Efficient Method of Solution of Large Scale Engineering Probelms with Interval Parameters

ANDRZEJ POWNUK
Department of Civil Engineering, Silesian University of Technology,Gliwice, Poland,
pownuk@zeus.polsl.gliwice.pl, http://zeus.polsl.gliwice.pl/∼pownuk

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Abstract. In this paper sensitivity analysis method [4] and first order Taylor expansion method will be applied to solution of finite element equations of truss structures and non-stationary diffusion equation with interval parameters. Only linear-elastic model of truss structures is considered.

In order to calculate the interval solution (i.e. displacement vector $\mathbf{u}$) it is necessary to calculate derivative $\frac{\partial \mathbf{u}}{\partial \mathbf{h}}$. According to many numerical experiments and some theoretical results it is convenient to assume that in some engineering applications the function $\mathbf{u}=\mathbf{u}(\mathbf{h})$ is monotone. Under such assumption it is possible to predict how to calculate the extreme solutions. Presented method gives quite accurate, however only approximate results.

Monotonicity assumption is not always true. Because of that the results are not always exact. The function $\mathbf{u}=\mathbf{u}(\mathbf{h})$ is highly nonlinear, because of that presented algorithm is better than first order Taylor expansion. On the following web page [1] it is possible to compare presented algorithm, the exact results and the first order Taylor expansion using appropriate web applications.

Keywords: uncertainty, interval equations, truss structures

1. Introduction to interval FEM

Many engineering problems can be described by parameter dependent system of equations in the following form [5]:

$$\mathbf{K}\left(\mathbf{h}\right)\mathbf{u} = \mathbf{Q}(\mathbf{h}) \tag{1}$$

where $\mathbf{K} \in R^{n\times n}$, $\mathbf{Q} \in R^n$, $\mathbf{u} \in R^n$, $\mathbf{h} \in R^m$. $\mathbf{h}$ is a vector of parameters of the structures (i.e. material characteristics, geometric characteristics, loads and other externals fields such as temperature. Very often we do not know the exact values of the parameters of the structure. Usually, if we do not know the exact values of the parameter h_i it is possible to estimate an upper and lower bound such that:

$$h_i^- \le h_i \le h_i^+ \quad \text{for} \quad i = 1, \ldots, m \tag{2}$$

in general we can write:

$$\mathbf{h} \in \hat{\mathbf{h}} \subset R^m \tag{3}$$

where $\hat{\mathbf{h}} = \left[h_1^-, h_1^+\right] \times \left[h_2^-, h_2^+\right] \times ... \times [h_m^-, h_m^+]$. Presented method can be applied, when it is not possible to obtain probabilistic characteristic of the structure.

Exact solution set of the equation (1) is very complicated and can be defined in the following way:

$$\mathbf{u}\left(\hat{\mathbf{h}}\right) = \{\mathbf{h} : \mathbf{K}\left(\mathbf{h}\right)\mathbf{u} = \mathbf{Q}(\mathbf{h}), \mathbf{h} \in \hat{\mathbf{h}}\} \tag{4}$$

Due to high complexity of the set $\mathbf{u}\left(\hat{\mathbf{h}}\right)$ in applications we can only find the smallest interval $\hat{\mathbf{u}}\left(\hat{\mathbf{h}}\right)$ which contains the set $\mathbf{u}\left(\hat{\mathbf{h}}\right)$ i.e.

$$\hat{\mathbf{u}}\left(\hat{\mathbf{h}}\right) = hull\ \mathbf{u}\left(\hat{\mathbf{h}}\right) \tag{5}$$

One can call the set $\hat{\mathbf{u}}\left(\hat{\mathbf{h}}\right)$ the interval solution.

Now some selected methods of finding the interval solution will be presented.

2. Endpoints combinations method

According to many numerical examples [4, 2] very the endpoint combination method

$$u_i^- = min\{\ u_i\left(h_1^{\alpha_1}, h_2^{\alpha_2}, ..., h_m^{\alpha_m}\right) : \alpha_1, ..., \alpha_m \in \{-,+\}\} \tag{6}$$

$$u_i^+ = max\{\ u_i\left(h_1^{\alpha_1}, h_2^{\alpha_2}, ..., h_m^{\alpha_m}\right) : \alpha_1, ..., \alpha_m \in \{-,+\}\} \tag{7}$$

give very good approximation of the solution, particularly when the intervals are relatively narrow. In some cases the results are exact (for example in the case of system of linear interval equations).

3. First order Taylor expansion method

We can approximate the value of the nonlinear function $\mathbf{u} = \mathbf{u}\left(\mathbf{h}\right)$ by using first order Taylor expansion method:

$$\mathbf{u}\left(\mathbf{h}\right) \approx \mathbf{u}_L\left(\mathbf{h}\right) = \mathbf{u}\left(\mathbf{h}_0\right) + \frac{\partial \mathbf{u}\left(\mathbf{h}_0\right)}{\partial \mathbf{h}}\left(\mathbf{h} - \mathbf{h}_0\right) \tag{8}$$

$\mathbf{h}_0$is a mid point of the interval vector $\hat{\mathbf{h}}$ (i.e. $\mathbf{h}_0 = mid\left(\hat{\mathbf{h}}\right)$).

The vector $\mathbf{u}\left(\mathbf{h}_0\right)$ is a solution of the following system of linear equations:

$$\mathbf{K}\left(\mathbf{h}_0\right)\mathbf{u}\left(\mathbf{h}_0\right) = \mathbf{Q}\left(\mathbf{h}_0\right) \tag{9}$$

If one calculate derivative of the equation (9) it is possible to get the matrix $\frac{\partial \mathbf{u}(\mathbf{h}_0)}{\partial \mathbf{h}}$

$$\mathbf{K}\left(\mathbf{h}_0\right)\frac{\partial \mathbf{u}\left(\mathbf{h}_0\right)}{\partial h_i} = \frac{\partial \mathbf{Q}\left(\mathbf{h}_0\right)}{\partial h_i} - \frac{\partial \mathbf{K}\left(\mathbf{h}_0\right)}{\partial h_i}\mathbf{u}\left(\mathbf{h}_0\right), \quad i = 1, \ldots, m, \tag{10}$$

when parameters $\mathbf{h}$ belong to the interval vector$\hat{\mathbf{h}}$, then the extreme values of the function $\mathbf{u}_L(\mathbf{h})$ can be calculated by using natural interval extension [3]

$$\hat{\mathbf{u}}_L\left(\hat{\mathbf{h}}\right) = \mathbf{u}\left(\mathbf{h}_0\right) + \frac{\partial \mathbf{u}\left(\mathbf{h}_0\right)}{\partial \mathbf{h}}\left(\hat{\mathbf{h}} - \mathbf{h}_0\right). \tag{11}$$

In calculation we can calculate upper and lower bounds in the following way:

$$u_{i,L}^{-}\left(\hat{\mathbf{h}}\right) = u_i\left(\mathbf{h}_0\right) - \sum_{\alpha=1}^{m}\left|\frac{\partial u_i\left(\mathbf{h}_0\right)}{\partial h_\alpha}\right|(h_\alpha^{+} - h_{\alpha,0}), \tag{12}$$

$$u_{i,L}^{+}\left(\hat{\mathbf{h}}\right) = u_i\left(\mathbf{h}_0\right) + \sum_{\alpha=1}^{m}\left|\frac{\partial u_i\left(\mathbf{h}_0\right)}{\partial h_\alpha}\right|(h_\alpha^{+} - h_{\alpha,0}). \tag{13}$$

4. High order Taylor expansion method

In order to find better approximation of the exact solution one can apply the following algorithm.

Algorithm 1

1) Approximate the value of the function $\mathbf{u}$=$\mathbf{u}(\mathbf{h})$ by high order Taylor expansion $\mathbf{u} = \mathbf{u}_{ap}\left(\mathbf{h}\right)$.

2) Find the points $\mathbf{h}_1^*, \mathbf{h}_2^*, ..., \mathbf{h}_p^*$ where the approximate function $\mathbf{u}_{ap}\left(\mathbf{h}\right)$ has extreme values.

3) Calculate the values of the function $\mathbf{u}_{ap}\left(\mathbf{h}\right)$ in the points $\mathbf{h}_1^*, \mathbf{h}_2^*, ..., \mathbf{h}_p^*$.

4) Calculate the interval solution.

$$u_i^{-} = min\left\{\ u_{ap,i}\left(\mathbf{h}_1^*\right), u_{ap,i}\left(\mathbf{h}_2^*\right), ..., u_{ap,i}\left(\mathbf{h}_p^*\right)\right\}, i = 1, \ldots, n, \tag{14}$$

$$u_i^{+} = max\left\{\ u_{ap,i}\left(\mathbf{h}_1^*\right), u_{ap,i}\left(\mathbf{h}_2^*\right), ..., u_{ap,i}\left(\mathbf{h}_p^*\right)\right\}, i = 1, \ldots, n. \tag{15}$$

In general, instead of Taylor expansion it is possible to apply any approximation method.

5. Monotonicity assumption and first order sensitivity analysis method

In engineering applications the function $\mathbf{u}$=$\mathbf{u}(\mathbf{h})$ is very often monotone (usually for narrow intervals $\hat{h}_1, \hat{h}_2, ..., \hat{h}_m$). In that case it is possible to calculate the extreme values of the displacements u_i by using appropriate endpoints of the intervals $\hat{h}_1, \hat{h}_2, ..., \hat{h}_m$. The algorithm of calculation is the following:

Algorithm 2

1) Calculate the mid point solution of the equation (9).
2) Calculate the derivatives of the solution in the mid point from the equation (4).
3) Calculate the sign vectors $\mathbf{S}_1, ..., \mathbf{S}_n$.

$$\mathbf{S}_i = \left[\; sign\left(\frac{\partial u_i(\mathbf{h}_0)}{\partial h_1}\right) \; sign\left(\frac{\partial u_i(\mathbf{h}_0)}{\partial h_2}\right) \; ... \; sign\left(\frac{\partial u_i(\mathbf{h}_0)}{\partial h_m}\right) \; \right]. \tag{16}$$

4) Calculate the independent sign vectors.
Two sign vectors $\mathbf{S}_i, \mathbf{S}_j$ are independent if

$$\mathbf{S}_i \neq \mathbf{S}_j, \quad \mathbf{S}_i \neq (-1) \cdot \mathbf{S}_j, \tag{17}$$

Independent sign vectors will be denoted as $\mathbf{S}_1^*, ..., \mathbf{S}_{n^*}^*$, where n^* is a number of independent sign vectors.

5) Calculate the extreme values of the solution for all independent sign vectors.

$$\mathbf{u}_i^{-,*} = \mathbf{u}\left(\mathbf{H}^-\left(\mathbf{S}_i^*, \hat{\mathbf{h}}\right)\right), \quad \mathbf{u}_i^{+,*} = \mathbf{u}\left(\mathbf{H}^+\left(\mathbf{S}_i^*, \hat{\mathbf{h}}\right)\right), \tag{18}$$

where

$$H_j^-\left(\mathbf{S}_i^*, \hat{\mathbf{h}}\right) = \begin{cases} h_j^-, \text{ if } S_{i,j}^* \geq 0 \\ h_j^+, \text{ if } S_{i,j}^* < 0 \end{cases}, \quad i = 1, \ldots, n, \tag{19}$$

$$H_j^+\left(\mathbf{S}_i^*, \hat{\mathbf{h}}\right) = \begin{cases} h_j^+, \text{ if } S_{i,j}^* \geq 0 \\ h_j^-, \text{ if } S_{i,j}^* < 0 \end{cases}, \quad i = 1, \ldots, n. \tag{20}$$

6) Calculate the interval solution $\hat{\mathbf{u}}$.

$$u_i^- = min\left\{ \; u_i\left(\mathbf{h}_1^-\right), ..., u_i\left(\mathbf{h}_{n^*}^-\right), u_i\left(\mathbf{h}_1^+\right), ..., u_i\left(\mathbf{h}_{n^*}^+\right)\right\}, \tag{21}$$

$$u_i^+ = max\left\{ \; u_i\left(\mathbf{h}_1^-\right), ..., u_i\left(\mathbf{h}_{n^*}^-\right), u_i\left(\mathbf{h}_1^+\right), ..., u_i\left(\mathbf{h}_{n^*}^+\right)\right\}, \tag{22}$$

where

$$\mathbf{h}_i^- = \mathbf{H}^-\left(\mathbf{S}_i^*, \hat{\mathbf{h}}\right), \quad \mathbf{h}_i^+ = \mathbf{H}^+\left(\mathbf{S}_i^*, \hat{\mathbf{h}}\right). \tag{23}$$

6. High order sensitivity analysis method

The first order sensitivity analysis method is based on the following general algorithm.

Algorithm 3

1) Approximate the value of the function by first order Taylor expansion.
2) Find the points $\mathbf{h}_1^*, \mathbf{h}_2^*, ..., \mathbf{h}_p^*$ where the Taylor expansion has extreme values.
3) Calculate the values of the exact solution in the points $\mathbf{h}_1^*, \mathbf{h}_2^*, ..., \mathbf{h}_p^*$.

4) Calculate the interval solution.

$$u_i^- = min\left\{\ u_i\left(\mathbf{h}_1^*\right), u_i\left(\mathbf{h}_2^*\right), ..., u_i\left(\mathbf{h}_p^*\right)\right\}, \quad i = 1, \ldots, n, \tag{24}$$

$$u_i^+ = max\left\{\ u_i\left(\mathbf{h}_1^*\right), u_i\left(\mathbf{h}_2^*\right), ..., u_i\left(\mathbf{h}_p^*\right)\right\}, \quad i = 1, \ldots, n. \tag{25}$$

In order to get better approximation of the exact solution one can approximate the function by high order Taylor expansion (or other approximation method).

Algorithm 4

1) Approximate the value of the function $\mathbf{u}$=$\mathbf{u}(\mathbf{h})$ by high order Taylor expansion.

2) Find the points $\mathbf{h}_1^*, \mathbf{h}_2^*, ..., \mathbf{h}_p^*$ where the approximate function (i.e. Taylor expansion) has extreme values.

3) Calculate the values of the exact solution in the points $\mathbf{h}_1^*, \mathbf{h}_2^*, ..., \mathbf{h}_p^*$.

4) Calculate the interval solution.

$$u_i^- = min\left\{\ u_i\left(\mathbf{h}_1^*\right), u_i\left(\mathbf{h}_2^*\right), ..., u_i\left(\mathbf{h}_p^*\right)\right\}, \tag{26}$$

$$u_i^+ = max\left\{\ u_i\left(\mathbf{h}_1^*\right), u_i\left(\mathbf{h}_2^*\right), ..., u_i\left(\mathbf{h}_p^*\right)\right\}. \tag{27}$$

In general, instead of Taylor expansion it is possible to apply any approximation method.

7. Comparison of First order Taylor expansion method and second order monotonicity test.

Let us consider a truss structure which is shown on Fig. 1.

In calculation the following numerical data was assumed P = 10 [kN], L = 1 [m], E = 210 [GPa] (Young modulus), A = 0.0025 [m^2] (area of cross-section).

Accuracy of the first order sensitivity analysis method and first order Taylor expansion method is expressed by using the following numbers:

$$du_{\rm i}^- = \frac{\left(u_i^{mid} - u_i^-\right) - \left(u_i^{mid} - u_{i,exact}^-\right)}{u_i^{mid} - u_{i,exact}^-} \cdot 100\% \tag{28}$$

$$du_{\rm i}^+ = \frac{\left(u_i^+ - u_i^{mid}\right) - \left(u_{i,exact}^+ - u_i^{mid}\right)}{u_{i,exact}^+ - u_i^{mid}} \cdot 100\% \tag{29}$$

where u_i^{mid} is a mid point solution, $u_{i,exact}^+$ is an exact value of upper bound, $u_{i,exact}^-$ is an exact value of upper bound.

Time of calculations is shown below. Calculation was done using computer with AMD Athlon XP 2600 with 512 MB RAM under RedHat Linux 9.0.

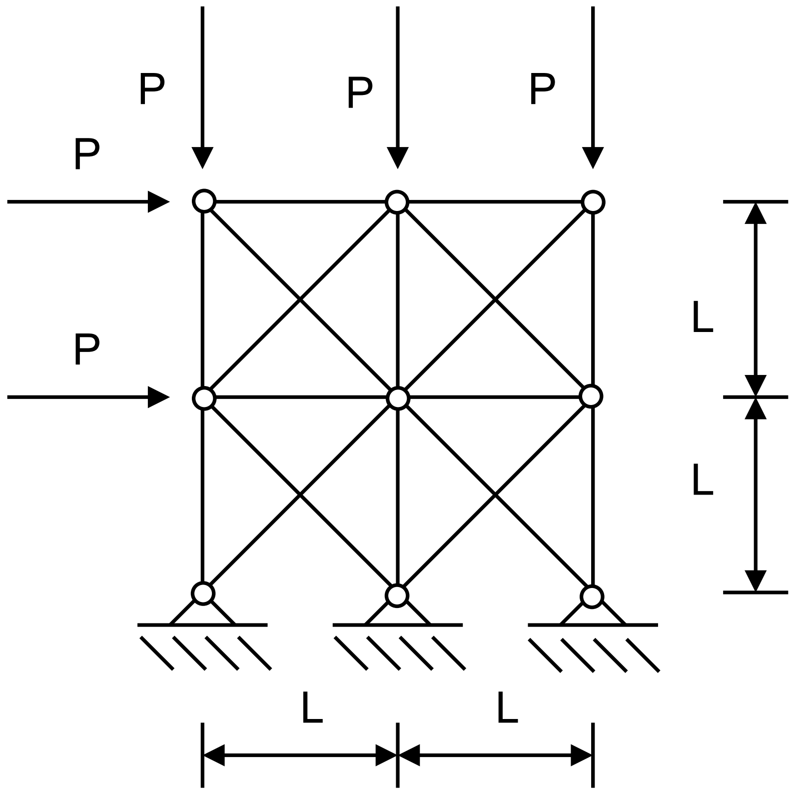

Figure 1. Truss structure

8. Interval solution of non-stationary diffusion equation

Let us consider non-stationary diffusion equation with interval parameters

$$\frac{\partial}{\partial x}\left(\beta\frac{k_x A_x}{\mu B}\frac{\partial p}{\partial x}\right)\Delta x+\frac{\partial}{\partial y}\left(\beta\frac{k_y A_y}{\mu B}\frac{\partial p}{\partial y}\right)\Delta y=\frac{V_b}{\alpha_c}\frac{\partial}{\partial t}\left(\frac{\phi}{B}\right) \tag{30}$$

where $k_x, A_x, k_y, A_y, \beta, \mu, B, \phi$ are some (interval) parameters, p is the pressure of oil, t is the time.

In order to get interval valued pressure of oil first order sensitivity analysis method was applied. The algorithm was implemented in C++ language using Borland C++ Builder compiler.

The program is able to take into account dependency of the parameters in different regions. The examples of the regions with different independent parameters are shown on Fig. (2).

Graphical representation of the interval solution (interval peruse of oil) for 7-th time step is shown in the Fig. (3).

Table I. Uncertainty of E and A 5%

Sensitivity		Taylor	
du_i^-	du_i^+	du_i^-	du_i^+
0.00	-0.50	5.34	-5.50
0.28	-0.37	5.03	-5.14
-1.31	-0.07	2.64	-4.36
-2.72	-2.82	-6.49	0.76
1.45	0.67	6.36	-4.31
1.11	-1.43	-3.42	3.10
-1.12	-0.34	3.15	-4.72
-0.19	0.00	-0.60	-0.47
-0.43	0.00	3.41	-4.52
0.21	0.00	-3.95	3.95
1.36	0.00	5.78	-4.61
-1.12	0.00	-5.74	4.57

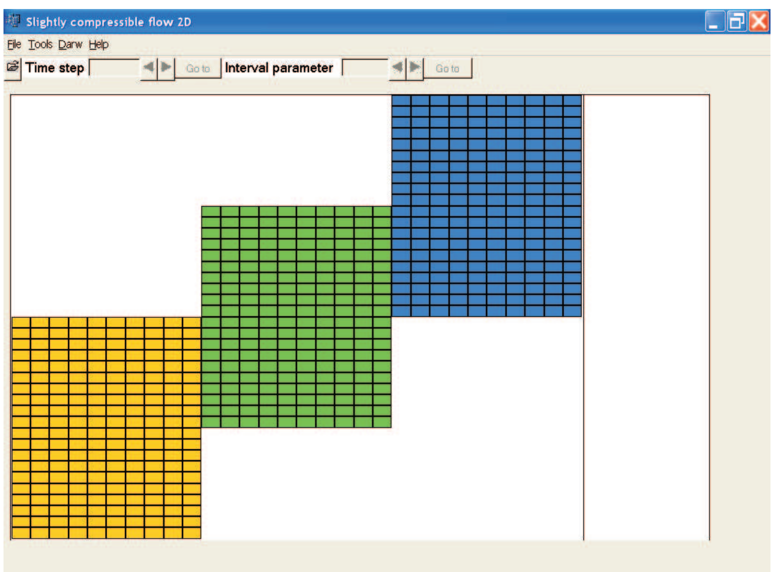

Figure 2. Graphical representation of dependences

Table II. Uncertainty of E and A 20%

Sensitivity		Taylor	
du_i^-	du_i^+	du_i^-	du_i^+
0.00	-1.37	18.67	-20.32
0.00	0.00	18.35	-19.30
-0.39	-0.08	14.88	-17.56
0.00	-0.03	-16.16	13.61
-0.18	-0.18	18.84	-19.80
0.00	0.00	-18.48	17.32
0.00	-0.76	16.06	-18.63
-0.34	-5.24	-4.57	-7.82
-1.66	-1.13	14.22	-19.03
-0.06	-1.94	-17.27	12.69
-0.31	-0.93	16.35	-19.22
-0.48	0.00	-19.07	17.91

In presented numerical example there were 10 interval parameters and 10 time steps. In presented example the system of parameter dependent equations (1) was generated by using finite difference method.

9. Conclusions

For truss structure:

1) Endpoint combination method gives exact results.

2) Endpoint combination method is very inefficient.

3) First order sensitivity analysis method is much more accurate than first order Taylor expansion method, particularly for large intervals.

4) Taylor expansion method is much more efficient than sensitivity analysis method.

5) The results of Taylor expansion method are acceptable for small intervals.

For diffusion equation example:

1) Sensitivity analysis method can be applied to solution of complex engineering problem.

Table III. Uncertainty of E and A 50%

Sensitivity		Taylor	
du_i^-	du_i^+	du_i^-	du_i^+
-0.03	-1.19	43.01	-48.34
-37.10	-0.39	-11.27	-46.95
-1.53	-0.24	28.41	-44.04
-0.25	-4.30	-41.91	21.75
-0.29	-0.28	43.11	-47.35
-0.33	-0.04	-45.43	38.26
0.00	-1.97	31.88	-45.78
-13.59	-15.68	-32.33	-30.86
0.00	-1.21	25.84	-46.34
-0.34	-7.88	-43.72	19.25
-1.68	-2.03	29.28	-46.56
-1.70	0.00	-46.31	40.78

Table IV. Endpoint combination method

Number of interval parameters	Time of calculations in seconds
10	0.02
15	1.86
18	17.18
20	124.69

2) It is using first order sensitivity analysis method possible to take into account different dependency of the parameters.

3) Parameter dependent system of equations (1) can be created using finite difference method.

Table V. First order sensitivity analysis

Number of interval parameters	Time of calculations in seconds
105	2
410	452
915	15208
1620	149554
2525	833782

Table VI. First order Taylor expansion

Number of interval parameters	Time of calculations in seconds
68	0.01
105	0.02
410	1.22
915	16.64
1314	50.04

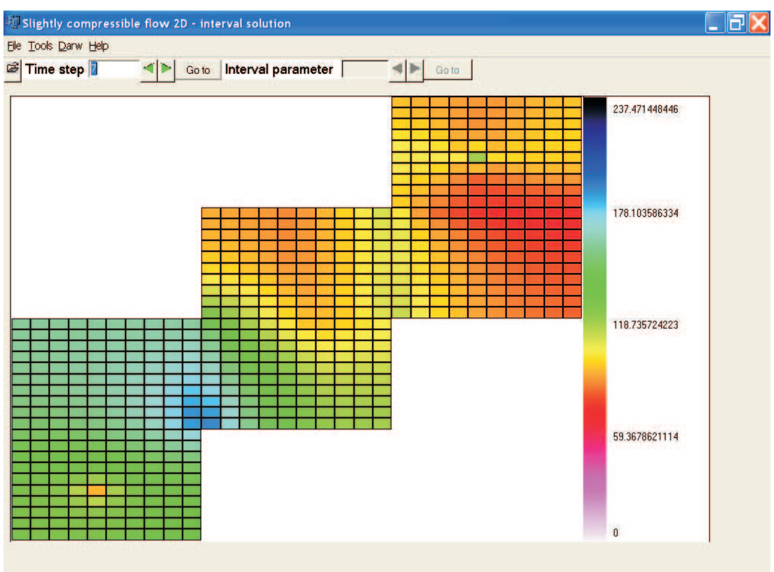

Figure 3. Interval pressure

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