2.2000	5.9471	0.9164	0.0000
0.90	0.10	1.0000	3.1923	2.2000	5.9471	0.9564	0.0000
1.00 [b]	***0.00***	***1.0000***	***3.1923***	***2.2000***	***5.9471***	***1.0000***	***0.0000***

a: Robust design; b: Reliable design

For comparison purposes, the same problem is solved using the weighted-sum approach. The following probabilistic optimization problem is solved instead of Problem (25)

$$\min_{\boldsymbol{\mu}_{\mathbf{x}}} f = \left(w_1 \frac{\mu_f}{\mu_f^*} + w_2 \frac{\Delta R_f}{\Delta R_f^*} \right)$$

$$\text{s.t. } P(G(\mathbf{x}) \geq 0) \geq R$$

$$P(1 \leq x_i \leq 10) \geq R, \quad i = 1,2 .$$

The results are summarized in Fig. 5 and Table 2. As shown in Fig. 5, the weighted-sum approach fails to identify a large portion of the Pareto set. It only identifies a small region around the robust optimum extreme.

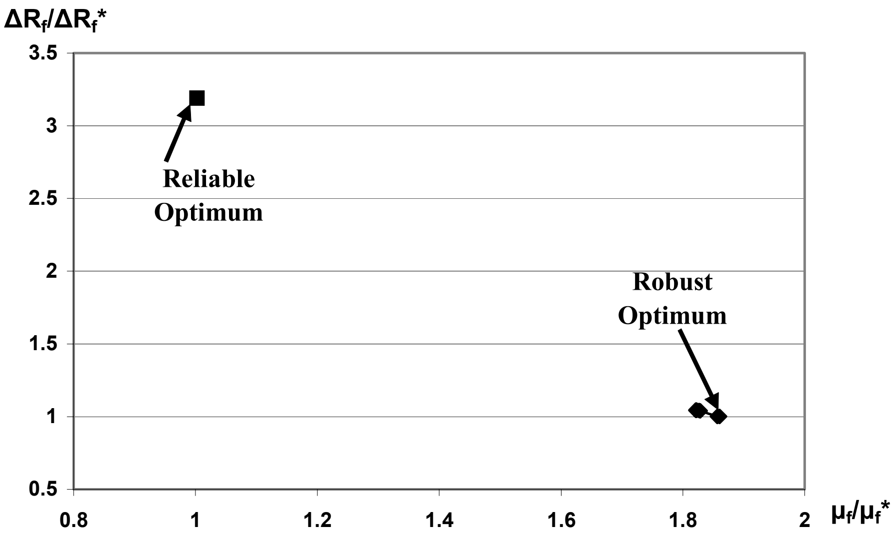

Figure 5. Pareto set for the mathematical example, using the weighted-sum method

Table 2. Pareto set details for the mathematical example, using the weighted-sum method

w_1	w_2	μ_f / μ_f^*	$\Delta R_f / \Delta R_f^*$	μ_{x_1}	μ_{x_2}	f	G
0.00[a]	***1.00***	***1.8600***	***1.0000***	***3.4666***	***5.5359***	***1.0000***	***0.8554***
0.10	0.90	1.8566	1.0002	3.4637	5.5213	1.0858	0.8379
0.20	0.80	1.8284	1.0403	3.4727	4.6744	1.1979	0.0000
0.30	0.70	1.8263	1.0410	3.4669	4.6802	1.2766	0.0000
0.40	0.60	1.8224	1.0433	3.4555	4.6916	1.3549	0.0000
0.41	0.59	1.8226	1.0431	3.4561	4.6909	1.3627	0.0000
0.42	0.58	1.8215	1.0440	3.4528	4.6943	1.3705	0.0000
0.43	0.57	1.8218	1.0437	3.4538	4.6933	1.3783	0.0000
0.44	0.56	1.8211	1.0442	3.4517	4.6954	1.3861	0.0000
0.45	0.55	1.8212	1.0441	3.4521	4.6950	1.3938	0.0000
0.46	0.54	1.0026	3.1901	2.2025	5.9446	2.1838	0.0000
0.47	0.53	1.0000	3.1923	2.2000	5.9471	2.1619	0.0000
0.48	0.52	1.0000	3.1923	2.2000	5.9471	2.1400	0.0000
0.49	0.51	1.0000	3.1923	2.2000	5.9471	2.1181	0.0000
0.50	0.50	1.0000	3.1923	2.2000	5.9471	2.0962	0.0000
0.60	0.40	1.0000	3.1923	2.2000	5.9471	1.8769	0.0000
0.70	0.30	1.0000	3.1923	2.2000	5.9471	1.6577	0.0000
0.80	0.20	1.0000	3.1923	2.2000	5.9471	1.4385	0.0000
0.90	0.10	1.0000	3.1923	2.2000	5.9471	1.2192	0.0000
1.00[b]	***0.00***	***1.0000***	***3.1923***	***2.2000***	***5.9471***	***1.0000***	***0.0000***

a: Robust design; b: Reliable design

5.2. A Cantilever Beam Example

In this example, a cantilever beam in vertical and lateral bending [37] is used (see Fig. 6). The beam is loaded at its tip by the vertical and lateral loads Y and Z, respectively. Its length L is equal to 100 in. The width *w* and thickness *t* of the cross-section are random design variables. The objective is to minimize the weight of the beam. This is equivalent to minimizing $w*t$, assuming that the material density and the beam length are constant.

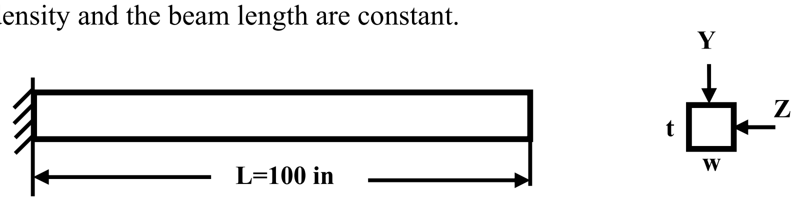

Figure 6. Cantilever beam under vertical and lateral bending.

One non-linear failure mode is used representing yielding at the fixed end of the cantilever. The RBDO problem is formulated as,

$$\min_{\mu_w,\mu_t} f = \mu_w * \mu_t$$
$$\text{s.t. } P(G_1(\mathbf{X}) \geq 0) \geq R$$
$$0 \leq \mu_w, \mu_t \leq 5$$

where the limit state $G_1(y,Z,Y,w,t) = y - (\frac{600}{wt^2} * Y + \frac{600}{w^2 t} * Z)$ represents the failure mode. The random design variables w and t are normally distributed with $\sigma_w = \sigma_t = 0.225$. Y, Z, y and E are normally distributed random parameters with Y~ N (1000, 100) lb, Z~ N (500,100) lb, y~ N (40000, 2000) psi and E~ N $(29*10^6, 1.45*10^6)$ psi; y is the random yield strength, Z and Y are mutually independent random loads in the vertical and lateral directions respectively, and E is the Young modulus. A reliability index $\beta = 3$ is used.

For the reliable/robust problem one more objective is added representing the variation of the beam tip displacement. The formulation is as follows,

$$\min_{\mu_w,\mu_t} f = \mu_w * \mu_t$$
$$\min_{\mu_w,\mu_t} \Delta R_\delta(w,t,E,Y,Z)$$
$$\text{s.t. } P(G_1(\mathbf{X}) \geq 0) \geq R$$
$$0 \leq \mu_w, \mu_t \leq 5$$

where the tip displacement δ is given by $\delta(w,t,E,Y,Z) = \frac{4L^3}{Ewt}\sqrt{(\frac{Y}{t^2})^2 + (\frac{Z}{w^2})^2}$.

Two objectives are simultaneously minimized subject to one probabilistic constraint. If the beam cross-sectional area is minimized, the beam stiffness is also minimized which usually leads to a large variation of the tip displacement. It is expected therefore, to have a trade-off between the two objectives. The percentile difference for the tip displacement is calculated as $\Delta R_\delta = \delta^{R_2} - \delta^{R_1}$ with $R_2 = 95\%$ and $R_1 = 5\%$. Similarly to the previous example, two separate single-objective optimization problems are first solved in order to establish the *utopia* point. The first problem (conventional RBDO) yields an optimum objective of $\mu_f^* = 11.2884$ for the design vector $[w^*, t^*] = [2.9421, 3.8369]$. The second single-objective, purely robust optimization problem yields a solution of $\Delta R_\delta^* = 0.1440$ for $[w^*, t^*] = [5,5]$.

For the calculation of the Pareto set, two linear preference functions h_1 and h_2 are used, corresponding to the two objectives f and ΔR_δ. They are all defined similarly to the previous example. Their "cut-off" values are $\mu_f = 3\mu_f^*$, $\Delta R_\delta = 15\Delta R_\delta^*$ for h_1 and h_2, respectively. The two objectives are aggregated using

$$h = \left(\frac{w_1 h_1^s + w_2 h_2^s}{w_1 + w_2} \right)^{1/s} ; w_1 + w_2 = 1,\ 0 \le w_1 \le 1$$

which is maximized by solving the following probabilistic optimization problem

$$\max_{\mu_w, \mu_t} h$$
$$\text{s.t. } P(G_1(\mathbf{X}) \ge 0) \ge R$$
$$0 \le \mu_w, \mu_t \le 5,$$

using the single-loop RBDO method of Section 3.2.

The results for this example are summarized in Figure 7 and Tables 3 and 4. A compensation level of $s = -1$ is assumed. The Pareto frontier of Fig. 7 shows the trade-off between μ_f / μ_f^* and $\Delta R_\delta / \Delta R_\delta^*$. The designs are almost equally spaced between the two extremes of reliable and robust designs.

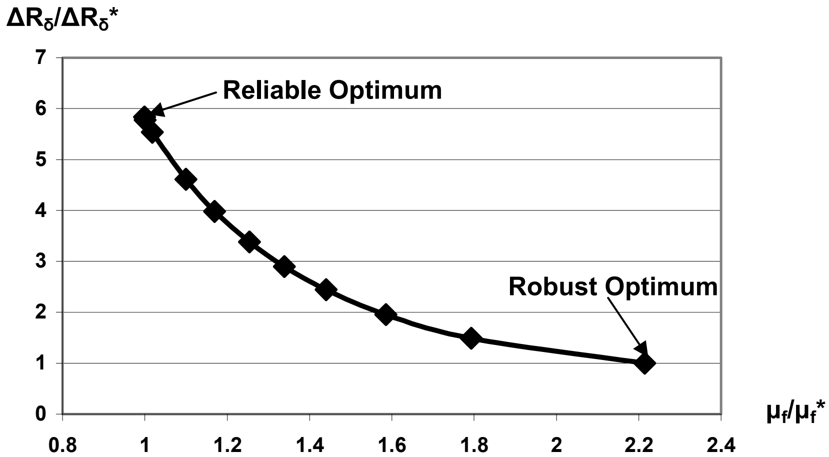

Figure 7. Pareto set for the beam example; trade-off between $\Delta R_\delta / \Delta R_\delta^*$ and μ_f / μ_f^*

Table 3 shows the exact values of the Pareto points and the corresponding values of constraint G_1 for ten equally spaced segments of the $0 \le w_1 \le 1$ domain. As indicated by their positive values, constraint G_1 is inactive for $0 \le w_1 \le 0.8$ and active for $0.9 \le w_1$.

Table 3. Pareto set details for the beam example

w_1	w_2	μ_f / μ_f^*	$\Delta R_\delta / \Delta R_\delta^*$	G_1
0.0[a]	***1.0***	***2.2147***	***0.9999***	***26054.0***
0.1	0.9	1.7930	1.4881	22672.9
0.2	0.8	1.5860	1.9506	19922.2
0.3	0.7	1.4403	2.4413	17111.9
0.4	0.6	1.3385	2.8985	14574.7
0.5	0.5	1.2541	3.3780	12053.2
0.6	0.4	1.1696	3.9830	8837.4
0.7	0.3	1.0994	4.6126	5635.8
0.8	0.2	1.0182	5.5360	1121.7
0.9	0.1	1.0002	5.7771	0.0
1.0[b]	***0.0***	***0.9993***	***5.8376***	***0.0***

a: Robust design; b: Reliable design

Table 4. Pareto set details for the beam example (Cont.)

w_1	w_2	h_1	h_2	h	w	t
0.0[a]	***1.0***	***0.3927***	***1.0000***	***1.0000***	***5.0000***	***5.0000***
0.1	0.9	0.6035	0.9651	0.9106	4.1995	4.8198
0.2	0.8	0.7070	0.9321	0.8763	3.7938	4.7189
0.3	0.7	0.7798	0.8970	0.8583	3.6199	4.4914
0.4	0.6	0.8307	0.8644	0.8506	3.4970	4.3207
0.5	0.5	0.8729	0.8301	0.8510	3.3765	4.1928
0.6	0.4	0.9152	0.7869	0.8592	3.2651	4.0437
0.7	0.3	0.9503	0.7420	0.8765	3.1734	3.9109
0.8	0.2	0.9909	0.6760	0.9065	3.0397	3.7810
0.9	0.1	0.9999	0.6588	0.9507	2.9978	3.7665
1.0[b]	***0.0***	***1.0000***	***0.6545***	***1.0000***	***2.9319***	***3.8477***

a: Robust design; b: Reliable design

The constraint G_1 is always inactive except for designs close to the reliable design where it becomes active. Table 4 shows the values of the optimal design vector $[w,t]$ and the corresponding preference functions h_1 and h_2 and the overall aggregate preference function h for the points on the Pareto set.

6. Summary and Conclusions

A computationally efficient unified method for reliability and robustness, based on a multi-objective optimization formulation, has been presented. The preference aggregation method is used to choose the "best" solution of the multi-objective optimization problem based on designer preferences. The proposed methodology addresses the shortcomings of the commonly used weighted-sum method which may fail to identify regions of the Pareto set of optimal solutions. Furthermore, it does not require the calculation of the entire Pareto set. It can identify the "best" design on the Pareto set, based on designer preferences and a rigorous, provable procedure of "indifference points."

The performance variation is assessed by a percentile difference method. The percentiles are efficiently calculated using a variation of the Advanced Mean Value method which provides much more accurate results compared with the commonly used Taylor series expansion for calculating the variance of a performance measure. The “best” design (optimal, reliable and robust) is calculated using an efficient single-loop probabilistic optimization method. Two examples illustrated the benefits of the proposed method and demonstrated its applicability.

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References

1. Reddy, M. V., Granhdi, R. V. and Hopkins, D. A., "Reliability Based Structural Optimization: A Simplified Safety Index Approach," *Computers and Structures*, **53**(6), 1407-1418, 1994.
2. Lee, J. O., Yang, Y. O. and Ruy, W. S., "A Comparative Study on Reliability Index andTarget Performance Based Probabilistic Structural Design Optimization," *Computers and Structures*, **80,** 257-269, 2002.
3. Wu, Y.-T., "Computational Methods for Efficient Structural Reliability and Reliability Sensitivity Analysis," *AIAA Journa*l, **32**(8), 1717-1723, 1994.
4. Wu, Y. –T. and Wang, W., "A New Method for Efficient Reliability-Based Design Optimization," Proceedings of 7th Special Conference on Probabilistic Mechanics & Structural Reliability, 274-277, 1996.
5. Zou, T., Mourelatos, Z. P., Mahadevan, S. and J. Tu, "Component and System Reliability Analysis Using an Indicator Response Surface Monte Carlo Approach," Proceedings of ASME Design Engineering Technical Conferences, DETC2003/DAC-48708, 2003.
6. Tu, J., Choi, K. K. and Park, Y. H., "A New Study on Reliability-Based Design Optimization", *ASME Journal of Mechanical Design*, **121**, 557-564, 1999.
7. Youn, B. D., Choi, K. K. and Park, Y. H., "Hybrid Analysis Method for Reliability-Based Design Optimization," *ASME Journal of Mechanical Design*, **125**(2), 221-232, 2003.
8. Wu, Y.-T. and Wang, W., "Efficient Probabilistic Design by Converting Reliability Constraints to Approximately Equivalent Deterministic Constraints, " *Journal of Integrated Design and Process Sciences*, **2**(4), 13-21, 1998.
9. Phadke, M. S., Quality Engineering Using Robust Design, Prentice Hall, NJ, 1989.
10. Taguchi, G., Elsayed, E. and Hsiang, T., *Quality Engineering in Production Systems*, McGraw-Hill, NY, 1989.
11. Chandra, M. J., Statistical Quality Control, *CRC Press*, Boca Raton, FL, 2001.
12. Parkinson, D. B., "Robust Design by variability Optimization," *Quality and Reliability Engineering International*, **13**, 97-102, 1997.
13. Bennett, G. and Gupta, L. C., "Least Cost Tolerance," I*nternational Journal of Production Research,* **8**(1), 65-74, 1969.
14. Forouraghi, B., "Worst-Case Tolerance Design and Quality Assurance via Genetic Algorithms," *Journal of Optimization Theory and Applications,* **113**(2), 251-268, 2000.
15. Jung, D. H. and Lee, B. C., "Development of a Simple and Efficient Method for Robust Optimization," *International Journal for Numerical Methods in Engineering,* **53,** 2201-2215, 2002.
16. Du, X. and Chen, W., "Towards a Better Understanding of Modeling Feasibility Robustness in Engineering Design," *ASME Journal of Mechanical Design*, **122**, 385-394, 2000.
17. Chen, W., Wiecek, M. M. And Zhang, J., "Quality Utility – A Compromise Programming Approach to Robust Design," *ASME Journal of Mechanical Design*, **121**(2), 179-187, 1999.
18. Youn, B. D. and Choi, K. K., "Performance Moment Integration Approach for Reliability-Based Robust Design Optimization," Proceedings of ASME Design Engineering Technical Conferences (DETC), Paper # DETC2004/DAC-57471, 2004.
19. Ramakrishnan, B. and Rao, S. S., "A Robust Optimization Approach using Taguchi's Loss Function for Solving Nonlinear Optimization Problems," Advances in Design Automation, ASME DE-32-1, 241-248, 1991.
20. Stoebner, A. M. and Mahadevan, S., "Robustness in Reliability-Based Design," Proceedings of the 41st AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics, and Materials Conference, Atlanta, GA, 2000.

21. Das, I. and Dennis, J., "A Closer Look at Drawbacks of Minimizing Sums of Objectives for Pareto Set Generation in Multicriteria Optimization Problems," *Structural Optimization,* **14**(1), 63-69, 1997.
22. Bras, B. A. and Mistree, F., "A Compromise Decision Support Problem for Robust and Axiomatic Design," *ASME Journal of Mechanical Design*, **117**(1), 10-19, 1995.
23. Chen, W., Allen, J. K., Mistree, F. And Tsui, K.-L., "A Procedure for Robust Design; Minimizing Variations Caused by Noise factors and Control Factors," *ASME Journal of Mechanical Design,* **118**(4), 478-485, 1996.
24. Dai, Z., Scott, M. J. and Mourelatos, Z. P., "Robust Design using Preference Aggregation Methods," Proceedings of ASME Design Engineering Technical Conferences (DETC), Paper # DETC2003/DAC-48715, 2003.
25. Du, X., Sudjianto, A. and Chen, W., "An Integrated Framework for Optimization under Uncertainty using Inverse Reliability Strategy," Proceedings of ASME Design Engineering Technical Conferences (DETC), Paper # DETC2003/DAC-48706, 2003.
26. Parkinson, A., Sorensen, C. and Pourhassan, N., "A General Approach for Robust Optimal Design," *ASME Journal of Mechanical Design*, **115**(1), 74-80, 1993.
27. Kokkolaras, M., Mourelatos, Z. P. and Papalambros P. Y., "Design Optimization of Hierarchically Decomposed Multilevel Systems under Uncertainty," Proceedings of ASME Design Engineering Technical Conferences (DETC), Paper # DETC2004/DAC-57357, 2004.
28. Otto, K. N. and Antonsson, E. K., "Trade-Off Strategies in Engineering Design," *Research in Engineering Design*, **3**(2), 87-104, 1991.
29. Scott, M. J. and Antonsson, E. K., "Aggregation Functions for Engineering Design Trade-Offs," *Fuzzy Sets and Systems*, **99**(3), 253-264, 1998.
30. Scott, M. J. and Antonsson, E. K., "Using Indifference Points in Engineering Decisions," Proceedings of ASME Design Engineering Technical Conferences, Paper# DETC2000/ DTM-14559, 2000.
31. Wu, Y.-T., Millwater, H. R. And Cruse, T. A., "Advanced Probabilistic Structural Analysis Method of Implicit Performance Functions," *AIAA Journal*, **28**(9), 1663-1669, 1990.
32. Liang, J., Mourelatos, Z. P., and Tu, J., "A Single-Loop Method for Reliability-Based Design Optimization," Proceedings of ASME Design Engineering Technical Conferences, Paper# DETC2004/ DAC-57255, 2004.
33. Du, X. and Chen, W., "Sequential Optimization and Reliability Assessment Method for Efficient Probabilistic Design," *ASME Journal of Mechanical Design*, **126**(2), 225-233, 2004.
34. Royset, J. O., Der Kiureghian, A. and Polak, E., "Reliability-based optimal structural design by the decoupling approach," *Reliability Engineering & System Safety*, **73**, 213-221, 2001.
35. Chen, X. and Hasselman, T. K. and Neill, D. J., "Reliability Based Structural Design Optimization for Practical Applications," Proceedings of the 38th AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics, and Materials Conference, 1997.
36. Hacker, K. and Lewis, K., "Robust Design Through the Use of a Hybrid Genetic Algorithm," Proceedings of ASME Design Engineering Technical Conferences (DETC), Paper # DETC2002/DAC-34108, 2002.
37. Wu, Y.-T., Shin, Y., Sues, R. and Cesare, M., "Safety – Factor Based Approach for Probabilistic – Based Design Optimization," 42nd AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics and Materials Conference, Seattle, WA, 2001.