A Search Algorithm for Calculating Validated Reliability Bounds

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Abstract. The search algorithm presented allows the CDF of a dependent variable to be bounded with 100% confidence, and allows for a guaranteed evaluation of the error involved. These reliability bounds are often enough to make decisions, and require a minimal number of function calls. The procedure is not intrusive, i.e. it can be equally applied when the function is a complex computer model (black box). The proposed procedure can handle input information consisting of probabilistic, interval-valued, set-valued, or random-set-valued information, as well as any combination thereof. The function as well as the joint pdf of the input variables can be of any type.

1. Introduction

Determining validated bounds for the Cumulative Distribution Function (CDF) of a function of random variables has attracted the attention of many scholars and a recent literature review may be found in [8]. Moore [24] and Moore [22] were probably the first ones to use interval analysis [23] to this end.

For example, Berleant and co-workers developed Statool [4-8], a computer program for obtaining bounds on the distributions of sums, products, and various other functions of random variables where the dependency relationship of the random variables need not be specified. Ferson [13] developed RiskCalc with similar capabilities. Independently, Lodwick and Jamison [19] presented a method for estimating and validating the cumulative distribution of a function of random variables (independent or dependent).

Dubois and Prade [12] firstly indicated how Random Set Theory might be used to bound the Cumulative Distribution Function (CDF) of a sum of two random variables. Tonon *et al.* [32] and Tonon [31] generalized this idea to a provide verified bounds to the CDF of a general function $y = f(u)$ where u is a generic random vector. Random Set Theory allowed the abovementioned procedures developed by different authors to be put in a rigorous light. They also showed that their procedure can be used equally well when some components of u are described as random variables, some others as intervals or Cartesian products, and some others as random sets. Additionally, a procedure was introduced to calculate the CDF of a particular value, y^*, of y; this procedure is meant to be used in reliability analyses and yields verified bounds on the reliability of a system. The motivation behind this procedure is that these bounds are often enough to make a decision, do not suffer from the shortcomings of Monte Carlo methods [14, 31], and often requires far less function calls than Monte Carlo methods.

In this paper, the procedure to calculate the CDF of a particular value, y^*, is advanced by introducing a searching algorithm with the aim of reducing the number of function calls. This is accomplished in Section 3. Before doing that and in order to establish common terminology and connect with [31] , Section 2 briefly restates the general procedure for calculating bounds on the entire CDF of y.

Section 4 specializes the searching algorithm to reliability analyses and Section 5 presents an application to the reliability analysis of a beam.

2. The entire CDF of $y=f(u)$ must be calculated

Let $pro(u)$ be the joint probability mass function of a discrete vector of input parameters $u = (u_1,...,u_p)$. Without loss of generality, it is assumed that the i-th parameter, u_i, belongs to interval

I_i, which may be infinite or semi-infinite. Consequently, u is constrained within a p-dimensional box $D = I_1 \times,...,\times I_p$, where $\times$ indicates Cartesian product. The procedure proposed [31, 32] consists of the following three steps.

2.1 Step 1

Let $\{A_j, j = 1,...,N\}$ be a partition of D and set

$$m\left(A_j\right) = \sum_{u:u\in A_j} pro\left(u\right) \tag{1}$$

If u is not discrete but continuous, $pro(u)$ is a joint probability density function (PDF) and Equation (1) becomes

$$m\left(A_j\right) = \int_{A_j} pro\left(u\right)du \tag{2}$$

Since this procedure is based on Random Set Theory, subsets A_j are called focal elements [31, 32]).

2.2 Step 2

Calculate the image $f(A_j)$ of each set A_j through function f. In general, this problem can be solved by applying twice the techniques of global optimization (e.g. [17, 28, 29, 33]).

However, if one divides the i-th interval I_i into n_i subintervals, then D is partitioned into $N = \prod_{i=1}^{p} n_i$ p-dimensional boxes A_j obtained as Cartesian products of p intervals (one per variable). As a consequence, A_j has 2^p vertices, which we indicate as v_k, k=1,..., 2^p. In this case, each parameter u_i varies in an interval $L_i=[{}_L L_i, {}_R L_i]$, and the methods of Interval Analysis can be used to efficiently calculate $f(A_j)$. These methods are continuously improving, and the reader is referred to the web page (http://cs.utep.edu/interval-comp/main.html) as well as to the Journal of Reliable Computing for up-to-date information and references.

If the function f is the response of a linearly elastic structure to static loads, then one can use the interval finite element formulation developed by Muhanna and Mullen [25, 26] to efficiently calculate $f(A_j)$. Finally, if f is an eigenvalue of a linearly elastic structure one can use the procedure developed by Modares and Mullen [21] to efficiently calculate $f(A_j)$.

2.3 Step 3

Calculate the upper, $F_{y,upp}$, and lower, $F_{y,low}$, bounds on the cumulative distribution function (CDF) of y, F_y, as follows:

$$F_{y,low}\left(y\right) \le F_y\left(y\right) \le F_{y,upp}\left(y\right) \tag{3.a}$$

where

$$F_{y,upp}\left(y\right) = \sum_{A_j:y\ge\inf\left(f\left(A_j\right)\right)} m\left(A_j\right) \tag{3.b}$$

$$F_{y,low}\left(y\right) = \sum_{A_j:y\ge\sup\left(f\left(A_j\right)\right)} m\left(A_j\right) \tag{3.c}$$

The proposed procedure allows for an explicit evaluation of the error involved in the calculation of the whole cumulative distribution function, i.e.:

$$\mathrm{Err}_{\max} = \max_{y}\left(F_{y,upp}(y) - F_{y,low}(y)\right) \tag{4}$$

or the error at a particular value y^*:

$$\mathrm{Err}(y^*) = F_{y,upp}(y^*) - F_{y,low}(y^*) \tag{5}$$

3. Only the CDF of a particular value *y** must be calculated

Consider the case in which the cumulative probability of only a particular element of Y, say y^*, is of interest, as is the case in reliability analyses (see Section 4). In this case, it is advisable to start off with a coarse partition of D into subsets A_i. Let:

$$S_1 = \{A_i : \sup f(A_i) < y^*\}, \tag{6.a}$$

$$S_2 = \{A_i : \inf f(A_i) > y^*\} \tag{6.b}$$

$$S_3 = \{A_i : \inf f(A_i) < y^* < \sup f(A_i)\} \tag{6.c}$$

$$C_k = \bigcup_{A_i : A_i \in S_k} A_i \tag{6.d}$$

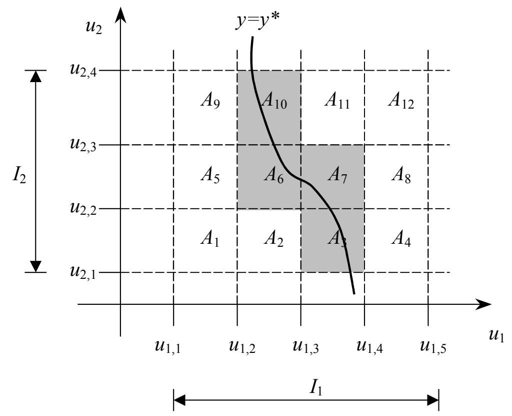

Figure 1. The image of focal elements A_3, A_6, A_7, and A_{10} straddle y^*.

Of course, $D = \bigcup_{k=1}^{3} C_k$.

To illustrate the procedure, consider a two-dimensional case ($p = 2$), the extension to larger dimensions being straightforward. It is assumed that the curve $f(u_1, u_2) = y^*$ intersect the boundary of D at two points (as is the case in most reliability analyses). Figure 1 illustrates an example, in which:

$$S_1 = \{A_1, A_2, A_5, A_9\}$$

$$S_2 = \{A_4, A_8, A_{11}, A_{12}\}$$

$$S_3 = \{A_3, A_6, A_7, A_{10}\}$$

As can be seen in Figure 1, elements in S_3 are those (and only those) focal elements intersected by the contour curve $f(u_1, u_2) = y^*$.

The procedure for bracketing $F_y(y^*)$ can be summarized as follows:

1) Determine the focal elements in S_3 as explained in Section 3.1 below.
2) Determine C_1 as explained in Section 3.2 below.
3) Calculate the bounds on the CDF of y^* as:

$$F_{y,low}(y^*) = \sum_{A_j \in S_1} m'(A_j) \leq F_y(y^*) \leq F_{y,upp}(y^*) = \sum_{\substack{A_j \in S_1 \\ A_j \in S_3}} m'(A_j) \tag{7}$$

where $m'(A_j)$ is given in Eqs. (1) or (2). It is to be noted that the weights $m'(A_j)$ do not have to be calculated for each focal element of partition D, but only for the focal elements in S_1 and S_3. Because the cumulative probability of y^* is in most cases small, the focal elements in S_1 and S_3 are less numerous than the focal elements in S_2. If the bounds in Eq. (7) are too large, the procedure in Section 3.3 below is followed.

3.1 Determining C_3

The determination of C_3 may be carried out in two parts:

1) Find a focal element belonging to S_3 along the boundary of D.
2) Find the remaining focal elements belonging to S_3.

Since region C_1 is in most cases smaller than region C_2, it is more efficient to start searching from the corner(s) belonging to C_1. Therefore, part one can be articulated into the following sequence:

1.1) WHILE $f(P) > y^*$, calculate function $f(P)$ at the corners P of D. Let $P^* = P$.

1.2) Consider the local numbering of points along the boundary and of focal elements as in Figure 2. Let j^* be the smallest j such that:
$f(u_{0,\mathrm{j}}) > y^*$.
IF such j^* does not exist, THEN GOTO point 1.3.
Calculate the image of A_{1,j^*} to check if it belongs to S_3.
IF A_{1,j^*} belongs to S_3:
THEN calculate the image of $A_{1,\mathrm{j}}$ for $j < j^*$ until $A_{1,\mathrm{j}} \in S_1$.
ELSE A_{1,j^*-1} belongs to S_3, and calculate the image of $A_{1,\mathrm{j}}$ for $j < j^*-1$ until $A_{1,\mathrm{j}} \in S_1$.
GOTO point 2.1.

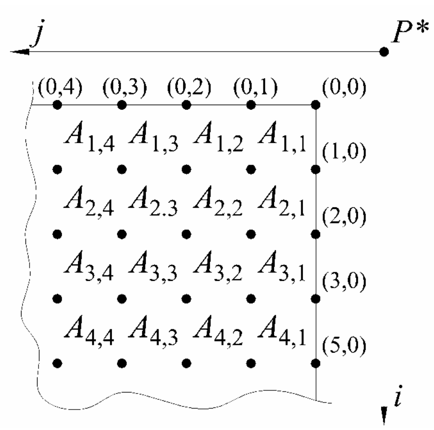

Figure 2. Local numbering of focal elements and points on the boundary of *D*.

1.3) If such j^* does not exist, then let i^* be the smallest i such that: $f(u_{i,0}) > y^*$.
Calculate the image of $A_{i^*,1}$ to check if it belongs to S_3.
IF $A_{i^*,1}$ belongs to S_3:
THEN calculate the image of $A_{i,1}$ for $i < i^*$ until $A_{i,1} \in S_1$.
ELSE $A_{i^*-1,1}$ belongs to S_3, and calculate the image of $A_{i,1}$ for $i < i^*-1$ until $A_{i,1} \in S_1$.
GOTO point 2.1.

If the focal element(s) determined in step 1 was (were) along a boundary edge parallel to the *x*-axis, then let $i = 1$, and the other focal elements in S_3 are found using the following procedure:

2.1) For the current i, let A_{i,j^*} be the focal element in S_3 with the largest j. Set $i = i+1$. Calculate the image of A_{i,j^*}.
IF $A_{i,j^*} \in S_3$,
THEN, calculate the image of $A_{i,j}$ for $j > j^*$ until $A_{i,j} \in S_2$;
calculate the image of $A_{i,j}$ for $j < j^*$ until $A_{i,j} \in S_1$.
ELSE, calculate the image of $A_{i,j}$ for $j < j^*$ until $A_{i,j} \in S_1$.
IF $i = n_2$,
THEN STOP
ELSE GOTO point 2.1.

If the curve $f(u_1, u_2) = y^*$ is known to be concave (resp. convex) toward P^*, then Point 2.1 simplifies as follows:

2.1) For the current i, let A_{i,j^*} be the focal element in S_3 with the largest j. Set $i = i+1$. Calculate the image of $A_{i,j}$ for $j \leq j^*$ (resp. $j \geq j^*$) until $A_{i,j} \in S_1$.
IF $i = n_2$,
THEN STOP
ELSE GOTO point 2.1.

If the focal element(s) determined in step 1 was (were) along a boundary edge parallel to the *y*-axis, then let $j = 1$, and the other focal elements in S_3 are found using the following procedure:

2.2) For the current j, let $A_{i^*,j}$ be the focal element in S_3 with the largest i. Set $j = j+1$. Calculate the image of $A_{i^*,j}$.
IF $A_{i^*,j} \in S_3$,
THEN, calculate the image of $A_{i,j}$ for $i > i^*$ until $A_{i,j} \in S_2$;

calculate the image of $A_{i,j}$ for $i < i^*$ until $A_{i,j} \in S_1$.
ELSE, calculate the image of $A_{i,j}$ for $i < i^*$ until $A_{i,j} \in S_1$.
IF $j = n_1$,
THEN STOP
ELSE GOTO point 2.2.

If the curve $f(u_1, u_2) = y^*$ is known to be concave (resp. convex) toward P^*, then Point 2.2 simplifies as follows:

2.2) For the current j, let $A_{i^*,j}$ be the focal element in S_3 with the largest i. Set $j = j + 1$. Calculate the image of $A_{i^*,j}$ for $i \le i^*$ (resp. $i \ge i^*$) until $A_{i,j} \in S_1$.
IF $j = n_1$,
THEN STOP
ELSE GOTO point 2.2.

3.2 Determining C_1

C_1 completely lies on one side of the curve $f(u_1, u_2) = y^*$, and consequently, of set C_3. C_3 will be on the side containing corner P^*, which was determined at Point 1.1.

3.3 Discretization refinement

If the bounds (7) on the CDF of y^* are too large, the discretization refinement is restricted to the focal elements in S_3. In fact, focal elements belonging to S_1 map to the left of y^* on the real line, and therefore do not need to be further discretized because their contribution to $F_y(y^*)$ is already known. Likewise, focal elements belonging to S_2 map to the right of y^* on the real line, and therefore do not need to be further discretized because they do not contribute to $F_y(y^*)$ altogether.

Additionally, it is useless to further discretize those focal elements in S_3 whose weight $m'(A_j)$ is very small as compared to the required precision on the CDF of y^* (e.g. $m'(A_j) = 10^{-10}$ if the required precision is 10^{-4}). Therefore, the number of focal elements that need to be further discretized is generally very small, which leads to drastic savings in the number of function calls.

Once the focal elements in S_3 have been further discretized into sub-elements, a procedure similar to that described in Section 3.1 can be used for determining the sub-elements belonging to S_3.

4. Reliability evaluation

Let $u=(u_1,...,u_p)$ be a vector of uncertain parameters that control the behavior of a given system. The safety of a system is quantified by the safety margin $z(u)$, such that [1, 2, 11, 16, 20]

- If $z<0$ the system is unsafe.
- If $z>0$ the system is safe.
- If $z=0$ the system is at a limit state condition.

The probability of failure of the system is defined as

$$\text{Pro}_{fail} = \text{Pro}(z < 0) \tag{8}$$

In general, a system is accepted if the probability of failure is smaller than a limit value

$$\text{Pro}_{fail} < \text{Pro}_{\text{lim}} \tag{9}$$

If the input u is known through its joint probability function, then the cumulative distribution function $F_z(z)$ of z can be calculated and

$$\text{Pro}_{fail} = F_z(0) \tag{10}$$

For complex systems, the calculation of $F_z(z)$ may very cumbersome, and a bracketing of the failure probability can be obtained using the procedures presented in Section 3, leading to Eq. (7), i.e.

$$F_{z,low}(0) \leq F_z(0) \leq F_{z,upp}(0) \tag{11}$$

where

- $F_{z,low}$ is the lower cumulative distribution function of z;
- $F_{z,upp}$ is the upper cumulative distribution function of z.

Three cases can be distinguished:

1) If $\text{Pro}_{\lim} < F_{z,low}(0)$, then the system is certainly unsafe, and it is not necessary to further increase the fineness of the discretization of the focal elements in S_3.
2) If $\text{Pro}_{\lim} > F_{z,upp}(0)$, then the system is certainly safe, and it is not necessary to further increase the fineness of the discretization of the focal elements in S_3.
3) If $F_{z,low}(0) < \text{Pro}_{\lim} < F_{z,upp}(0)$, then it is necessary to further increase the fineness of the discretization. As discussed in Section 3.3, one only needs to further discretize focal elements in set S_3.

As an alternative to the procedure proposed here, once the joint pdf of the uncertain input u has been discretized as described in Section 2, one can use the efficient approximation technique developed by Bae *et al.* [3] to calculate approximate reliability bounds. However, these bounds do not offer a guaranteed envelope because Bae's procedure uses the Multi-Point Approximation method to construct a surrogate for the original safety margin using the Two-Point Adaptive Non-linear Approximation [34] as a local approximation.

5. Numerical example

Consider a beam of length $l = 5$ m, fixed at one end, and subjected to a random concentrated load u_1 at the free end, and to a random distributed load u_2 along all its length. Let us assume $u_1 \sim N(10, 1)$ kN, and $u_2 \sim N(1, 0.3)$ kN/m, with a correlation coefficient of 0.5; a similar example is proposed by Ang and Tang ([1], Problem 4.18). The resistant moment at the fixed end is equal to $M = 90$ kN·m. The safety margin of the bending resistance at the fixed end reads

$$z = M - (5 \cdot u_1 + 12.5 \cdot u_2) \tag{12}$$

and it is a normal variate with mean equal to 27.5 kN·m, and standard deviation $\sigma = 7.603$ kN·m. This closed-form solution allows for a handy check of the results obtained with the proposed procedures because the exact probability of failure is $1.49 \cdot 10^{-4}$. Let us assume that the limit probability of failure is 10^{-5}, and that one wants to determine whether the beam is safe or not under the given random loads.

Let us use the domain D and the discretization shown in Figure 3. Following the procedure outlined in Section 3.2 (Point 1.1), function z is evaluated at the corners of D in clockwise order starting from $P = (-10, -1)$, until corner $P = (30, 3)$ yields $z(P) < 0$. We set $P^* = (30, 3)$, and (Point 1.2) by marching along $u_2 = 3$, we get $j^* = 4$ (absolute coordinates: (10.5, 3)). By calculating the images of focal elements $A_{1,j}$ with $j \leq j^* = 4$, it is found that $A_{1,4}$ and $A_{1,3}$ (absolute numbering A_{70} and A_{80} in Figure 3) belong to S_3, whereas $A_{1,2}$ (absolute numbering A_{90} in Figure 3) does not, and the search for focal elements of S_3 along the boundary is finished.

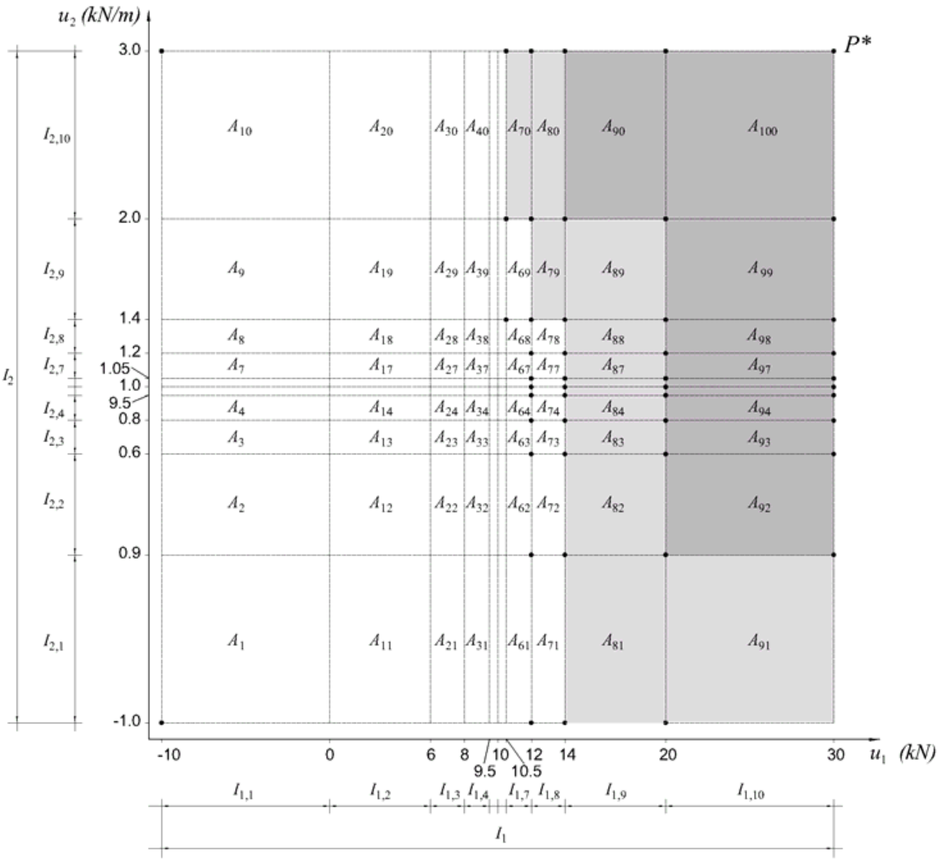

Figure 3. First discretization of set $D=I_1\times I_2$ into focal elements for the calculation of the system reliability. A dot marks the points at which function *f* must be evaluated. A light gray hatch identifies focal elements belonging to set S_3, whereas a dark gray hatch identifies focal elements belonging to set S_1.

Following Point 2.1, one gets $i = 1$, and $(i, j^*) = (1, 4)$. Let us set $i = 1+1 = 2$, and calculate the image of $A_{2,4}$; since $A_{2,4}$ belongs to S_2, the procedure does not calculate the images of $A_{2,j}$ for $j > j^*$ because all of these focal elements belong to S_2. The procedure calculates the images of $A_{2,j}$ for $j < j^*$ until $A_{2,j} \in S_1$, which occurs for $j = 1$, i.e. $A_{2,4} \in S_2$, $A_{2,3} \in S_3$, $A_{2,2} \in S_3$, $A_{2,1} \in S_1$. Since $i = 2 \neq n_2 = 10$, the procedure goes back to Point 2.1 with $i = 2$, and so on.

The results of the procedure are illustrated in Figure 3, in which a light gray hatch is used for the focal elements belonging to S_3, and a dark gray hatch is used for the focal elements belonging to S_1. Table 1 gives the weights $m'(A)$ for the elements of S_3. A total of 48 function evaluations were necessary to determine sets C_1, C_2, and C_3.

The calculated probability of failure is in the range

$$F_{z,low}(0) = 1.01\cdot 10^{-2} < F_z(0) = 1.49\cdot 10^{-4} < F_{z,upp}(0) = 2.57\cdot 10^{-6}$$

Table 1. Focal elements in set S_3.

Focal element (global numbering as in Figure 3)	Weight $m'(A_i)$
70	$2.24 \cdot 10^{-4}$
79	$9.66 \cdot 10^{-3}$
80	$1.72 \cdot 10^{-4}$
81	$6.29 \cdot 10^{-15}$
82	$1.21 \cdot 10^{-9}$
83	$2.14 \cdot 10^{-8}$
84	$1.19 \cdot 10^{-7}$
85	$1.02 \cdot 10^{-7}$
86	$1.62 \cdot 10^{-7}$
87	$1.13 \cdot 10^{-6}$
88	$4.35 \cdot 10^{-6}$
89	$2.32 \cdot 10^{-5}$
91	$1.57 \cdot 10^{-15}$

Since $F_{z,low}(0) < \text{Pro}_{\text{lim}} < F_{z,upp}(0)$, it is necessary to increase the fineness of the discretization (Section 3.3). The weights of the focal elements in Table 1, are negligible except for A_{70}, A_{79}, A_{80}, and A_{89}; therefore, only the latter four focal elements are further discretized into $5 \times 5 = 25$ sub-focal elements each as depicted in Figure 4. A light gray hatch is used for the focal elements belonging to S_3, and a dark gray hatch is used for the focal elements belonging to S_1.
The calculated probability of failure is in the range

$$F_{z,low}(0) = 4.51 \cdot 10^{-5} < F_z(0) = 1.49 \cdot 10^{-4} < F_{z,upp}(0) = 3.25 \cdot 10^{-4}$$

This range gives a guarantee that $\text{Pro}_{\text{lim}} < F_{z,low}(0)$, and therefore the beam is unsafe.

Additional 58 function evaluations were used in the discretization refinement. It is remarkable that only 48 + 58 = 106 function evaluations were necessary to perform a reliability analysis with 100% confidence, despite the very low value of the probability of failure.

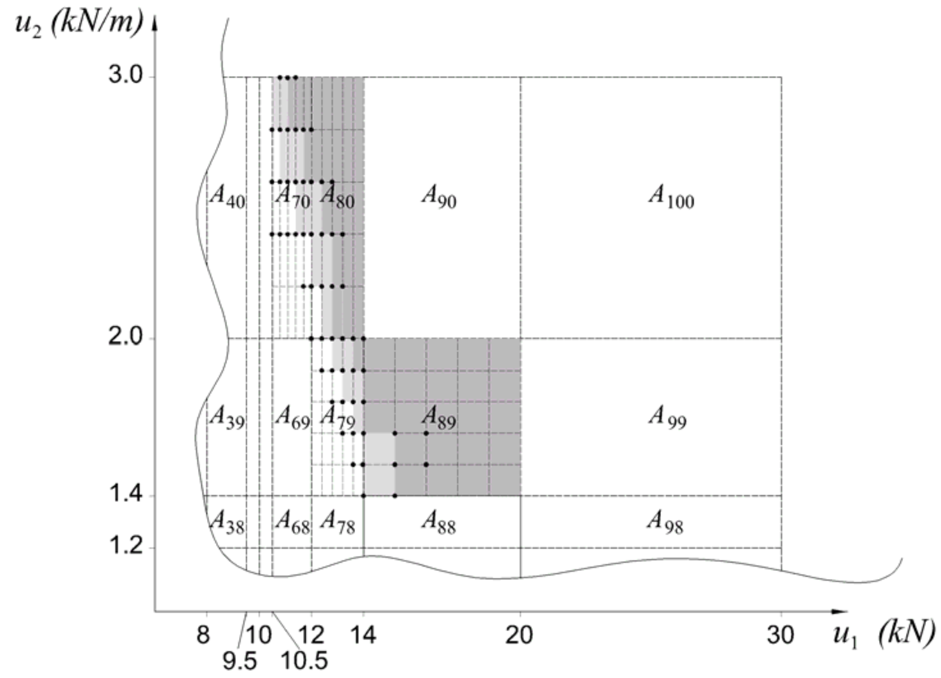

Figure 4. Second discretization of focal elements A_{70}, A_{70}, A_{79}, and A_{89}, into sub-focal elements for the refinement of the calculation of the system reliability. A dot marks the points at which function *f* must be evaluated. A light gray hatch identifies sub-focal elements belonging to set S_3, whereas a dark gray hatch identifies sub-focal elements belonging to set S_1.

On the other hand, the number of function evaluations necessary to achieve an error $e = F_{z,upp}(0) - F_{z,low}(0) = 2.8 \cdot 10^{-4}$ with confidence 1-δ with crude Monte Carlo is equal to [15]

$$n_C(e,\delta) = \frac{1}{4 \cdot \delta \cdot e^2}$$

Table 2 presents the number of Monte Carlo simulations for several values of the confidence level. It is evident that the procedure proposed leads to substantial computational savings. For example, if one requires that the reliability of reliability calculations be at least equal to the reliability of the structure being analyzed, then one should require a confidence level of 99.999%, which yields some 90 million function calls.

Table 2. Number of Monte Carlo simulations vs. confidence level.

Confidence level (%)	n_c
90	8,929
95	17,857
99	89,286
99.9	892,857
99.99	8,928,571
99.999	89,285,710

6. Conclusions

The searching procedure presented allows one to bound the CDF of a dependent variable with 100% confidence, and allows for a guaranteed evaluation of the error involved in the calculations. These bounds are often enough to make decisions, and require a minimal number of function calls. The procedure is not intrusive, i.e. it can be equally applied when the function is a complex computer model (black box). The proposed procedure can handle input information consisting of probabilistic, interval-valued, set-valued, or random-set-valued information, as well as any combination thereof. The function as well as the joint pdf of the input variables can be of any type.

The application to a beam subjected to two random loads showed that the number of function calls is drastically reduced as compared to Monte Carlo methods. For example, only 106 function calls were necessary to conclude with 100% confidence that the CDF was greater than the specified limit value of the probability of failure, and that the beam was unsafe.

The drawback of the procedure presented is that it suffers from a dimensionality effect. However, Monte Carlo methods are not free from dimensionality effects. For example, Davis' and Rabinowitz's comment as follows on multiple integration by sampling when more than 12 variables are involved ([10], page 417): "Sophisticated methods of variance reduction appear to exhibit a dimensional effect and are probably ruled out in this range. Some authors feel that the dimensional effect may even play a role in crude [sampling] methods inasmuch as it may occur in the constant in the asymptotic error term." Indeed, Sloan and Wozniakowski [30] have shown that Monte Carlo may depend polynomially or even exponentially on the number of variables.

Current research is focusing on improving the efficiency of the procedures presented here by incorporating Bayesian philosophies and procedures [9, 18, 27] into the method's algorithm, with the aim of reducing the random variables in the problem to those that appreciably influence the output. Furthermore, adaptive techniques are being investigated in order to locally refine the discretization of those focal elements whose image straddles y^*.

Current applications aim at integrating the procedure presented with interval finite element formulations [21, 25, 26] for the efficient reliability analysis of structures.

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