Interval Finite Element as a Basis for Generalized Models of Uncertainty in Engineering Mechanics

Rafi L. Muhanna
School of Civil and Environmental Engineering, Georgia Institute of Technology, Atlanta, GA 30332, USA, e-mail: rafi.muhanna@gtrep.gatech.edu

Robert L. Mullen
Department of Civil and Environmental Engineering, Case Western Reserve University, Cleveland, OH 44106, USA, e-mail: rlm@po.cwru.edu

Hao Zhang
School of Civil and Environmental Engineering, Georgia Institute of Technology, Atlanta, GA 30332, USA, e-mail: hao.zhang@ce.gatech.edu

Abstract. Latest scientific and engineering advances have started to recognize the need of defining multiple types of uncertainty. Probabilistic modeling cannot handle situations with incomplete or little information on which to evaluate a probability, or when that information is nonspecific, ambiguous, or conflicting [46, 11, 43]. Many interval-based models of uncertainty have been developed to treat such situations.

This paper presents an interval approach for the treatment of parameter uncertainty for linear static problems of mechanics. Uncertain parameters are introduced in the form of unknown but bounded quantities (intervals). Interval analysis is applied to Finite Element Method to analyze the system response due to uncertain stiffness and loading. To avoid overestimation, the formulation is based on an element-by-element (EBE) technique. Element matrices are formulated, based on the physics of materials, and the Lagrange multiplier method is applied to impose the necessary constraints for compatibility and equilibrium. Earlier EBE formulation provided sharp bounds only on displacements [29]. Based on the developed formulation, the bounds on the system's displacement and forces are obtained simultaneously and have the same level of accuracy. Very sharp enclosures for the exact system responses are obtained. A number of numerical examples are introduced and scalability is illustrated.

1. Introduction

An important issue faced by real life engineering practice is how to deal with variables and parameters of uncertain values. For a proper performance assessment, these uncertainties must be accounted for appropriately. There are various ways in which the types of uncertainty might be classified. One is distinguish between "aleatory" (or stochastic) uncertainty and "epistemic" uncertainty. The first refers to underlying, intrinsic variabilities of physical quantities and the latter refers to uncertainty which might be reduced with additional data or information, or better modeling and better parameter estimation [23]. Probability theory is the traditional approach to handle uncertainty. This approach requires sufficient statistical data to justify the assumed statistical distributions. Analysts agree that, given sufficient statistical data, the probability theory describes the stochastic uncertainty well. However, probabilistic modeling cannot handle situations with incomplete or little information on which to evaluate a probability, or when that information is nonspecific, ambiguous, or conflicting [46, 11, 43]. Many generalized models of uncertainty have been developed to treat such situations, which includes imprecise probabilities [46], Dempster-Shafer theory of evidence [9, 44] and random set [19], fuzzy sets [47], possibility theory [8], probability bounds [12], convex model [5], and others.

These set-based uncertainty models have a variety of mathematical descriptions, however, they are all closely connected with interval arithmetic [25]. For example, a fuzzy number [47] can be viewed as a set of valued intervals with different confidence of given level of presumptions (α cuts). Thus fuzzy arithmetic can be performed as interval arithmetic on α cuts. A Dempster-Shafer structure [9, 44] with interval focal elements can be viewed as a set of intervals with probability mass assignments, where the computation is carried out using the interval focal sets. Probability bounds analysis [12] is a combination of the methods of standard interval analysis and probability theory. Uncertain variables are decomposed into a list of pairs of the form (interval, probability). In this sense, interval arithmetic serves as the calculation tool for the generalized models of uncertainty.

Recently, various generalized models of uncertainty have been applied to finite element method (FEM) to solve a partial differential equation with uncertain parameters. Regardless what model is adopted, the proper interval solution will represent the first requirement for any further rigorous formulation. Finite element method with interval valued parameters results in Interval Finite Element Method (IFEM). The numerical solution of IFEM is the focus of this paper. Different formulations of IFEM have been developed. However, the used solution techniques can be reduced to two main approaches; optimization-based and anti-optimization. In the optimization approaches [20, 38, 1, 24], optimization algorithm is employed to search for the extrema (max/min) of the system response in the interval parameter domain. This optimization approach often encounters practical difficulties. Firstly it requires sophisticated optimization algorithm, where the objective function is implicit and complicated in most structural engineering problems, thus often only approximate solution is achievable. Secondly, this approach is computationally expensive. For each response quantity, two optimization problems must be solved to find the extreme lower and the extreme upper bounds. This will be a huge computational effort, especially in the case of practical engineering problems.

More recently, anti-optimization approaches for the interval finite element analysis have been developed in a number of works. For linear elastic problems, this approach leads to a system of linear interval equations, then the solution is sought using interval methods developed for this purpose. The major difficulty associated with this approach is the so-called "dependency problem" [26, 34, 17, 29]. The dependency in interval arithmetic leads to an overestimation of the system response. A straightforward replacement of the system parameters with interval ones without taking care of the dependency problem is known as a naïve application of interval arithmetic in finite element method (naïve interval FEM), and usually such a use results in meaningless wide and even catastrophic results [29].

In the anti-optimization category, a number of developments can be presented. A combinatorial approach (based on an exhaustive combination of the extreme values of the interval parameters) was used in [37]. This approach gives exact solution in linear elastic problems. However, it is computationally tedious and expensive, and is limited to the solutions of small-scale problems only. Convex modeling and superposition approach was proposed to analyze load uncertainty in [35], and exact solution was obtained. However, the superposition is only applicable to load uncertainty. Combinatorial approach was used in [14] to treat interval modulus of elasticity. Chen et al. [6] have developed static displacement bounds analysis using matrix perturbation theory. The first-order perturbation was used and second-order term had been neglected. The result is approximate and not guaranteed to contain the exact bounds. McWilliam [22] proposed two methods for determining the static displacement bounds of structures with interval parameters. The first method is a modified version of perturbation analysis. The second method is based on the assumption that the displacement surface is monotonic. However, for the general case, the validity of monotonicity is difficult to verify. Dessombz [10] have introduced an interval FEM in which the interval parameters were factorized during the assemblage process of the stiffness matrix, then Rump's iterative algorithm [40] was employed for solving the linear interval equation. In this work, the overestimation control becomes more difficult with the increase of the number of the interval parameters, which does not lead to useful results for practical problems. In the works of Muhanna and Mullen [27], Mullen and Muhanna [30, 31], an interval-based fuzzy finite element has been developed for treating uncertain loads in static structural problems. Load dependency was eliminated and the exact solution was obtained. Also, Muhanna and Mullen [29] have developed an interval finite element method based on element-by-element technique and Lagrange multiplier. Uncertain modulus of elasticity was considered. Most sources of overestimation were eliminated, and a sharp result for displacement was obtained.

In this paper a new formulation for interval finite element analysis of linear elastic structures will be introduced. Material and load uncertainties are handled simultaneously and sharp enclosures on the system's displacement and forces are obtained efficiently. A brief review of interval arithmetic is presented, the formulation is described, and numerical examples are given.

2. Short review of interval arithmetic

For simplicity and better clarity, all interval quantities will be introduced in bold face. Detailed information about interval arithmetic can be found in series of books and publications such as [16, 25, 2, 34, 42, 45].

2.1. Basic Definition

An interval number is a closed set in $\mathbb{R}$ that includes the possible range of an unknown real number, where $\mathbb{R}$ denotes the set of real numbers. Therefore, a real interval is a set of the form

$$\mathbf{x} = [\underline{x},\ \overline{x}] \tag{1}$$

where $\underline{x}$ and $\overline{x}$ are the lower and upper bounds (endpoints) of the interval number $\mathbf{x}$ respectively. The midpoint $\check{x}$ of $\mathbf{x}$ is introduced as

$$\check{x} \equiv \text{mid}(\mathbf{x}) := \frac{\underline{x} + \overline{x}}{2} \tag{2}$$

Sometimes it is convenient to write the interval in the midpoint form

$$\mathbf{x} = \check{x}(1 + \boldsymbol{\alpha}) \tag{3}$$

in which $\boldsymbol{\alpha}$ is a 0-midpoint interval. For example, when we say $\mathbf{x}$ has 4% uncertainty, it means $\boldsymbol{\alpha} = [-0.02, 0.02]$, and $\mathbf{x} = \check{x}(1 + [-0.02, 0.02])$.

The set of real intervals will be denoted by $\mathbb{IR}$. Operations with at least one interval operand are by definition interval operations. It is easy to see that the set of all possible results for $x \in \mathbf{x}$ and $y \in \mathbf{y}$ forms a closed interval (for 0 not in a denominator interval), and the endpoints can be calculated by

$$\mathbf{x} \circ \mathbf{y} = [\min\ (x_i \circ y_i),\ \max\ (x_i \circ y_i)] \quad \text{for} \circ \in \{+, -, \cdot, /\} \tag{4}$$

2.2. Dependency Problem in Interval Arithmetic

The interval-system quality is measured by the width of the interval results, and a sharp enclosure for the exact solution is desirable. However, the width of results may be unnecessarily wide in some occasions due to dependency effect. For example, if the interval function $f(\mathbf{x}) = \mathbf{x} - \mathbf{x}$ is evaluated with $\mathbf{x} = [a, b] = [1, 2]$, the interval subtraction rule (Appendix A) gives the result: $f(\mathbf{x}) = [a - b, b - a] = [-1, 1]$, which is containing the exact solution $[0, 0]$, but much wider. The interval arithmetic implicitly made the assumption that all intervals are independent, namely it treats $\mathbf{x} - \mathbf{x}$ as if evaluating the intervals $\mathbf{x} - \mathbf{y}$, and $\mathbf{x}, \mathbf{y}$ are two independent interval quantities that happen to have the same bounds. This phenomenon is referred as overestimation due to "dependency" of the variables [26, 34, 17, 29]. Reducing the overestimation is a central issue to a successful interval analysis. In general, sharp results are obtained with the proper understanding of the physical nature of the problem and reduction of the dependence. In the above example, the exact solution could be achieved in evaluating $\mathbf{x} - \mathbf{x}$ as $\mathbf{x}(1 - 1) = 0$.

2.3. Interval Vectors and Matrices

An interval matrix $\mathbf{A} \in \mathbb{IR}^{n \times k}$ is interpreted as a set of real $n \times k$ matrices by the convention $\mathbf{A} = \{A \in \mathbb{R}^{n \times k} \mid A_{ij} \in \mathbf{A_{ij}} \quad \text{for } i = 1, \ldots, n;\ j = 1, \ldots, k\}$. The set of $n \times k$ interval matrices is denoted by $\mathbb{IR}^{n \times k}$. An $n \times 1$ interval matrix is an interval vector, denoted by $\mathbb{IR}^n$. Operations on interval matrices are extended naturally from the corresponding deterministic matrices operations. Algebraic properties of interval matrix operations are provided in [34, 3, 21].

2.4. Linear Interval Equations

A linear interval equation with coefficient matrix $\mathbf{A} \in \mathbb{IR}^{n\times n}$ and right-hand side $\mathbf{b} \in \mathbb{IR}^n$ is defined as the family of linear equations

$$Ax = b \quad (A \in \mathbf{A},\ b \in \mathbf{b}) \tag{5}$$

Therefore, a linear interval equation represents systems of equations in which the coefficients are unknown numbers ranging in certain intervals. The solution set of (5) is given by:

$$\Sigma(\mathbf{A},\, \mathbf{b}) = \{x \in \mathbb{R}^n \mid \exists A \in \mathbf{A},\ \exists b \in \mathbf{b} : Ax = b\} \tag{6}$$

The solution set $\Sigma(\mathbf{A},\, \mathbf{b})$ usually is not an interval vector, and does not need even to be convex; in general, $\Sigma(\mathbf{A},\, \mathbf{b})$ has a very complicated structure. In order to guarantee that the solution set $\Sigma(\mathbf{A},\, \mathbf{b})$ is bounded, it is required that the matrix $\mathbf{A}$ be regular, i.e. that every matrix $A \in \mathbf{A}$ is nonsingular. The hull of the solution set $\Sigma(\mathbf{A},\, \mathbf{b})$ is an interval vector which has the narrowest possible interval components, denoted as

$$\mathbf{A}^H\mathbf{b} := \Diamond\Sigma(\mathbf{A}, \mathbf{b}) \tag{7}$$

where

$$\mathbf{A}^H\mathbf{b} = \Diamond\{A^{-1}b | A \in \mathbf{A}, b \in \mathbf{b}\} \quad \text{for } \mathbf{b} \in \mathbb{IR}^n \tag{8}$$

In fact, computing the hull of the solution set for the general case is NP-Hard problem [39]. The solution of interest is seeking an enclosure, i.e., an interval vector $\mathbf{x}$ containing $\mathbf{A}^H\mathbf{b}$, while narrow enough to be practically useful:

$$\mathbf{A}^H\mathbf{b} \ \subseteq \mathbf{x} \tag{9}$$

A number of methods have been developed to find $\mathbf{x}$ for the general linear interval equations such as Interval Gauss elimination, Interval Gauss-Seidel iteration, Krawczyk's iteration, and fixed-point iteration [15, 32, 34, 18, 40, 41]. These algorithms usually involve a preconditioning of the coefficient matrix, and then iterations are performed to get the enclosure. The present work is using Brouwer's fixed point theorem and Krawczyk's operator. This method has been discussed in the works of [15, 32, 33, 18, 40, 41].

One typical approach to find the solution of a linear system $Ax = b$, is to transform it into a fixed point equation $g(x) = x$, in which

$$g(x) = x - R(Ax - b) = Rb + (I - RA)x \tag{10}$$

and R is a nonsingular matrix. From Brouwer's fixed point theorem, it follows that for some interval vector $\mathbf{x} \in \mathbb{IR}^n$

$$Rb + (I - RA)x \in \mathbf{x} \quad \forall x \in \mathbf{x} \tag{11}$$

implies

$$\exists x \in \mathbf{x} : Ax = b \tag{12}$$

To verify condition (11) is a range determination problem, and can be reduced to interval arithmetic:

$$Rb + (I - RA)\mathbf{x} \subseteq \mathbf{x} \tag{13}$$

If an interval vector $\mathbf{x}$ satisfying (13) can be found, then $\mathbf{x}$ contains the solution of $Ax = b$. The result can be extended to find the enclosure of the solution set of linear interval equation $\mathbf{Ax} = \mathbf{b}$ [34, 42]. The following theorem can be presented:

THEOREM 1 (Rump 2001). *Let* $\mathbf{A} \in \mathbb{IR}^{n\times n}$, $R \in \mathbb{R}^{n\times n}, \mathbf{b}, \mathbf{x} \in \mathbb{IR}^n$ *be given, if*

$$R\mathbf{b} + (I - R\mathbf{A})\mathbf{x} \subseteq \mathrm{int}(\mathbf{x}) \tag{14}$$

then R *and every matrix* $A \in \mathbf{A}$ *is nonsingular, and*

$$\Sigma(\mathbf{A}, \mathbf{b}) = \{x \in \mathbb{R}^n \mid \exists A \in \mathbf{A},\ \exists b \in \mathbf{b} : Ax = b\} \subseteq \mathbf{x} \tag{15}$$

where $\mathrm{int}(\mathbf{x})$ denotes the interior of $\mathbf{x}$. Expression (15) provides a guaranteed enclosure to the solution set of the linear interval equation $\mathbf{Ax} = \mathbf{b}$. The residual form of (14) can be given in the form [34]:

$$R\mathbf{b} - R\mathbf{A}x_0 + (I - R\mathbf{A})\mathbf{x}^* \subseteq \mathrm{int}(\mathbf{x}^*) \tag{16}$$

where $\mathbf{x} = x_0 + \mathbf{x}^*$ and x_0 is a deterministic vector, in particular, $\check{A}^{-1}$ is a good choice for R, and $x_0 = R\check{b}$. Assigning $\mathbf{z} = R\mathbf{b} - R\mathbf{A}x_0$, $\mathbf{C} = (I - R\mathbf{A})$, iteration could be constructed [40] in the following form

$$\mathbf{x}^{*n+1} = \mathbf{z} + \mathbf{C}(\boldsymbol{\varepsilon}\mathbf{x}^{*n}) \qquad (\text{for } n = 0, 1, 2, \ldots) \tag{17}$$

and the stopping criteria (16) becomes

$$\mathbf{x}^{*n+1} \subseteq \mathrm{int}(\mathbf{x}^{*n}) \tag{18}$$

In Eq. (17) $\boldsymbol{\varepsilon}$ is a constant interval number, and it serves as an “inflation parameter” to enforce finite termination of the algorithm. If the condition (18) is satisfied after n iterations, then $\mathbf{x}^{*n+1} + x_0$ is an enclosure of the solution set of $\mathbf{Ax} = \mathbf{b}$. The quality (how sharp the enclosure is) of the enclosure provided in (17) depends mainly on the width of the iterative matrix $\mathbf{C}$ and is crucial for the solution convergence the condition that the spectral radius $\rho(|\mathbf{C}|) < 1$ [41].

It is noticeable, however, that the above algorithm is designed for the non-parametric linear interval equations, i.e., the coefficients in the system are assumed to vary independently between their bounds. For many engineering problems, the coefficients have complex dependency relations. For example, the stiffness matrix in FEM is symmetric and positive definite. To account for the dependency effect, one approach is to adapt the solver for non-parametric interval equation. This approach usually involves reformulation of the coefficient matrix and right hand side vector. It has been shown a sharp or even exact enclosure could be obtained in some cases [28, 29, 10].

3. Interval finite element analysis

3.1. Overestimation in IFEM

A naïve use of interval arithmetic in FEM (naïve IFEM), i.e., replacing deterministic numbers in conventional FEM with interval numbers and solving the system as non-parametric interval equation will result in meaningless wide results [29, 10]. Let us consider the two step bar shown in Fig. 1. The structure is subjected to a unit load at node 3. The conventional FEM gives the equilibrium equations:

$$Ku = p \tag{19}$$

or

$$\begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{20}$$

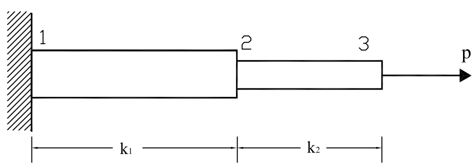

Figure 1. Original two-step bar

If the stiffness terms k_1 and k_2 are introduced as the interval parameters $\mathbf{k_1}$ and $\mathbf{k_2}$, and the interval numbers of $[0.99, 1.01]$ and $[1.98, 2.02]$ are assigned for $\mathbf{k_1}$ and $\mathbf{k_2}$ respectively, the naïve IFEM takes the following form:

$$\begin{pmatrix} [2.97, 3.03] & [-2.02, -1.98] \\ [-2.02, -1.98] & [1.98, 2.02] \end{pmatrix} \begin{pmatrix} \mathbf{u_1} \\ \mathbf{u_2} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{21}$$

Solving (21) using theorem 1, the value of $\mathbf{u_1}$ and $\mathbf{u_2}$ are obtained as:

$$\mathbf{u_1} = [0.876, 1.123]$$

and

$$\mathbf{u_2} = [1.349, 1.651] \tag{22}$$

On the other hand, the exact solution can be achieved by solving (20) symbolically

$$\mathbf{u_1} = \frac{1}{\mathbf{k_1}} = \frac{1}{[0.99, 1.01]} = [0.990, 1.010]$$

and

$$\mathbf{u_2} = \frac{\mathbf{k_1} + \mathbf{k_2}}{\mathbf{k_1}\mathbf{k_2}} = \frac{1}{\mathbf{k_1}} + \frac{1}{\mathbf{k_2}} = \frac{1}{[0.99, 1.01]} + \frac{1}{[1.98, 2.02]} = [1.485, 1.515] \tag{23}$$

The above-presented results for the interval solution of a simple two-step bar problem provide an insight about some aspects of the interval finite element formulation and reveal the most important sources of overestimation. The main two factors that lead for overestimation are the element coupling and multiple occurrences of the interval variables. The four parametric coefficients $\mathbf{k_2}$ in (20) represent the same physical quantity. In the computational process, interval arithmetic treats this physical quantity as four independent interval variables of equal endpoints. Evidently, the same physical quantity cannot have two different values at the same time. It is critical to the formulation of interval finite element analysis, the way the sources of overestimation are handled.

3.2. Present Formulation

In order to reduce the overestimation in the interval finite element solutions, the issues of coupling and multiple occurrences of interval variables have to be handled properly.

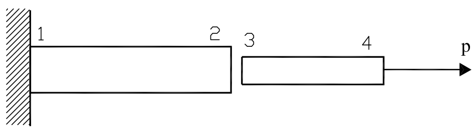

Figure 2. EBE two-step bar model

In this work, an element-by-element technique (EBE) is used to circumvent the element coupling problem [29]. The EBE technique can be illustrated by the two-step bar problem in Fig. 1. The elements are disjointed as shown in Fig. 2, thus the system stiffness matrix K takes a block-diagonal structure with dimension of $a \times a$, and $a =$ degrees of freedom per element $\times$ number of elements in the structure. EBE approach adds to the number of degree of freedom (DOF) in the system but avoids the element coupling. The system stiffness matrix K in EBE approach is singular, and Lagrange multiplier method will be used to ensure the compatibility conditions and eliminate the singularity of K.

In steady-state analysis, the variational formulation for a deterministic case of a discrete structural model is given in the following form [13, 4]

$$\Pi = \frac{1}{2} u^T K u - u^T p \tag{24}$$

with the conditions

$$\frac{\partial \Pi}{\partial u_i} = 0 \quad \text{for all } i \tag{25}$$

where Π, K, u, and p are total potential energy, stiffness matrix, displacement vector, and load vector respectively. Assume that we want to impose onto the solution the m linearly independent discrete constraints $Cu - t = 0$ where C and t contain constants. To impose constraints by Lagrange multipliers, we premultiply $Cu - t$ by a row vector λ that contains as many Lagrange multipliers λ_i as there are constraint equations, and add this to the potential energy (24) [7]. Thus

$$\Pi^* = \frac{1}{2} u^T K u - u^T p + \lambda^T (Cu - t) \tag{26}$$

Invoking the stationarity of Π^*, i.e., $\partial\Pi^*/\partial u = 0$ and $\partial\Pi^*/\partial\lambda = 0$ we obtain

$$\begin{pmatrix} K & C^T \\ C & 0 \end{pmatrix} \begin{pmatrix} u \\ \lambda \end{pmatrix} = \begin{pmatrix} p \\ t \end{pmatrix} \tag{27}$$

Considering the compatibility conditions in the present case takes the form $Cu = t = 0$, (27) reduces to

$$\begin{pmatrix} K & C^T \\ C & 0 \end{pmatrix} \begin{pmatrix} u \\ \lambda \end{pmatrix} = \begin{pmatrix} p \\ 0 \end{pmatrix} \tag{28}$$

Equation (28) stands for the deterministic FEM formulation. In the interval case, where the material and the load are considered to be interval numbers, the deterministic linear equation (28) becomes the interval linear equation

$$\begin{pmatrix} \mathbf{K} & C^T \\ C & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \mathbf{p} \\ 0 \end{pmatrix} \tag{29}$$

The coefficient matrix in (29) represents the combination of two parts: the interval element-by-element stiffness matrix $\mathbf{K}$ and the constant deterministic Lagrange multipliers matrix C.

The linear interval equation (29) can be solved by theorem 1. However, theorem 1 is used with the implicit assumption that the coefficients of $\mathbf{A}$ are independent among themselves and as well as the components of $\mathbf{b}$ vary independently. Special treatment has to be applied to reduce the dependency effect.

For an element with interval parameters of modulus of elasticity $\mathbf{E}$, the interval parameter could be factorized out from the element stiffness matrix. Consider the ith finite element in the structure, assume the uncertainty in the modulus of elasticity is $\boldsymbol{\alpha}_i$, i.e., $\mathbf{E}_i = \check{E}_i(1+\boldsymbol{\alpha}_i)$, the element stiffness matrix $\mathbf{K}_i$ can be expressed in the form $\mathbf{K}_i = \check{K}_i(I + \mathbf{d}_i)$. $\check{K}_i$ is the midpoint of $\mathbf{K}_i$, I is identity matrix, and $\mathbf{d}_i$ is an interval diagonal matrix containing the interval quantity $\boldsymbol{\alpha}_i$. Let us take a truss element for example, its element stiffness matrix can be written as

$$\begin{pmatrix} \frac{\check{E}A}{L} & -\frac{\check{E}A}{L} \\ -\frac{\check{E}A}{L} & \frac{\check{E}A}{L} \end{pmatrix} \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \boldsymbol{\alpha} & 0 \\ 0 & \boldsymbol{\alpha} \end{pmatrix} \right) \tag{30}$$

Later in the formulation, care will be taken of the multiple occurrence of $\boldsymbol{\alpha}$ in (30).

Following the same procedure for each element, the system stiffness matrix $\mathbf{K}$ constructed by EBE model can be expressed as:

$$\mathbf{K} = \check{K}(I + \mathbf{d}) \tag{31}$$

$\check{K}$ is the midpoint of $\mathbf{K}$, and $\mathbf{d}$ is an interval diagonal matrix; their submatrices are $\check{K}_i$ and $\mathbf{d}_i$, respectively, $i = 1, 2, \ldots, m$, where m is the number of elements in the structure.

Applying this factorization, the system equation (29) can be written as

$$\left(\begin{pmatrix} \check{K} & C^T \\ C & 0 \end{pmatrix} + \begin{pmatrix} \check{K}\mathbf{d} & 0 \\ 0 & 0 \end{pmatrix} \right) \begin{pmatrix} \mathbf{u} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \mathbf{p} \\ 0 \end{pmatrix} \tag{32}$$

To utilize the theorem 1 in the present formulation, (32) is introduced as

$$\mathbf{A}\mathbf{x} = \mathbf{b} \tag{33}$$

with

$$\mathbf{A} = \left(\begin{pmatrix} \check{K} & C^T \\ C & 0 \end{pmatrix} + \begin{pmatrix} \check{K}\mathbf{d} & 0 \\ 0 & 0 \end{pmatrix} \right), \quad \mathbf{x} = \begin{pmatrix} \mathbf{u} \\ \boldsymbol{\lambda} \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} \mathbf{p} \\ 0 \end{pmatrix} \tag{34}$$

$\mathbf{A}$ can be decomposed furthermore

$$\begin{aligned} \mathbf{A} &= \begin{pmatrix} \check{K} & C^T \\ C & 0 \end{pmatrix} + \begin{pmatrix} \check{K} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{d} & 0 \\ 0 & 0 \end{pmatrix} \\ \mathbf{A} &= \check{A} + S\mathbf{D} \end{aligned} \tag{35}$$

Using the residual form (16) to construct fixed point iteration (17)

$$\mathbf{x}^{*n+1} = \mathbf{z} + \mathbf{C}(\boldsymbol{\varepsilon}\mathbf{x}^{*n}) \qquad (\text{for } n = 0, 1, 2, \ldots) \tag{36}$$

in which $\mathbf{z} = R\mathbf{b} - R\mathbf{A}x_0$, $\mathbf{C} = (I - R\mathbf{A})$, $R = \check{A}^{-1}$, $x_0 = R\check{b}$. By substituting $\mathbf{z}$ and $\mathbf{C}$, the iteration (36) becomes

$$\begin{aligned} \mathbf{x}^{*n+1} &= (R\mathbf{b} - R(\check{A} + S\mathbf{D})x_0) + (I - R(\check{A} + S\mathbf{D}))(\boldsymbol{\varepsilon}\mathbf{x}^{*n}) \\ \mathbf{x}^{*n+1} &= R\mathbf{b} - x_0 - RS\mathbf{D}x_0 - RS\mathbf{D}(\boldsymbol{\varepsilon}\mathbf{x}^{*n}) \\ \mathbf{x}^{*n+1} &= R\mathbf{b} - x_0 - RS\mathbf{D}(x_0 + \boldsymbol{\varepsilon}\mathbf{x}^{*n}) \\ \mathbf{x}^{*n+1} &= R\mathbf{b} - x_0 - RS\mathbf{M}^n\boldsymbol{\delta} \end{aligned} \tag{37}$$

In the problems with deterministic right hand side, we have $\mathbf{b} = \check{b}$, and (37) reduces to a even simpler form

$$\mathbf{x}^{*n+1} = -RS\mathbf{M}^n\boldsymbol{\delta} \tag{38}$$

A key point in the formulation (37) is that $\mathbf{D}(x_0 + \boldsymbol{\varepsilon}\mathbf{x}^{*n})$ has been introduced as $\mathbf{M}^n\boldsymbol{\delta}$ using the $\mathbf{M}$ matrix concept [31, 29] to handle the dependency problem in $\mathbf{D}(x_0 + \boldsymbol{\varepsilon}\mathbf{x}^{*n})$. $\mathbf{M}$ is an interval matrix with the dimensions $(n \times m)$, and $n =$ dimensions of the system. It contains the components from $(x_0 + \boldsymbol{\varepsilon}\mathbf{x}^{*n})$, it will be update with each iteration. $\boldsymbol{\delta}$ is an constant interval vector with the dimensions of m, and the components are the uncertainties $\boldsymbol{\alpha}_i$ of the modulus of elasticity of each element, $i = 1, \ldots, m$. Every interval parameter $\boldsymbol{\alpha}_i$ associated with element i occurs only once in $\boldsymbol{\delta}$. The following example shows how generally $\mathbf{D}\mathbf{x}$ could be rewritten as $\mathbf{M}\boldsymbol{\delta}$. Suppose there are two interval parameters $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$

$$\begin{pmatrix} \boldsymbol{\alpha_1} & 0 & 0 & 0 \\ 0 & \boldsymbol{\alpha_1} & 0 & 0 \\ 0 & 0 & \boldsymbol{\alpha_2} & 0 \\ 0 & 0 & 0 & \boldsymbol{\alpha_2} \end{pmatrix} \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \\ \mathbf{x}_4 \end{pmatrix} = \begin{pmatrix} \mathbf{x}_1 & 0 \\ \mathbf{x}_2 & 0 \\ 0 & \mathbf{x}_3 \\ 0 & \mathbf{x}_4 \end{pmatrix} \begin{pmatrix} \boldsymbol{\alpha_1} \\ \boldsymbol{\alpha_2} \end{pmatrix} \tag{39}$$

This treatment eliminates the multiple occurrences of $\boldsymbol{\alpha}_i$ in $\mathbf{D}$, thus reduces the overestimation due to dependency problem. If the condition (18) is satisfied after n iterations, the enclosure $\mathbf{x}$ is given by

$$\mathbf{x} = \mathbf{x}^{*n+1} + x_0 \tag{40}$$

The obtained interval vector $\mathbf{x}$ contains two parts: $\mathbf{x} = [\mathbf{u} \quad \boldsymbol{\lambda}]$. The first part, $\mathbf{u}$, is the enclosure for the system's displacement response.

In conventional deterministic FEM, element forces in global coordinate can be calculated by

$$F_i = K_i u_i \tag{41}$$

in which K_i, u_i are element stiffness matrix and element nodal displacement in global coordinate. The element forces in local coordinate can be obtained by premultiply a transformation matrix T_i. In the interval FEM, however, following the same procedure to calculate element force will bring in overestimation, making the bounds of the element forces unnecessarily wide. The reason is that both $\mathbf{K}_i$ and $\mathbf{u}_i$ are functions of the same interval parameter $\boldsymbol{\alpha}_i$, this multiple occurrences of $\boldsymbol{\alpha}_i$ should be eliminated. In the present IFEM formulation, element forces are calculated from Lagrange multipliers. From (29), it follows

$$\mathbf{K}\mathbf{u} = \mathbf{p} - C^T \boldsymbol{\lambda} \tag{42}$$

Because of its element-by-element structure, (42) produces the element forces directly (in global coordinate). Instead of calculating the left hand side of (42), we will calculate its right hand side to handle dependence problem. Suppose the enclosure $\mathbf{x}$ has been achieved after n iterations, then $\boldsymbol{\lambda}$ can be obtained from $\mathbf{x}$ by a boolean matrix L, i.e.,

$$\boldsymbol{\lambda} \;=\; L\mathbf{x} \tag{43}$$

The interval load $\mathbf{p}$ can be rewritten as

$$\mathbf{p} = N\mathbf{b} \tag{44}$$

in which N is a boolean matrix for $\mathbf{p}$. Substitute (37), (43) and (44) into $\mathbf{p} - C^T\boldsymbol{\lambda}$

$$\begin{aligned}
\mathbf{p} - C^T\boldsymbol{\lambda} \;&=\; \mathbf{p} - C^T L(\mathbf{x}^{*n+1} + x_0) \\
\mathbf{p} - C^T\boldsymbol{\lambda} \;&=\; N\mathbf{b} - C^T L(R\mathbf{b} - RS\mathbf{M}^n\boldsymbol{\delta}) \\
\mathbf{p} - C^T\boldsymbol{\lambda} \;&=\; (N - C^T LR)\mathbf{b} + C^T LRS\mathbf{M}^n\boldsymbol{\delta}
\end{aligned} \tag{45}$$

Equation (45) may be premultiplied by a transformation matrix T to get the element forces in local coordinate, i.e.,

$$\mathbf{F} = T(\mathbf{p} - C^T\boldsymbol{\lambda}) = T(N - C^T LR)\mathbf{b} + TC^T LRS\mathbf{M}^n\boldsymbol{\delta} \tag{46}$$

In (46), the multiple occurrences of the interval load $\mathbf{b}$ and interval material parameter $\boldsymbol{\delta}$ has been minimized, and a very sharp results for element force response are obtained.

4. Examples

The present interval-based finite element method is illustrated by numerical solutions for three problems with stiffness and load uncertainty.

Table I. Solutions for displacements of two-bay truss

	$\underline{\mathbf{v}}_2$(m)	$\overline{\mathbf{v}}_2$(m)	$\underline{\mathbf{u}}_4$(m)	$\overline{\mathbf{u}}_4$(m)	$\underline{\mathbf{v}}_4$(m)	$\overline{\mathbf{v}}_4$(m)
Comb$\times 10^{-5}$	-21.0342	-18.8416	3.7029	4.2043	-1.04833	-0.92828
Present$\times 10^{-5}$	-21.0429	-18.822	3.6942	4.2075	-1.04886	-0.92657
Naïve$\times 10^{-5}$	-22.7616	-17.1033	3.2221	4.6796	-1.16246	-0.81297
Present error	0.04%	0.10%	0.23%	0.08%	0.05%	0.18%
Naïve error	8.21%	9.23%	12.98%	11.30%	10.89%	12.42%

Table II. Solutions for axial forces of two-bay truss [compression(−)]

	$\underline{\mathbf{N}}_2$(kN)	$\overline{\mathbf{N}}_2$(kN)	$\underline{\mathbf{N}}_4$(kN)	$\overline{\mathbf{N}}_4$(kN)	$\underline{\mathbf{N}}_4$(kN)	$\overline{\mathbf{N}}_4$(kN)
Comb	-8.3470	-7.4613	11.4479	12.7533	-14.2587	-12.7992
Present	-8.3513	-7.4522	11.4390	12.7576	-14.2635	-12.7891
Naïve	-9.691	-6.127	-10.336	34.542	-15.910	-11.164
Present error	0.05%	0.12%	0.08%	0.03%	0.03%	0.08%
Naïve error	16.10%	17.88%	190.29%	170.85%	11.58%	12.78%

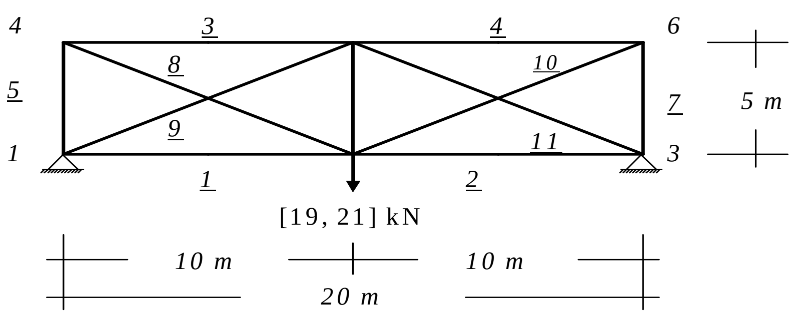

Figure 3. Two-bay truss

The first example is a two-bay truss as shown in Fig. 3. The truss is subjected to a concentrated load, applied at the middle lower joint. The variation in the loading is 10% of the midpoint value, and the used interval value is [19, 21]kN. Each element has a cross-sectional area $A_i = 0.01$ m^2, and an uncertain modulus of elasticity $\mathbf{E}_i = [199, 201]$ GPa, $i = 1, \ldots, 11$. The modulus of elasticity of each element are assumed to be varied independently.

The results for displacements and element forces are given in Table I and Table II, respectively. The present approach captured the bounds of the system response with errors within a range of 0.03% to 0.23%. However, the naïve IFEM overestimated the bounds of displacements by a range of 8.21% to 12.98%, and the errors escalated to as big as 190% in element force calculation.

Table III. Solutions for displacements of two-bay two-floor frame

	$\underline{\mathbf{v}}_4$(m)	$\overline{\mathbf{v}}_4$(m)	$\underline{\mathbf{v}}_9$(m)	$\overline{\mathbf{v}}_9$(m)	$\underline{\boldsymbol{\theta}}_9$(rad)	$\overline{\boldsymbol{\theta}}_9$(rad)
Comb$\times 10^{-6}$	−6.7640	−6.1548	−13.0697	−11.9207	5.6331	6.2691
Present$\times 10^{-6}$	−6.7660	−6.1485	−13.0760	−11.9076	5.6219	6.2767
Naïve$\times 10^{-6}$	−8.7042	−4.2104	−15.3237	−9.6599	3.7305	8.1681
Present error	0.03%	0.10%	0.05%	0.11%	0.20%	0.12%
Naïve error	28.68%	31.59%	17.25%	18.97%	33.78%	30.29%

Table IV. Solutions for axial forces (**N**), shear forces (**V**) and bending moment (**M**) of column 1 in two-bay two-floor frame

	$\underline{\mathbf{N}}_1$(kN)	$\overline{\mathbf{N}}_1$(kN)	$\underline{\mathbf{V}}_1$(kN)	$\overline{\mathbf{V}}_1$(kN)	$\underline{\mathbf{M}}_1$(kN·m)	$\overline{\mathbf{M}}_1$(kN·m)
Comb	−149.676	−137.3503	5.2608	5.8790	−14.1977	−12.5250
Present	−149.721	−137.2694	5.2408	5.8941	−14.2345	−12.4775
Naïve	−194.393	−93.097	−30.381	41.572	−83.892	57.047
Present error	0.03%	0.06%	0.38%	0.26%	0.26%	0.38%
Naïve error	29.88%	32.22%	677.50%	607.13%	490.89%	555.47%

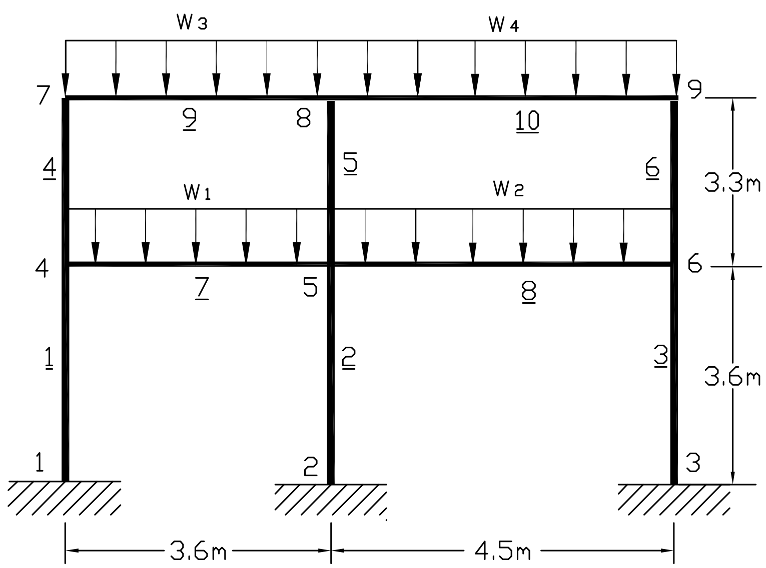

Figure 4. Two-bay two-floor frame

The second example is a two-bay two-floor frame as shown in Fig. 4. The columns have cross-sectional area $A_i = 0.4\text{m}^2$, moment of inertia $I_i = 0.036\text{m}^4$, interval modulus

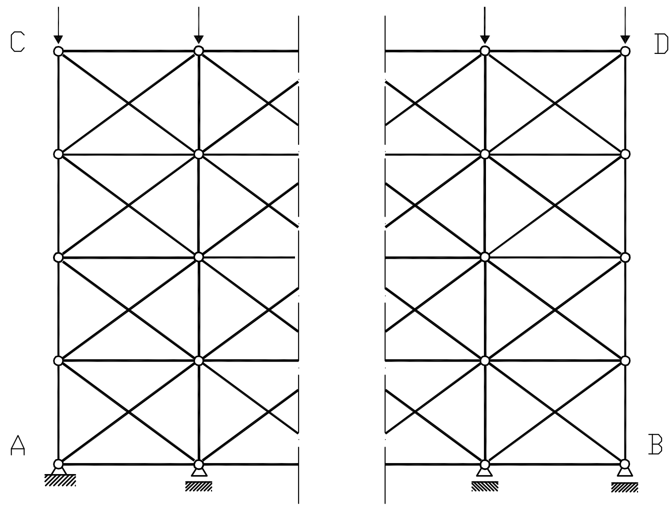

Figure 5. Large scale truss

Table V. Solutions for displacement (corner D) of twenty-bay truss

	$\underline{\mathbf{u}}$(m)	$\overline{\mathbf{u}}$(m)	$\underline{\mathbf{v}}$(m)	$\overline{\mathbf{v}}$(m)
Pownuk solution$\times10^{-5}$	7.54868	7.84538	-5.82393	-5.65726
Present solution$\times10^{-5}$	7.50621	7.88574	-5.84312	-5.63686

of elasticity $\mathbf{E}_i = [199, 201]$GPa, $i = 1, \ldots, 6$. The beams have cross-sectional area $A_i = 0.6\text{m}^2$, moment of inertia $I_i = 0.08\text{m}^4$, interval modulus of elasticity $\mathbf{E}_i = [199, 201]$GPa, $i = 7, \ldots, 10$. The frame is loaded by uniform loads $\mathbf{w}_i$ ($i = 1, 2, 3, 4$). Each load has 8% uncertainty, and the following data were used: $\mathbf{w}_1 = [24,\ 26]$kN/m, $\mathbf{w}_2 = [24,\ 26]$kN/m, $\mathbf{w}_3 = [48,\ 52]$kN/m, $\mathbf{w}_4 = [48,\ 52]$ kN/m. All the uncertain quantities are varied independently.

The results for displacements of selected nodes are given in Table III. The shear force, axial force and bending moment (at node 4) of column 1 is listed in Table IV. The present algorithm leads to sharp bounds of the exact solution of displacements and element forces, with errors within a range of 0.03% to 0.38%. Whereas, the naïve IFEM solution overestimates the bounds of element forces by 30% to 677%, it could not even get the correct sign for some terms.

To investigate problem size effect on the present formulation, a series of large-scale truss problems were analyzed. The configuration of the structures is shown in Fig. 5. Each element has 1% uncertain modulus of elasticity $\mathbf{E}_i = [2.0895, 2.1105]$GPa, and 1% uncertain cross-sectional area $\mathbf{A}_i = [0.0024875, 0.0025125]\text{m}^2$. Assume all interval parameters are varied independently. Table V lists the displacement results for a 20 bay truss (648 interval parameters). In this example the naïve method failed to converge and the combinatorial method is computationally prohibitive due to the large number of interval parameters.

Table VI. Truss problems with interval parameters

number of interval parameters	iteration number	iteration time (sec)	total computation time (sec)	variation in typical displacement*
246	5	0.172	1.04	2.24%
392	5	0.453	3.97	2.47%
648	6	1.484	15.05	2.67%
890	7	3.704	40.69	2.92%
1192	7	8.031	95.8	3.23%
1452	8	14.329	171.7	3.38%
1932	8	26.078	381.5	3.79%

*defined as ratio of radius to midpoint value
(horizontal displacement at corner D of the truss)

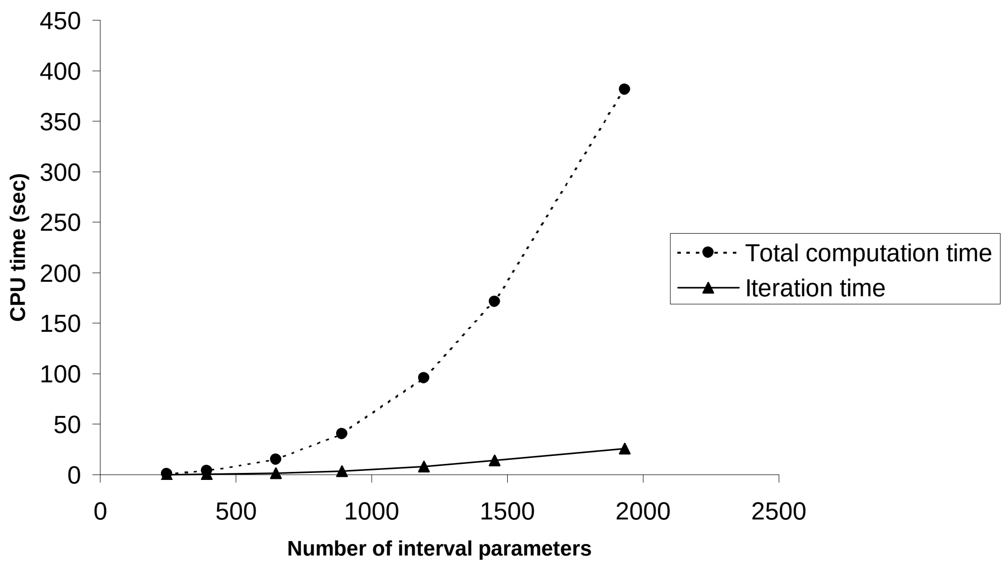

Figure 6. Computation time vs problem scale

Pownuk sensitivity analysis method [36] was used as an approximate solution to compare our results with. However, this sensitivity approach is based on the monotonicity assumption and does not provide a solution enclosure, but a good narrower estimate can be obtained when uncertainty is small enough.

Table VI lists the problem size, required number of iterations, iteration CPU time and total computational CPU time. The ratio of the radius of a typical displacement (the horizontal displacement at corner D) to its midpoint value is also listed in Table VI.

The computations were carried out on a PC with Intel Pentium4 2.4GHz CPU with 1GB RAM. The calculations show that the sharpness of the results maintains the same level despite of the increase of the problem size. Fig. 6 shows the relationship of problem size vs iteration CPU time, and problem size vs total computational CPU time. It can be seen the computational time does not increase exponentially with the increase of the problem size. In the current stage, most computation time is spent on calculating the preconditioning matrix R. It is important to note that system (29) is very sparse, and we expect a major computation time reduction when the sparsity is fully exploited. This will be a future work.

5. Conclusion

In this paper a new interval finite element formulation is presented. Uncertain loads and stiffness are introduced as interval numbers. The major difficulty associated with the IFEM is the overestimation due to dependency effect: the computed range of the response is much wider than the actual range. For engineering application, the physical nature of the problem must be considered to control the overestimation. In the present approach an element-by-element technique is used and the compatibility conditions are ensured by the Lagrange multiplier method. The resulting linear interval equation is solved using the Brouwer's fixed point theory with Krawczyk's operator and a newly developed overestimation control. The numerical examples show the naïve interval FEM produces meaningless wide results. The present approach, however, eliminates most sources of overestimation and a very sharp enclosure for the system's displacement and forces are obtained simultaneously and have the same level of accuracy. The numerical examples also illustrated the present formulation's scalability.

References

1. Akpan, U. O., T. S. Koko, I. R. Orisamolu, and B. K. Gallant: 2001, 'Practical fuzzy finite element analysis of structures'. *Finite Elem. Anal Des* **38**, 93–111.
2. Alefeld, G. and J. Herzberger: 1983, *Introduction to interval computations*. New York: Academic Press.
3. Apostolatos, N. and U. Kulisch: 1968, 'Grundzüge einer Intervallrechtung für Matrizen und einige Anwwendungen'. *Elektron. Rechenanlagen* **10**, 73–83. (in German).
4. Bathe, K.: 1996, *Finite element procedures*. Upper Saddle River, NJ: Prentice-Hall.
5. Ben-Haim, Y. and F. Elishakoff: 1990, *Convex models of uncertainty in applied mechanics*. Amsterdam: Elsevier Science.
6. Chen, S. H., H. D. Lian, and X. W. Yang: 2002, 'Interval static displacement analysis for structures with interval parameters'. *Int. J Numer. Methods Engrg.* **53**, 393–407.
7. Cook, R. D., D. S. Malkus, and M. E. Plesha: 1989, *Concepts and applications of finite element analysis*. John Wiley & Sons.
8. de Cooman, G., D. Ruan, and E. E. Kerre (eds.): 1995, *Foundations and applications of possibility theory*. Singapore: World Scientific.
9. Dempster, A. P.: 1967, 'Upper and lower probabilities induced by a multi-valued mapping'. *Ann. Mat. Stat.* **38**, 325–339.

10. Dessombz, O., F. Thouverez, J.-P. Laîné, and L. Jézéquel: 2001, 'Analysis of mechanical systems using interval computations applied to finite elements methods'. *J. Sound. Vib.* **238**(5), 949–968.
11. Ferson, S. and L. R. Ginzburg: 1996, 'Different methods are needed to propagate ignorance and variability'. *Reliab. Engng. Syst. Saf.* **54**, 133–144.
12. Ferson, S., V. kreinovich, L. Ginzburg, D. S. Myers, and K. Sentz: 2003, 'Constructing probability boxes and Dempster-Shafer structures'. Technical Report SAND2002-4015, Sandia National Laboratories.
13. Gallagher, R. H.: 1975, *Finite element analysis fundamentals*. Englewood Cliffs, N.J.: Printice Hall.
14. Ganzerli, S. and C. P. Pantelides: 1999, 'Load and resistance convex models for optimum design'. *struct. Optim.* **17**, 259–268.
15. Gay, D. M.: 1982, 'Solving interval linear equations'. *SIAM J. Numer. Anal.* **19**(4), 858–870.
16. Hansen, E.: 1965, 'Interval arithmetic in matrix computation'. *SIAM J. Numer. Anal.* **I**(2), 308–320.
17. Hansen, E.: 1992, *Global optimization using interval analysis*. New York: Marcel Dekker.
18. Jansson, C.: 1991, 'Interval linear system with symmetric matrices, skew-symmetric matrices, and dependencies in the right hand side'. *Computing* **46**, 265–274.
19. Kendall, D. G.: 1974, *Foundations of a theory of random sets*, pp. 322–376. New York: Wiley.
20. Koyluoglu, U., S. Cakmak, N. Ahmet, and R. K. Soren: 1995, 'Interval algebra to deal with pattern loading and structural uncertainty'. *J. Engrg. Mech.* **121**(11), 1149–1157.
21. Mayer, O.: 1970, 'Algebraische und metrische Strukturen in der Intervallrechung und eingine Anwendungen'. *Computing* **5**, 144–162. (in German).
22. McWilliam, S.: 2000, 'Anti-optimisation of uncertain structures using interval analysis'. *Comput. Struct.* **79**, 421–430.
23. Melchers, R. E.: 1999, *Structural reliability analysis and prediction*. West Sussex, England: John Wiley & Sons, 2 edition.
24. Möller, B., W. Graf, and M. Beer: 2000, 'Fuzzy structural analysis using level-optimization'. *Comput. Mech.* **26**(6), 547–565.
25. Moore, R. E.: 1966, *Interval Analysis*. Englewood Cliffs, N. J.: Prentice-Hall, Inc.
26. Moore, R. E.: 1979, *Methods and applications of interval analysis*. Philadelphia: SIAM.
27. Muhanna, R. L. and R. L. Mullen: 1995, 'Development of interval based methods for fuzziness in continuum mechanics'. In: *Proc. ISUMA-NAFIPS'95*. pp. 23–45.
28. Muhanna, R. L. and R. L. Mullen: 1999, 'Formulation of fuzzy finite element methods for mechanics problems'. *Compu.-Aided Civ. Infrastruct. Engrg.* **14**, 107–117.
29. Muhanna, R. L. and R. L. Mullen: 2001, 'Uncertainty in mechanics problems-interval-based approach'. *J. Engrg. Mech.* **127**(6), 557–566.
30. Mullen, R. L. and R. L. Muhanna: 1996, 'Structural analysis with fuzzy-based load uncertainty'. In: *Proc. 7th ASCE EMD/STD Joint Spec. Conf. on Probabilistic Mech. and Struct. Reliability*. Mass., pp. 310–313.
31. Mullen, R. L. and R. L. Muhanna: 1999, 'Bounds of structural response for all possible loadings'. *J. Struct. Engrg., ASCE* **125**(1), 98–106.
32. Neumaier, A.: 1987, 'Overestimation in linear interval equations'. *SIAM J. Numer. Anal.* **24**(1), 207–214.
33. Neumaier, A.: 1989, 'Rigorous sensitivity analysis for parameter-dependent systems of equations'. *J. Math. Anal. Appl.* **144**, 14–25.
34. Neumaier, A.: 1990, *Interval methods for systems of equations*. Cambridge University Press.
35. Pantelides, C. P. and S. Ganzerli: 2001, 'Comparison of fuzzy set and convex model theories in structural design'. *Mech. Systems Signal Process.* **15**(3), 499–511.
36. Pownuk: 2004, 'Calculation of the extreme values of displacements in truss structures with interval parameters'. http://s212.bud.polsl.gliwice.pl/ andrzej/php/apdl2interval/apdl2interval_ init.php.
37. Rao, S. S. and L. Berke: 1997, 'Analysis of uncertain structural systems using interval analysis'. *AIAA J.* **35**(4), 727–735.
38. Rao, S. S. and L. Chen: 1998, 'Numerical solution of fuzzy linear equations in engineering analysis'. *Int. J. Numer. Meth. Engng.* **43**, 391–408.

39. Rohn, J.: 1995, 'Linear interval equations: computing sufficiently accurate enclosures is NP-Hard'. Technical Report 621, Institute of computer science, Academy of Sciences of the Czech Republic.
40. Rump, S. M.: 1983, 'Solving algebraic problems with high accuracy'. In: U. Kulisch and W. Miranker (eds.): *A new approach to scientific computation*. New York: Academic Press.
41. Rump, S. M.: 1992, 'On the solution of interval linear systems'. *Computing* **47**, 337–353.
42. Rump, S. M.: 2001, 'Self-validating methods'. *Linear Algebra Appl.* **324**, 3–13.
43. Sentz, K. and S. Ferson: 2002, 'combination of evidence in Dempster-Shafer theory'. Technical Report SAND2002-0835, Sandia National Laboratories.
44. Shafer, G.: 1976, *A mathematical theory of evidence*. Princeton,N.J.: Princeton University Press.
45. Sun microsystems: 2002, 'Interval arithmetic in high performance technical computing'. Sun microsystems. (A White Paper).
46. Walley, P.: 1991, *statistical reasoning with imprecise probabilities*. London: Chapman and Hall.
47. Zadeh, L. A.: 1978, 'Fuzzy Sets as a Basis for a Theory of Possibility'. *Fuzzy Sets and Systems* **1**, 3–28.