Non-Probabilistic Design Optimization with Insufficient Data using Possibility and Evidence Theories

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Abstract: Early in the engineering design cycle, it is difficult to quantify product reliability or compliance to performance targets due to insufficient data or information for modeling the uncertainties. Design decisions are therefore, based on fuzzy information that is vague, imprecise qualitative, linguistic or incomplete. The uncertain information is usually available as intervals with lower and upper limits. In this paper, the possibility and evidence theories are used to account for uncertainty in design with incomplete information. The formal theories to handle uncertainty are first introduced using the theoretical fundamentals of fuzzy measures. The first part of the paper highlights how the possibility theory can be used in design. A computationally efficient and accurate hybrid (global-local) optimization approach is used to calculate the confidence level of "fuzzy" response combining the advantages of the commonly used vertex and discretization methods. A possibility-based design optimization method is proposed where all design constraints are expressed possibilistically. It is shown that the method gives a conservative solution compared with all conventional reliability-based designs obtained with different probability distributions. Also, a general possibility-based design optimization method is presented which handles a combination of random and possibilistic design variables. The second part of the paper describes a design optimization method using evidence theory. The method can be used when limited and often conflicting, information is available from "expert" opinions. A computationally efficient design optimization formulation is presented, which can handle a mixture of epistemic and random uncertainties. It quickly identifies the vicinity of the optimal point and the active constraints by moving a hyper-ellipse in the original design space, using a reliability-based design optimization (RBDO) algorithm. Subsequently, a derivative-free optimizer calculates the evidence-based optimum, starting from the close-by RBDO optimum, considering only the identified active constraints. The computational cost is kept low by first moving to the vicinity of the optimum quickly and subsequently using local surrogate models of the active constraints only. Two numerical examples demonstrate the application of possibility and evidence theories in design and highlight the trade-offs among reliability-based, possibility-based and evidence-based designs.

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1. INTRODUCTION

Engineering design under uncertainty has recently gained a lot of attention. Uncertainties are usually modeled using probability theory. In Reliability-Based Design Optimization (RBDO), variations are represented by standard deviations which are typically assumed constant, and a mean performance is optimized subject to probabilistic constraints (Tu, Choi and Park, 1999; Liang, Mourelatos and Tu, 2004; Wu, Shin, Sues and Cesare, 2001; Lee, Yang and Ruy, 2002; Youn, Choi and Park, 2001). In general, probability theory is very effective when sufficient data is available to quantify uncertainty using probability distributions. However, when sufficient data is not available or there is lack of information due to ignorance, the classical probability methodology may not be appropriate. For example, during the early stages of product development, quantification of the product's reliability or compliance to performance targets is practically very difficult due to insufficient data for modeling the uncertainties. A similar problem exists when the reliability of a complex system is assessed in the presence of incomplete information on the variability of certain design variables, parameters, operating conditions, boundary conditions etc.

Uncertainties can be classified in two general types; aleatory (stochastic or random) and epistemic (subjective) (Oberkampf, Helton, and Sentz, 2001; Sentz and Ferson, 2002; Klir and Yuan, 1995; Klir and Filger, 1988; Yager, Fedrizzi and Kacprzyk, 1994) Aleatory or irreducible uncertainty is related to inherent variability and is efficiently modeled using probability theory. However, when data is scarce or there is lack of information, the probability theory is not useful because the needed probability distributions cannot be accurately constructed. In this case, epistemic uncertainty, which describes subjectivity, ignorance or lack of information, can be used. Epistemic uncertainty is also called reducible because it can be reduced with increased state of knowledge or collection of more data.

Formal theories to handle uncertainty have been proposed in the literature including evidence theory (or Dempster – Shafer theory) (Klir and Filger, 1988; Yager, Fedrizzi and Kacprzyk, 1994], possibility theory [Dubois and Prade, 1988) and interval analysis (Moore, 1966). Two large classes of fuzzy measures, called belief and plausibility measures, respectively, characterize the mathematical theory of evidence. They are mutually dual in the sense that one of them can be uniquely determined from the other. Evidence theory uses plausibility and belief (upper and lower bounds of probability) to measure the likelihood of events. When the plausibility and belief measures are equal, the general evidence theory reduces to the classical probability theory. Therefore, the classical probability theory is a special case of evidence theory.

Possibility theory handles epistemic uncertainty if there is no conflicting evidence among experts (Klir and Filger, 1988). It uses a special subclass of dual plausibility and belief measures, called possibility and necessity measures, respectively. In possibility theory, a fuzzy set approach is common, where membership functions characterize the input uncertainty (Zadeh, 1965). Even if a probability distribution is not available due to limited information, lower and upper bounds (intervals) on uncertain design variables are usually known. In this case, interval analysis (Moore 1966; Muhanna and Mullen, 2001; Mullen and Muhanna, 1999) and fuzzy set theory (Zadeh, 1965) have been extensively used to characterize and propagate input uncertainty in order to calculate the interval of the uncertain output. An efficient method for reliability estimation with a combination of random and interval variables is presented in (Penmetsa and Grandhi, 2002). However, it is not implemented in a design optimization framework. A few design optimization studies have been also reported, where some or all of the uncertain design variables are in interval form (Du and Sudjianto, 2003; Rao and Cao, 2002; Gu and batill, 1998).

Optimization with input ranges has also been studied under the term anti-optimization (Elishakoff, Haftka and Fang, 1994; Lombardi and Haftka, 1998). Anti-optimization is used to describe the task of finding the “worst-case” scenario for a given problem. It solves a two-level (usually nested) optimization problem. The outer level performs the design optimization while the inner level performs the anti-optimization. The latter seeks the worst condition under the interval uncertainty (Lombardi and Haftka, 1998). A decoupled approach is suggested in (Lombardi and Haftka, 1998) where the design optimization alternates with the anti-optimization rather than nesting the two. It was mentioned that this method takes longer to converge and may not even converge at all if there is strong coupling between the interval design variables and the rest of the design variables. A “worst-case” scenario approach using interval variables has also been considered in multidisciplinary systems design [(Gu and Batill, 1998; Du and Chen, 2000).

Very recently, possibility-based design algorithms have been proposed (Mourelatos and Zhou, 2005; Choi, Du and Youn, 2004) where a mean performance is optimized subject to possibilistic constraints. It was shown that more conservative results are obtained compared with the probability-based RBDO. A comprehensive comparison of probability and possibility theories is given in (Nikolaidis, Chen, Cudney, Haftka and Rosca, 2004) for design under uncertainty.

Evidence theory is more general than probability and possibility theories, even though the methodologies of uncertainty propagation are completely different (Oberkampf and Helton, 2002; Bae, Grandhi and Canfield, 2004). It can be used in design under uncertainty if limited, and even conflicting, information is provided from experts. Furthermore, the basic axioms of evidence theory allow to combine aleatory (random) and epistemic uncertainty in a straightforward way without any assumptions (Bae, Grandhi and Canfield, 2004). Evidence theory however, has been barely explored in engineering design. One of the reasons may be its high computational cost due mainly to the discontinuous nature of uncertainty quantification. Evidence-based methods have been only recently used to propagate epistemic uncertainty (Bae, Grandhi and Canfield, 2004; Bae, Grandhi and Canfield, 2004) in large-scale engineering systems. Although a computationally efficient method is proposed in (Bae, Grandhi and Canfield, 2004; Bae, Grandhi and Canfield, 2004], the design issue is not addressed. We are aware of only one study which propagates epistemic uncertainty using evidence theory and also performs a design optimization (Agarwal, Renaud, Preston and Padmanabhan, 2004). The optimum design is calculated for multidisciplinary systems under uncertainty using a trust region sequential approximate optimization method with surrogate models representing the uncertain measures as continuous functions.

In this paper, the possibility and evidence theories are used to account for uncertainty in design with incomplete information. The formal theories to handle uncertainty are first introduced using the theoretical fundamentals of fuzzy measures. The first part of the paper highlights how the possibility theory can be used in design. A computationally efficient and accurate hybrid (global-local) optimization approach is presented for calculating the confidence level of “fuzzy” response, combining the advantages of the commonly used vertex and discretization methods. A possibility-based design optimization method is subsequently described where all design constraints are expressed possibilistically. The method gives a conservative solution compared with all conventional reliability-based designs obtained with different probability distributions.

Also, a general possibility-based design optimization method is presented which handles a combination of random and possibilistic design variables.

In the second part of the paper, a computationally efficient design optimization method is proposed based on evidence theory, which can handle a mixture of epistemic and random uncertainties. The method can be used when limited and often conflicting, information is available from “expert” opinions. The algorithm quickly identifies the *vicinity* of the optimal point and the active constraints by moving a hyper-ellipse in the original design space, using an RBDO algorithm. Subsequently, a derivative-free optimizer calculates the evidence-based optimum, starting from the close-by RBDO optimum, considering only the identified active constraints. The computational cost is kept low by first moving to the vicinity of the optimum quickly and subsequently using local surrogate models of the active constraints only.

The paper is organized as follows. Section 2 gives an introduction to fuzzy measures. Section 3 describes the fundamentals of possibility theory based on fuzzy measures as well as some numerical methods for propagating non-probabilistic uncertainty, which are essential in possibility-based design. A detailed formulation of Possibility-Based Design Optimization (PBDO) where design constraints are satisfied possibilistically, is presented in section 4. Section 5 presents a detailed formulation of an Evidence-Based Design Optimization (EBDO) method and its implementation. All principles are demonstrated with examples in section 6. Results are compared among deterministic optimization, RBDO, PBDO and EBDO. Finally, a summary and conclusions are given in section 7.

2. FUZZY MEASURES

The evidence and possibility theories are based on the mathematical foundation of fuzzy measures which provide the foundation of fuzzy set theory. Before we introduce the basics of fuzzy measures, it is helpful to review the used notation on set representation. A universe X represents the entire collection of elements having the same characteristics. The individual elements in the universe X are denoted by x, which are usually called singletons. A set A is a collection of some elements of X. All possible sets of X constitute a special set called the power set $\wp(X)$.

A fuzzy measure is defined by a function $g: \wp(X) \rightarrow [0,1]$ which assigns to each crisp [Ross 1995] subset of X a number in the unit interval [0,1]. The assigned number in the unit interval for a subset $A \in \wp(X)$, denoted by g(A), represents the degree of available evidence or belief that a given element of X belongs to the subset A.

In order to qualify as a fuzzy measure, the function g must obey the following three axioms:

Axiom 1 (boundary conditions): $g(\varnothing)=0$ and $g(X)=1$.

Axiom 2 (monotonicity): For every $A, B \in \wp(X)$, if $A \subseteq B$, then $g(A) \leq g(B)$.

Axiom 3 (continuity): For every sequence ($A_i \in \wp(X)$, $i=1,2,\dots$) of subsets of $\wp(X)$, if either $A_1 \subseteq A_2 \subseteq \dots$ or $A_1 \supseteq A_2 \supseteq \dots$ (i.e., the sequence is monotonic), then $\lim_{i\to\infty} g(A_i) = g(\lim_{i\to\infty} A_i)$.

A belief measure is a function *Bel*: $\wp(X) \rightarrow [0,1]$ which satisfies the three axioms of fuzzy measures and the following additional axiom [9]:

$$Bel(A_1 \cup A_2) \geq Bel(A_1) + Bel(A_2) - Bel(A_1 \cap A_2)\,. \tag{1}$$

The axiom (1) can be expanded for more than two sets. For $A \in \wp(X)$, *Bel*(A) is interpreted as the degree of belief, based on available evidence, that a given element of X belongs to the set A.

A plausibility measure is a function

$$Pl : \wp(X) \Rightarrow [0,1] \tag{2}$$

which satisfies the three axioms of fuzzy measures and the following additional axiom [(Klir and Filger, 1988)

$$Pl(A_1 \cap A_2) \leq Pl(A_1) + Pl(A_2) - Pl(A_1 \cup A_2) \tag{3}$$

Every belief measure and its dual plausibility measure can be expressed with respect to the non-negative function

$$m : \wp(X) \Rightarrow [0,1] \tag{4}$$

such that $m(\varnothing) = 0$ and

$$\sum_{A \in \wp(X)} m(A) = 1 \quad . \tag{5}$$

The function m is called Basic Probability Assignment (BPA) due to the resemblance of Eq. (5) with a similar equation for probability distributions. The basic probability assignment *m*(A) is interpreted either as the degree of evidence supporting the claim that a specific element of X belongs to the set A or as the degree to which we believe that such a claim is warranted. At this point, it should be noted that the BPA is very different from the probability distribution function. Basic probability assignments are defined on sets of the power set (i.e., on $A \in \wp(X)$), whereas the probability distribution functions are defined on the singletons x of the power set (i.e., on $x \in \wp(X)$). Every set $A \in \wp(X)$ for which *m*(A)>0 is called a focal element of *m*. Focal elements are subsets of X on which the available evidence focuses; i.e. available evidence exists.

Given a BPA *m*, a belief measure and a plausibility measure are uniquely determined by

$$Bel(A) = \sum_{B \subseteq A} m(B) \tag{6}$$

and

$$Pl(A) = \sum_{B \cap A \neq 0} m(B)\,. \tag{7}$$

which are applicable for all $A \in \wp(X)$.

In Eq. (6), *Bel*(A) represents the total evidence or belief that the element belongs to A as well as to various subsets of A. The *Pl*(A) in Eq. (7) represents not only the total evidence or belief that the element in question belongs to set A or to any of its subsets but also the additional evidence or belief associated with sets that overlap with A. Therefore,

$$Pl(A) \geq Bel(A)\,. \tag{8}$$

Probability theory is a subset of evidence theory. When the additional axiom of belief measures (see Eq. (1)) is replaced with the stronger axiom

$$Bel(A \cup B) = Bel(A) + Bel(B) \text{ where } A \cap B = \varnothing\,, \tag{9}$$

we obtain a special type of belief measures which are the classical probability measures. In this case, the right hand sides of Eq. (6) and (7) become equal and therefore,

$$Bel(A) = Pl(A) = \sum_{x \in A} m(x) = \sum_{x \in A} p(x) \tag{10}$$

for all $A \in \wp(X)$, where *p(x)* is the classical probability distribution function (PDF). Note that the BPA *m(x)* is equal to *p(x)*. Therefore with evidence theory, we can simultaneously handle a mixture of input parameters. Some of the inputs can be described probabilistically (random uncertainty) and some can be described through expert opinions (epistemic uncertainty with incomplete data). In the second case, the range of each input parameter will be discretized using a finite number of intervals. The BPA value for each interval must be equal to the PDF area within the interval.

It should be noted that according to evidence theory, the *Bel(A)* and *Pl(A)* bracket the true probability *P(A)* [9], i.e.

$$Bel(A) \le P(A) \le Pl(A). \tag{11}$$

Evidence obtained from independent sources or experts must be combined. If the BPA's m_1 and m_2 express evidence from two experts, the combined evidence *m* can be calculated by the following Dempster's rule of combining (Sentz and Ferson, 2002)

$$m(A) = \frac{\sum_{B \cap C = A} m_1(B) m_2(C)}{1 - K} \quad \text{for} \quad A \neq 0 \tag{12}$$

where

$$K = \sum_{B \cap C = 0} m_1(B) m_2(C) \tag{13}$$

represents the *conflict* between the two independent experts. Dempster's rule filters out any conflict, or contradiction among the provided evidence, by normalizing with the complementary degree of conflict. It is usually appropriate for relatively small amounts of conflict where there is some consistency or sufficient agreement among the opinions of the experts. Yager (Yager, Fedrizzi and Kacprzyk, 2004) has proposed an alternative rule of combination where all degrees of contradiction are attributed to total ignorance. Other rules of combining can be found in (Sentz and Ferson, 2002).

The possibility theory is a subcase of the general evidence theory. It can be used to characterize epistemic uncertainty, when incomplete data is available. It applies only when there is *no conflict* in the provided body of evidence. In such a case, the focal elements of the body of evidence are nested and the associated belief and plausibility measures are called consonant. In contrary, when there is conflicting evidence, the belief and plausibility measures are dissonant. A family of subsets of the universal set is nested if they can be ordered in such a way that each is contained within the next. Thus, $A_1 \subset A_2 \subset \cdots \subset A_n$ are nested sets. Consonant belief and plausibility measures are usually known as necessity measures *n* and possibility measures π, respectively. Therefore, if there is no conflicting information, $n(A) = Bel(A)$ and $\pi(A) = Pl(A)$. The necessity and possibility are dual measures, related by

$$n(A) = 1 - \pi(\overline{A}). \tag{14}$$

where $\overline{A}$ is the complement of set *A*.

3. FUNDAMENTALS OF POSSIBILITY THEORY

This section highlights the fundamentals of possibility theory as it was originally introduced in the context of fuzzy set theory (Zadeh, 1978). In the fuzzy set approach to possibility theory, focal elements are represented by *a*-cuts of the associated fuzzy set. Focal elements are subsets that are assigned nonzero degrees of evidence. The possibility theory can be used to bracket the true probability based on the fuzzy set approach at various confidence intervals (*a*-cuts). The advantage of this is that as the design progresses and the confidence level on the input parameter bounds increases, the design need not be reevaluated to obtain the new bounds of the response.

Similarly to the probability measures, which are represented by the probability distribution functions, the possibility measures can be represented by the possibility distribution function $r : X \Rightarrow [0,1]$ such that

$$\pi(A) = \max_{x \in A} r(x). \qquad (15)$$

It can be shown that possibility measures are formally equivalent to fuzzy sets. In this equivalence, the membership grade of an element x corresponds to the plausibility of the singleton consisting of that x. Therefore, a consonant belief structure is equivalent to a fuzzy set of X.

A fuzzy set is an imprecisely defined set that does not have a crisp boundary. It provides instead, a gradual transition from “belonging” to “not belonging” to the set. A function can be defined such that the values assigned to the elements of the set are within a specified range and indicate the membership grade of these elements in the set. Larger values denote higher degrees of set membership. Such a function is called a membership function and the set defined by it a fuzzy set.

The membership function μ_A by which a fuzzy set A is usually defined has the form $\mu_A : X \to [0, 1]$ where [0, 1] denotes the interval of real numbers from 0 to 1, inclusive. Given a fuzzy subset A of X with membership function μ_A, Zadeh (Zadeh, 1978) defines a possibility distribution function *r* associated with A as numerically equal to μ_A, i.e. $r(x) = \mu_A(x)$ for all $x \in X$. Then, he defines the corresponding possibility measure π as

$$\pi(A) = \sup_{x \in A} r(x) \text{ for each } A \in \wp(X). \qquad (16)$$

Eq. (16) is equivalent to Eq. (15) when X is finite. In the fuzzy set approach to possibility theory, focal elements are represented by *a*-cuts of the associated fuzzy set. For the remaining of this discussion, we will follow the fuzzy set approach to possibility theory.

Eq. (11) states that the true probability is bracketed by the belief and plausibility measures. If we know the possibility distribution function $\mu_Y(y)$ of the response *Y*, then the true probability *P(Y)* can be also bracketed as

$$n(Y) \le P(Y) \le \pi(Y) \qquad (17)$$

where the necessity $n(Y)$ and possibility $\pi(Y)$ measures are calculated from Eqs (14) and (16), respectively. The “extension principle” (Klir and Filger, 1988; Yager, Fedrizzi and Kacprzyk, 1994; Ross, 1995) is used to calculate the possibility distribution function $\mu_Y(y)$ of the response.

3.1. FUZZIFICATION PROCESS AND EXTENSION PRINCIPLE

The process of quantifying a fuzzy variable is known as fuzzification. If any of the input variables is imprecise, it is considered fuzzy and must be therefore, fuzzified in order for the uncertainty to be propagated using fuzzy calculus. The fuzzification is done by constructing a possibility distribution, or membership function, for each imprecise (fuzzy) variable. Details can be found in (Ross, 1995). The membership function takes values in the [0,1] interval. Here, we use convex normal possibility distributions to characterize the fuzzy variables. An example of a convex normal triangular possibility distribution is shown in Fig. 1. The point for which the possibility is equal to one is called normal point. The possibility distribution is convex since it is strictly decreasing to the left and right of the normal point. At each confidence level, or *a*-cut, a set X_a is defined as

$$X_a = \{x : x_L^a \leq x \leq x_R^a, a \in [0,1]\}, \tag{18a}$$

which is a monotonically decreasing function of *a*; i.e.

$$a_1 > a_2 \Rightarrow X_{a_1} \subset X_{a_2} \text{ for every } a_1, a_2 \in [0,1]. \tag{18b}$$

Due to the convexity of the possibility distribution function, all sets generated at different *a*-cuts are nested according to Eq. (18b). Therefore, the convexity and normality of the possibility distribution function satisfies the basic requirement of nested sets (no conflicting evidence) in possibility theory.

After the fuzzification of the imprecise input variables, the “extension principle” is used to propagate the epistemic uncertainty through the transfer function in order to calculate the fuzzy response. The “extension principle” calculates the possibility distribution of the fuzzy response from the possibility distributions of the fuzzy input variables. In particular, given the transfer function $y = f(\underline{x})$, where the output y depends on the N independent fuzzy inputs $\underline{x} = \{x_1, \cdots, x_N\}$, the “extension principle” states that the possibility distribution μ_Y of the output is given by

$$\mu_Y[y = f(\underline{x})] = \sup_y \{\min_j [\mu_X(f(\underline{x}_j))]\} \tag{19}$$

where “sup” denotes the suprenum operator that gives the least upper bound. The above equation can be interpreted as follows. For a crisp value of the output *y*, there may exist more than one combination of crisp values of input variables $\underline{x}$ resulting in the same output.

The possibility of each combination is given by the smallest possibility value for all fuzzy input variables. The possibility that $y = f(\underline{x})$, is given by the maximum possibility for all these combinations. Note that in probability theory, the probability of an outcome is equal to the product of the probabilities of the constituent events. In fuzzy set theory however, the possibility of an outcome is equal to the minimum possibility of the constituent events.

If the outcome can be reached in many ways, then the outcome probability, in probability theory, is given by the sum of the probabilities of all the ways. In fuzzy theory, the possibility of the outcome is given by the maximum possibility of all the possibilities (Ross, 1995).

The direct ("brute force") solution of Eq. (19) is practically intractable except for simple cases involving one or two fuzzy variables. The computational effort increases exponentially with increasing number of fuzzy input variables. For this reason, approximate numerical techniques have been proposed, among which the discretization method (Akpan, Rushton and Koko, 2002) and the vertex method (Penmetsa and Grandhi, 2002) are the most popular ones.

In the discretization method, the domain of each fuzzy variable $i; 1 \leq i \leq N$ is discretized with M_i discrete values at each *a*-cut. Then the output *y* is evaluated at all possible combinations $\prod_{i=1}^{N} M_i$ for each *a*-cut. Subsequently, Eq. (19) is used to calculate the possibility distribution of the output. The range of the output is defined by the minimum and maximum response from all combinations. Although this method can be very accurate, the associated computational cost is practically prohibitive.

In the vertex method, all the binary combinations of only the extreme values of the fuzzy variables at an *a*-cut are fed into the deterministic transfer function. The bounds of the fuzzy response are then obtained at the *a*-cut, by choosing the maximum and minimum responses. The procedure is repeated for all *a*-cuts of interest. The method has the potential to give accurate

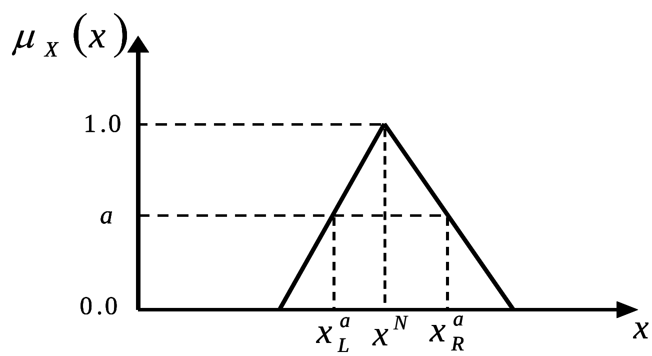


Figure 1. Triangular possibility distribution for a fuzzy variable.

bounds of the response based on the bounded input. However, when the transfer function exhibits minima or maxima within the domain defined by the extreme values of the input variables, the vertex method is inaccurate. This is due to the fact that the function is evaluated only at the binary combinations of the input variable bounds. For a problem with N fuzzy input variables, the required number of function evaluations for the vertex method is $A * 2^N$, where *A* is the number of *a*-cuts.

In general, the vertex method is computationally more efficient compared with the discretization method. However, the required computational effort grows exponentially with the number of input fuzzy variables (Ross, 1995). For this reason, most of the reported applications are restricted to very few fuzzy variables (Mullen and Muhanna, 1999; Chen and Rao, 1997; Rao and Sawyer, 1995).

A hybrid (global-local) optimization method has been reported in (Mourelatos and Zhou, 2005], which ensures computational efficiency without loss of accuracy. An optimization algorithm is used to calculate the minimum and maximum values of the response at each *a*-cut. Because the global minimum and maximum values of the response are needed, a derivative free, global optimizer called DIRECT (DIvisions of RECTangles), is used in order to avoid being trapped at a local optimum and obtain therefore, an inaccurate solution. DIRECT is a modification of the standard Lipschitzian approach that eliminates the need to specify a Lipschitz constant (Jones, Perttunen and Stuckman, 1993). Although global optimizers may get close to the global optimum quickly, it takes them longer to achieve a high degree of accuracy because they usually have a slow rate of convergence. This suggests that the best performance can be obtained by combining DIRECT with a gradient-based local optimizer in a hybrid approach. In this work, DIRECT is first used, followed by a local optimizer based on Sequential Quadratic Programming (SQP). DIRECT provides a converged global optimum based on "loose" convergence criteria. Subsequently, the DIRECT solution is used as starting point for SQP, which identifies the optimum accurately and efficiently.

3.2. A MATHEMATICAL EXAMPLE

The following two-variable, six-hump camel function (Wang, 2003) is used

$$y(x_1, x_2) = 4x_1^2 - 2.1x_1^4 + \frac{1}{3}x_1^6 + x_1x_2 - 4x_2^2 + 4x_2^4, \quad x_{1,2} \in [-2,2].$$

to illustrate the accuracy and efficiency of the hybrid optimization method of the previous section and compare it with the vertex and discretization methods. For demonstration reasons, the

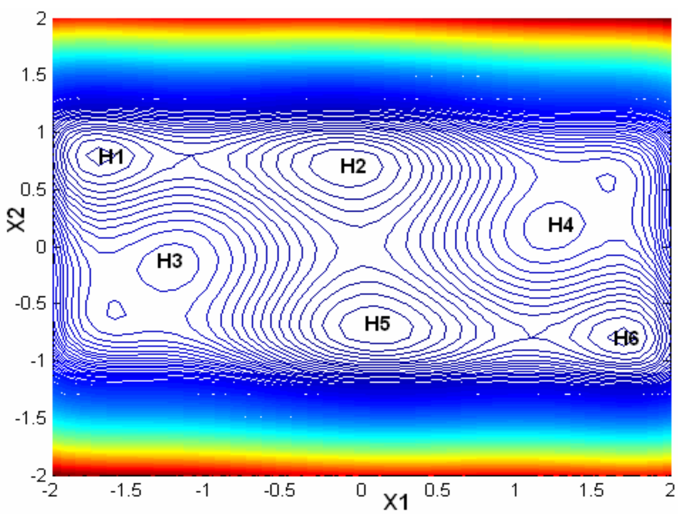


Figure 2. Contour plot for mathematical example.

following simple triangular membership functions are used for the two input variables x_1 and x_2

$$\mu_{X_i}(x_i) = \begin{cases} -\dfrac{x_i}{2}+1, & 0 \le x_i \le 2 \\ \dfrac{x_i}{2}+1, & -2 \le x_i \le 0 \end{cases} \qquad i = 1,2 \cdot$$

Fig. 2 shows the contour plot of the six hump camel function. The H's indicate all extreme points. Points H2 and H5 with coordinates (0.0898, -0.7127) and (-0.0898, 0.7127) respectively, are two global optima with an equal function value of y_{min}= -1.0316. The calculated membership

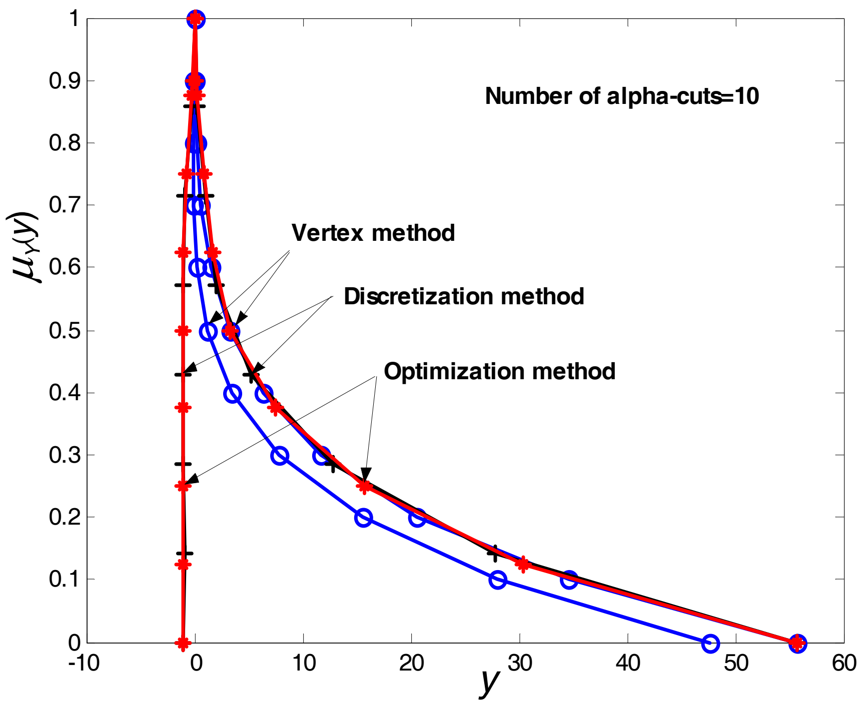


Figure 3. Response membership function for mathematical example.

functions of the response y using the vertex, discretization and hybrid optimization methods are plotted in Fig. 3. Ten *a*-cuts are used for all three methods. For the discretization method, the range of each input fuzzy variable, at each *a*-cut, is equally split in 15 divisions. It is known that if the input membership functions are convex normal, the response membership function must also be convex normal. The justification is that when the input uncertainty increases (low *a*-cut values), the uncertainty of the response must remain the same or increase. As shown in Fig. 3, the response membership function obtained by the vertex method is not convex and therefore, it is wrong.

As explained in section 3.1, the discretization method evaluates the function not only at the upper and lower limits of the input variables at each alpha cut but also between the bounds. Thus, it can capture the extreme points that might be present in between the upper and lower bounds. At each alpha cut, all combinations are obtained and the minimum and maximum response values are calculated in order to get the response membership function. It is clear that the response becomes more accurate as the number of divisions per alpha cut increases. As shown in Fig. 3, the response membership function calculated with the discretization method, is convex and

Table 1. Accuracy and efficiency comparison of vertex, discretization and hybrid optimization methods

	Vertex	**Discretization**	**Hybrid Optimization**
Lower Bound	47.73	-1.01	-1.03
Upper Bound	55.73	55.73	55.73
No of F.E.	4	256	140

normal. The uncertainty decreases as the level of confidence increases (increasing *a*-cut values). The major disadvantage of this method is that as the number of design variables increases and the number of divisions per *a*-cut also increases, the method becomes computationally very expensive. In this example, the number of *a*-cuts is 10 and the number of divisions per *a*-cut is 15. Therefore, the number of function evaluations is $10*(15+1)^2=2560$. The response membership function of the six hump camel function is also calculated using the proposed hybrid optimization method. The result is identical with that obtained with the discretization method (see Fig. 3).

Table 1 summarizes the lower and upper bound values of the response at the zero *a*-cut, as calculated by the vertex, discretization and hybrid optimization methods. The vertex method is very efficient but inaccurate. The hybrid optimization method however, has the same accuracy with the “brute force” discretization method but it is much more efficient.

4. POSSIBILITY-BASED DESIGN OPTIMIZATION

In deterministic design optimization, an objective function is minimized subject to satisfying a set of constraints. In Reliability-Based Design Optimization (RBDO), where all design variables are characterized probabilistically, an objective function is usually minimized subject to the probability of satisfying each constraint being greater than a specified high reliability level.

In this section, a methodology is presented on how to use possibility theory in design. We will show that the possibility-based design is conservative compared with all RBDO designs obtained with different probability distributions. In RBDO, some optimality is usually sacrificed in order to accommodate the random uncertainty. The possibility-based design sacrifices a little more optimality in order to accommodate the lack of probability distribution information. It therefore, encompasses all RDBO designs obtained with different distributions.

According to Eq. (11), the probability *P(A)* of event *A* is bracketed by the belief *Bel(A)* and plausibility *Pl(A)*; i.e. $Bel(A) \le P(A) \le Pl(A)$. We have also mentioned that for consonant (no conflicting evidence) belief structures, the plausibility measures are equal to the possibility measures, resulting in $\eta(A) \le P(A) \le \pi(A)$, where η and π are the necessity and possibility measures, respectively (see Eq. 17). This means that *the possibility* $\pi(A)$ *provides an upper bound to the probability P(A).* From the design point of view, we can thus conclude (Klir and

Filger, 1988; Ross, 1995; Zadeh, 1978) that *what is possible may not be probable*, and *what is impossible is also improbable*.

Note that for an impossible event *A*, the possibility $\pi(A)$ is zero. If we therefore, make sure that the possibility of violating a constraint is zero, then the probability of violating the same constraint will be also zero. If feasibility of a constraint *g* is expressed with the positive null form $g \geq 0$, the constraint is always satisfied if

$$\pi(g \leq 0) = 0 . \tag{20}$$

The possibility π in Eq. (20) is calculated using Eq. (16). Fig. 4 shows the membership function $\mu_G(g)$ of constraint g. The possibility of set $A = \{g : g_{\min} \leq g \leq g_{\min}^{\alpha}, \alpha \in [0,1]\}$ is $\pi(A) = \alpha$ and the possibility of set $B = \{g : g_{\min}^{\alpha} \leq g \leq g_{\max}^{\alpha}, \alpha \in [0,1]\}$ is $\pi(B) = 1$. Similarly, the possibility of constraint violation is $\pi(g \leq 0) = \alpha_1$. Eq. (20) can be relaxed as

$$\pi(g \leq 0) \leq \alpha \tag{21}$$

where the a-cut level is small; i.e. $\alpha << 1$. Based on Fig. 4, the relation (21) is satisfied if

$$g_{\min}^{\alpha} \geq 0 \tag{22}$$

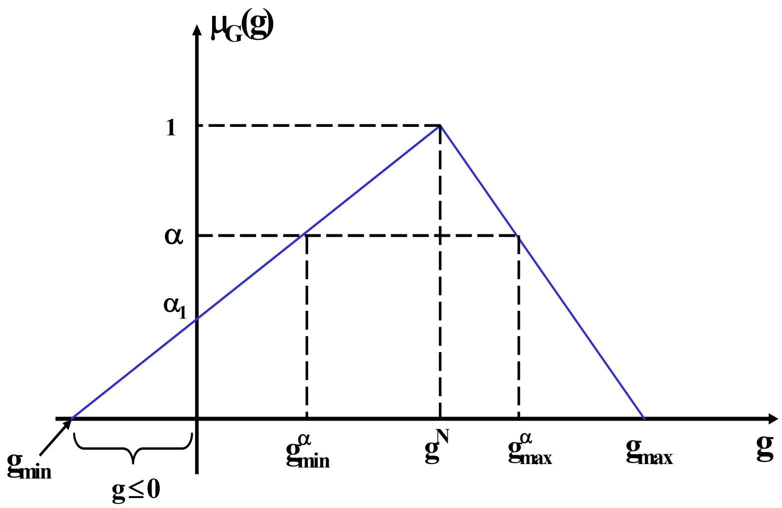


Figure 4. Used notation in possibility-based design optimization

where $g_{\min}^{\alpha}$ is the <u>global</u> minimum of g at the *a*-cut. Eq. (22) is analogous to the R-percentile formulation [1] of a probabilistic constraint in RBDO. The possibilistic constraint of Eqs (21) or (22) becomes active if $g_{\max}^{\alpha} = 0$.

Based on this discussion, a possibility-based design optimization (PBDO) problem can be formulated as

$$\min_{\mathbf{d},\mathbf{x}^N} f\left(\mathbf{d}, \mathbf{x}^N, \mathbf{p}^N\right)$$

$$\text{s.t.} \quad \pi\left(g_i(\mathbf{d}, \mathbf{X}, \mathbf{P}) \le 0\right) \le \alpha\,, \quad i = 1,\ldots,n \tag{23}$$
$$\mathbf{d}_L \le \mathbf{d} \le \mathbf{d}_U \;,\; \mathbf{x}_L \le \mathbf{x}^N \le \mathbf{x}_U$$

where $\mathbf{d} \in R^k$ is the vector of deterministic design variables, $\mathbf{X} \in R^m$ is the vector of possibilistic design variables, $\mathbf{P} \in R^q$ is the vector of possibilistic design parameters and $\mathbf{x}^N$ and $\mathbf{p}^N$ are the normal point vectors for the possibilistic design variables and parameters, respectively. According to the used notation, a bold letter indicates a vector, an upper case letter indicates a possibilistic variable or parameter and a lower case letter indicates a deterministic variable or a realization of a possibilistic variable or parameter. Feasibility of the i^{th} deterministic constraint is expressed with the positive null form $g_i \ge 0$.

The possibilistic design variables are represented with convex normal possibility distributions (membership functions). Note that they may not be necessarily triangular. The superscript *N* denotes the normal point of each distribution where the membership function value is equal to one. Subscripts *L* and *U* denote lower and upper bounds, respectively. In PBDO, we will assume that the membership functions of the possibilistic design variables have a constant shape and that their normal points are design variables moving within predetermined bounds. This is analogous to RBDO where the PDF of each random design variable stays constant while its mean value is a design variable.

Based on Eq. (22), the PBDO formulation (23) is equivalent to

$$\min_{\mathbf{d},\mathbf{x}^N} f\left(\mathbf{d}, \mathbf{x}^N, \mathbf{p}^N\right)$$
$$\text{s.t.} \quad g^{\alpha}_{i_{\min}} \ge 0 \quad i = 1,\ldots,n \tag{24}$$
$$\mathbf{d}_L \le \mathbf{d} \le \mathbf{d}_U \;,\; \mathbf{x}_L \le \mathbf{x}^N \le \mathbf{x}_U \,.$$

The PBDO formulation (23) or (24) is a double-loop optimization problem where an optimization is performed (inner loop) when the design optimization (outer loop) calls for a possibilistic constraint evaluation. It should be noted that the PBDO optimum at *a*=1 coincides with the deterministic optimum.

4.1. PBDO with a Combination of Random and Possibilistic Variables

Reliability-based design optimization (RBDO) provides optimum designs in the presence of only random (or aleatory) uncertainty (Tu, Choi and Park, 1999; Liang, Mourelatos and Tu, 2004; Wu, Shin, Sues and Cesare, 2001). A typical RBDO problem is formulated as (Liang, Mourelatos and Tu, 2004]

$$\min_{\mathbf{d},\boldsymbol{\mu}_\mathbf{Y}} f\left(\mathbf{d}, \boldsymbol{\mu}_\mathbf{Y}, \boldsymbol{\mu}_\mathbf{Z}\right)$$
$$\text{s.t.} \quad P\left(g_i(\mathbf{d}, \mathbf{Y}, \mathbf{Z}) \ge 0\right) \ge R_i = 1 - p_{f_i}\,, \quad i = 1,\ldots,n \tag{25}$$
$$\mathbf{d}^L \le \mathbf{d} \le \mathbf{d}^U \;,\; \boldsymbol{\mu}^L_Y \le \boldsymbol{\mu}_\mathbf{Y} \le \boldsymbol{\mu}^U_\mathbf{Y}$$

where $\mathbf{Y} \in R^\ell$ is the vector of random design variables and $\mathbf{Z} \in R^r$ is the vector of random design parameters.

For a variety of practical applications however, there may not be enough information to characterize all design variables and parameters probabilistically. A subset of them can be therefore, characterized possibilistically using membership functions. A possibility-based design optimization problem with a combination of random and possibilistic (or fuzzy) variables can be formulated as

$$\min_{\mathbf{d},\mathbf{x}^N,\boldsymbol{\mu}_Y} f\left(\mathbf{d},\boldsymbol{\mu}_\mathbf{Y},\boldsymbol{\mu}_\mathbf{Z},\mathbf{x}^N,\mathbf{p}^N\right) \tag{26}$$

$$\begin{aligned}
\text{s.t.}\quad & g^{\alpha}_{i_{\min}} \geq 0,\ i=1,\ldots,n \\
& \mathbf{d}_L \leq \mathbf{d} \leq \mathbf{d}_U\ ,\ \boldsymbol{\mu}^L_\mathbf{Y} \leq \boldsymbol{\mu}_\mathbf{Y} \leq \boldsymbol{\mu}^U_\mathbf{Y} \\
& \mathbf{x}_L \leq \mathbf{x}^N \leq \mathbf{x}_U \\
\text{with}\quad & g^{\alpha}_{i\min} = \min_{\mathbf{X}}(\beta_i - \beta_{t_i}) \geq 0,\ i=1,\ldots,n, \\
& \mathbf{x}^{\alpha}_{L} \leq \mathbf{x} \leq \mathbf{x}^{\alpha}_{U}\ ,\ \mathbf{p}^{\alpha}_{L} \leq \mathbf{p} \leq \mathbf{p}^{\alpha}_{U} \\
\text{and}\quad & \beta = \min_{\mathbf{U}}\|\mathbf{U}\| \\
\text{s.t.}\quad & G(\mathbf{U}) = 0
\end{aligned}$$

where β_t is the target reliability index. Note that $\mathbf{x}^{\alpha}_{L}$ and $\mathbf{x}^{\alpha}_{U}$ are the lower and upper limits of **X** at an *a*-cut.

Problem (26) represents a triple-loop optimization sequence. The design optimization of the outer loop calls a series of possibilistic constraints in the middle loop. Each possibilistic constraint is in general, a global optimization problem. Finally, each possibilistic constraint is a function of the corresponding reliability index β which represents the third loop of the optimization sequence. For computational purposes, two out of the three nested loops can be easily combined.

5. EVIDENCE-BASED DESIGN OPTIMIZATION (EBDO)

In this section, a methodology is presented on how to use evidence theory in design. We will show that the evidence theory-based design is more conservative compared with all RBDO designs obtained with different probability distributions and less conservative compared with the PBDO design.

If feasibility of a constraint g is expressed with the non-negative null form $g \geq 0$, we have shown that $Bel(g \geq 0) \leq P(g \geq 0) \leq Pl(g \geq 0)$ where $P(g \geq 0)$ is the probability of constraint satisfaction. Therefore,

$$P(g < 0) \leq p_f \text{ is satisfied if } Pl(g < 0) \leq p_f \tag{27}$$

where p_f is the probability of failure which is usually a small prescribed value. The above statement is equivalent to

$$P(g \geq 0) \geq R \text{ is satisfied if } Bel(g \geq 0) \geq R \tag{28}$$

where $R = 1 - p_f$ is the corresponding reliability level.

Hence, an evidence theory-based design optimization (EBDO) problem can be therefore, formulated as

$$\begin{aligned}
&\min_{\mathbf{d},\mathbf{x}^N} f\left(\mathbf{d},\mathbf{x}^N,\mathbf{p}^N\right) \\
\text{s.t.}\quad & Pl\left(g_i\left(\mathbf{d},\mathbf{X},\mathbf{P}\right)<0\right)\le p_{f_i},\ i=1,\dots,n \\
& \mathbf{d}_L \le \mathbf{d} \le \mathbf{d}_U\ ,\ \mathbf{x}_L^N \le \mathbf{x}^N \le \mathbf{x}_U^N
\end{aligned} \tag{29}$$

where $\mathbf{X}\in R^m$ and $\mathbf{P}\in R^q$ are the vectors of uncertain design variables and parameters. The superscript "N" indicates nominal value of uncertain variables or parameters. The uncertainty is provided by expert opinions.

It should be noted that the plausibility measure is used instead of the equivalent belief measure, in Problem (29). The reason is that at the optimum, the failure domain for each active constraint is usually much smaller than the safe domain over the frame of discernment (FD) (domain of all focal elements with nonzero combined BPA; see next section). As a result, the computation of the plausibility of failure is much more efficient than the computation of the belief of safe region.

5.1. ASSESSING *BEL* AND *PL* WITH DEMPSTER-SHAFER THEORY

Evidence theory can quantify epistemic uncertainty, even when the experts provide conflicting evidence. This section shows how to propagate epistemic uncertainty through a given model (transfer function) which is necessary in calculating the plausibility of constraint violation in Problem (29). The uncertainty propagation will be illustrated using the following simple transfer function

$$y = f\left(a,b\right) \tag{30}$$

where $a\in A, b\in B$ are two independent input parameters and *y* is the output. The combined BPA's for both *a* and *b* are obtained from Dempster's rule of combining of Eq. (12) if multiple experts have provided evidence for either *a* or *b*. With combined information for each input parameter, we define a vector $c=\lfloor a_{ci},b_{cj}\rfloor$, needed to calculate the output *y* as

$$C = A\times B = \left\{c=\lfloor a_{ci},b_{cj}\rfloor, a_{ci}\in A, b_{cj}\in B\right\} \tag{31}$$

where subscript *c* stands for "combined" and *i,j* indicate focal elements.

Taking advantage of assumed parameter independency, the BPA for *c* is

$$m_c\left(h_{ij}\right)=m\left(a_{ci}\right)m\left(b_{cj}\right) \tag{32}$$

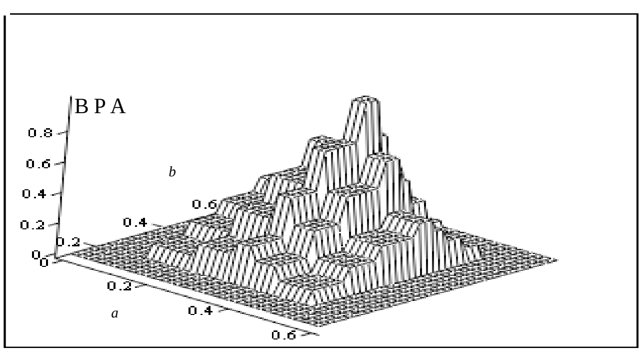


Figure 5. Representative BPA structure for two parameters *a* and *b*.

where $h_{ij} = \lfloor a_{ci}, b_{cj} \rfloor$ and a_{ci}, b_{cj} denote intervals such that $a \in a_{ci}$ and $b \in b_{cj}$. Eq. (32) can be used to calculate the combined BPA structure for the entire domain *C*. For every $(a,b) \in c \ |c \in C$, needed to evaluate the output *y*, the combined BPA m_c is used. A representative combined BPA structure is shown in Fig. 5.

The Cartesian product *C* of Eq. (31) is also called frame of discernment (FD) in the literature. It consists of all focal elements (rectangles in Fig. 5 with nonzero combined BPA) and can be viewed as the finite sample space in probability theory.

If a domain *F* is defined as

$$F = \{g : g = f(a,b) - y_0 > 0, (a,b) \in c, c = [a_c, b_c] \subset C\} \tag{33}$$

where y_0 is a specified value. According to evidence theory,

$$Bel(F) \leq p_f \leq Pl(F). \tag{34}$$

where $p_f = P(g > 0)$ is the true probability.

The *Bel (F)* and *Pl (F)* are calculated using Eqs (6) and (7) where set *A* is equal to set *F* of Eq. (33) and *B* is a rectangular domain (focal element) such that $B \subseteq A$ for Eq. (6) and $B \cap A \neq 0$ for Eq. (7). $B \subseteq A$ means that the focal element must be entirely within the domain *g>0* and $B \cap A \neq 0$ means that the focal element must be entirely or partially within the domain *g>0* (see Fig. 6). In order to identify if a focal element *B* satisfies $B \subseteq A$ or $B \cap A \neq 0$, the following minimum and maximum values of *g* must be calculated

$$[g_{\min}, g_{\max}] = \left\lfloor \min_{\mathbf{x}} g(\mathbf{x}), \max_{\mathbf{x}} g(\mathbf{x}) \right\rfloor \tag{35}$$

for $\mathbf{x}^L \leq \mathbf{x} \leq \mathbf{x}^U$ where $(\mathbf{x}^L, \mathbf{x}^U)$ defines the focal element domain. For monotonic functions, the vertex method [34] can be used to calculate the minimum and maximum values in Eq. (35) by simply identifying the minimum and maximum values among all vertices of the focal element domain. If for a focal element, $g_{\min}$ and $g_{\max}$ are both positive, the focal element will contribute to the calculation of belief and plausibility. On the other hand, if $g_{\min}$ and $g_{\max}$ are both negative, the focal element will not contribute to the calculation of belief or plausibility. If however, $g_{\min}$ is negative and $g_{\max}$ is positive, the focal element will not contribute to the belief but it will contribute to the plausibility calculation. This is shown schematically in Fig. 6.

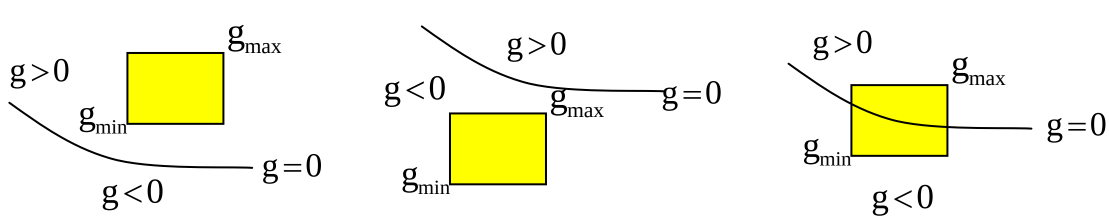


Figure 6. Schematic illustration of focal element contribution to belief and plausibility measures.

In summary the following tasks are performed in order to calculate the belief and plausibility of the failure region:

1) For each input parameter, combine the evidence from the experts by combining the individual BPA's from each expert using Dempster's rule of combining (Eq. (12)).
2) Construct the BPA structure for the *m*-dimensional frame of discernment, where *m* is the number of input parameters. Assuming independent input parameters, Eq. (32) is used.
3) Identify the failure region space (set *F* of Eq. (33)).
4) Use Eqs (6) and (7) to calculate the belief and plausibility measures of the failure region. The failure region must be identified only within the frame of discernment. The true probability of failure is bracketed according to Eq. (34).

5.2. Implementation of the EBDO Algorithm

A computationally efficient solution of Problem (29) is presented here. As a geometrical interpretation of it, we can view the design point (**d**,**x**) moving within the feasible domain so that the objective *f* is minimized (see Fig. 7). If the entire FD is in the feasible domain, the constraints are satisfied and are inactive. A constraint becomes active if part of the FD is in the "failure" region so that the plausibility of constraint violation is equal to p_f. In general, Problem (29) represents movement of a hyper-cube (FD) within the feasible domain.

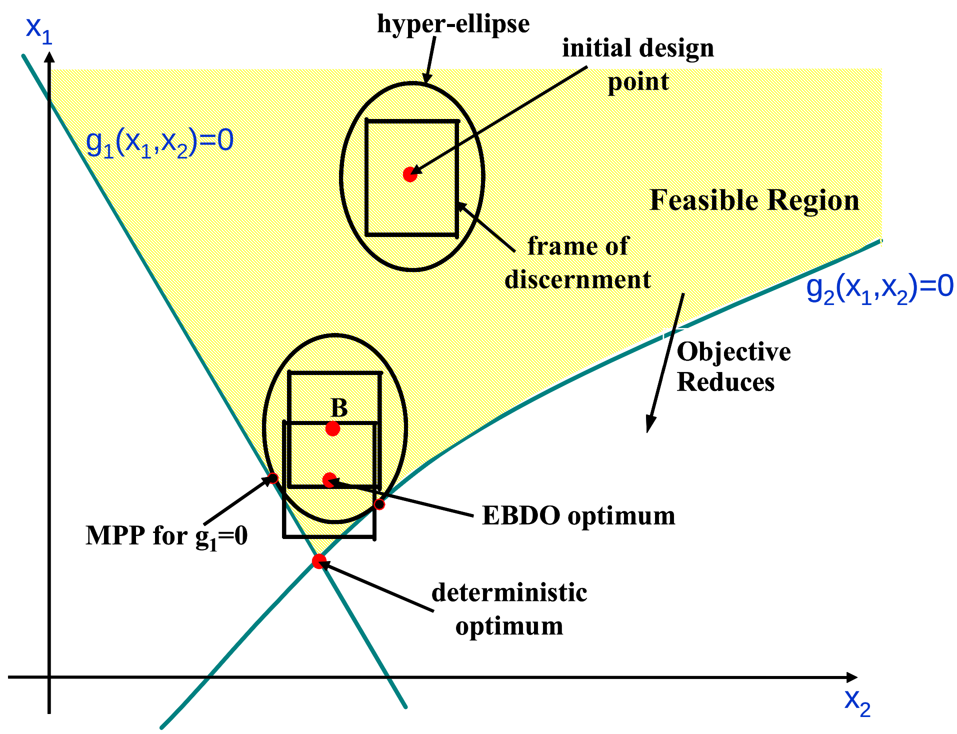


Figure 7. Geometrical interpretation of the EBDO algorithm

In order to save computational effort, the bulk of the FD movement, from the initial design point to the *vicinity* of the optimal point (point B of Fig. 7), can be achieved by *moving a hyper-ellipse which contains the FD*. The center of the hyper-ellipse is the "approximate" design point and each axis is arbitrarily taken equal to three times the standard deviation of a hypothetical normal distribution. This assumes that each dimension of the FD hyper-cube is equal to six times the standard deviation of the hypothetical normal distribution. The hyper-ellipse can be easily moved in the design space by solving a RBDO problem. The RBDO optimum (point B of Fig. 7) is in the vicinity of the solution of Problem (29) (EBDO optimum). The RBDO solution also identifies all active constraints and their corresponding most probable points (MPP's). The maximal possibility search algorithm (Choi, Du and Youn, 2004) can also be used to move the FD hyper-cube in the feasible domain. It should be noted that the 3-sigma axes hyper-ellipse is arbitrary. The size of the hyper-ellipse is not however, crucial because it is only used to calculate the initial point (point B of Fig. 7) of the EBDO algorithm. The latter calculates the true EBDO optimum accurately. From our experience, a 3 to 4-σ size works fine.

At this point, we generate a *local* response surface of each active constraint around its MPP. In this work, the Cross-Validated Moving Least Squares (CVMLS) [39] method is used based on an Optimum Symmetric Latin Hypercube (OSLH) [40] "space-filling" sampling.

A derivative-free optimizer calculates the EBDO optimum. It uses as initial point the previously calculated RBDO optimum which is close to the EBDO optimum. Problem (29) is solved, considering only the identified active constraints. For the calculation of the plausibility of failure $Pl(g < 0)$ of each active constraint, an algorithm presented in (Mourelatos and Zhou, 2005) is used. It identifies all focal elements which contribute to the plausibility of failure. The computational effort is significantly reduced because accurate local response surfaces are used for the active constraints. The cost can be much higher if the optimization algorithm evaluates the actual active constraints instead of their efficient surrogates (response surfaces). It should be noted that a derivative-free optimizer is needed due to the discontinuous nature of the combined BPA structure. The DIRECT derivative-free, global optimizer is used (Jones, Perttunen and Stuckman, 1993).

6. EXAMPLES

In this section, the possibility-based and evidence-based design algorithms are demonstrated with a cantilever beam example and a pressure vessel example. For both examples, comparisons are made with deterministic design and reliability-based design results. It should be noted that theoretically, the possibility and reliability-based results can not be compared because the possibility and reliability theories are based on different axioms. However for practical purposes, we attempt to compare them by arbitrarily using membership functions which "resemble" the probability density functions used in the reliability-based results.

6.1. A Cantilever Beam Example

In this example, a cantilever beam in vertical and lateral bending (Wu, Shin, Sues and Cesare, 2001) is used (see Fig. 8). The beam is loaded at its tip by the vertical and lateral loads Y and Z, respectively. Its length L is equal to 100 in. The width *w* and thickness *t* of the cross-section are deterministic design variables. The objective is to minimize the weight of the beam. This is equivalent to minimizing $f = w*t$, assuming that the material density and the beam length are

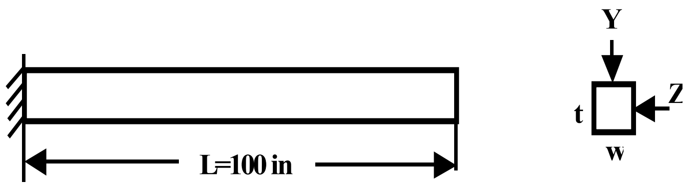


Figure 8. Cantilever beam under vertical and lateral bending

constant.

Two non-linear failure modes are used. The first failure mode is yielding at the fixed end of the cantilever; the other failure mode is that the tip displacement exceeds the allowable value of $D_0 = 2.5''$. The PBDO problem is formulated as,

$$\begin{aligned} &\min_{w,t} f = w*t \\ \text{s.t.} \quad & g^{\alpha}_{j_{\min}} \geq 0 \qquad j = 1,2 \\ & g_1(y,Z,Y,w,t) = y - (\frac{600}{wt^2}*Y + \frac{600}{w^2t}*Z) \\ & g_2(E,Z,Y,w,t) = D_0 - \frac{4L^3}{Ewt}\sqrt{(\frac{Y}{t^2})^2 + (\frac{Z}{w^2})^2} \\ & 0 \leq w,t \leq 5 \end{aligned} \tag{40}$$

where g_1 and g_2 are the limit states corresponding to the two failure modes. The design variables w and t are deterministic. In the RBDO study of [2], Y, Z, y and E are normally distributed random parameters with Y~ N (1000, 100) lb, Z~ N (500,100) lb, y~ N (40000,2000) psi and E~ N ($(29*10^6, 1.45*10^6)$ psi; y is the random yield strength, Z and Y are mutually independent random loads in the vertical and lateral directions respectively, and E is the Young modulus. A reliability index $\beta = 3$ has been used in [2] for both constraints.

For the PBDO case, Y, Z, y and E are possibilistic parameters described with the triangular membership functions $\left(x^N - 3*\sigma, x^N, x^N + 3*\sigma\right)$ where x^N is the normal point of each variable and σ is the used standard deviation in the RBDO study. The frame of discernment defined by the $\left(x^N - 3*\sigma, x^N + 3*\sigma\right)$ coordinates is also used in EBDO.

Table 2. Comparison of PBDO, EBDO and RBDO optima for the cantilever beam example

	Determ. Optimum	**Reliability Optimum**	**Possibility Optimum**		**Evidence Optimum**	
Design Variables			**α=0.1**	**α=0**	$\mathbf{p_f = 0.1}$	$\mathbf{p_f = 0.0013}$
w	2.0470	2.4781	2.5298	2.5901	2.4534	2.5028
t	3.7459	3.8421	4.1726	4.210	3.6162	3.9902
Objective						
f(w,t)	7.6679	9.5212	10.556	10.901	8.8721	9.9868
Constraints						
$g_1(\mathbf{x}) / \bar{y}$	0	0	0	0	0	0.0032
$g_2(\mathbf{x}) / D_0$	0	0.1436	0.15	0.168	0.00428	0.0835

Table 2 compares the deterministic optimization, RBDO, PBDO and EBDO results. The PBDO optimum (objective function) with a=0 is higher than the RBDO optimum. Because it represents the worst case design, it provides an upper bound of all RBDO optima obtained with different distributions, as long as these distributions have similar variability ranges (e.g. different beta distributions defined over the same range). For a higher a-cut (a=0.1), the PBDO optimum reduces. It should be noted that the PBDO optimum at a=1 coincides with the deterministic optimum. The last two rows of Table 2 show the normalized values of the two constraints at the

Table 3. BPA structure for y, Y, Z and E

Z			**y** ($\times 10^3$)	
Interval	**BPA**		**Interval**	**BPA**
[200 300]	2.2%		[35 37]	6.1%
[300 400]	13.6%		[37 38]	9.2%
[400 450]	15%		[38 39]	15%
[450 500]	19.2%		[39 40]	19.2%
[500 550]	19.2%		[40 41]	19.2%
[550 600]	15%		[41 42]	15%
[600 700]	13.6%		[42 43]	9.2%
[700 800]	2.2%		[43 45]	7.1%

Y			**E** ($\times 10^6$)	
Interval	**BPA**		**Interval**	**BPA**
[700 800]	2.2%		[26.5 27.5]	10%
[800 900]	13.6%		[27.5 28.5]	21%
[900 1000]	34.1%		[28.5 29]	13.5%
[1000 1100]	34.1%		[29 29.5]	13.5%
[1100 1200]	13.6%		[29.5 30.5]	21%
[1200 1300]	2.4%		[30.5 31.3]	21%

optimum. The first constraint is normalized by the mean yield strength $\bar{y} = 40000$ and the second constraint is normalized by the allowable tip displacement $D_0 = 2.5$. Although both constraints are active at the deterministic optimum, only the first constraint is active for both the RBDO and PBDO optima.

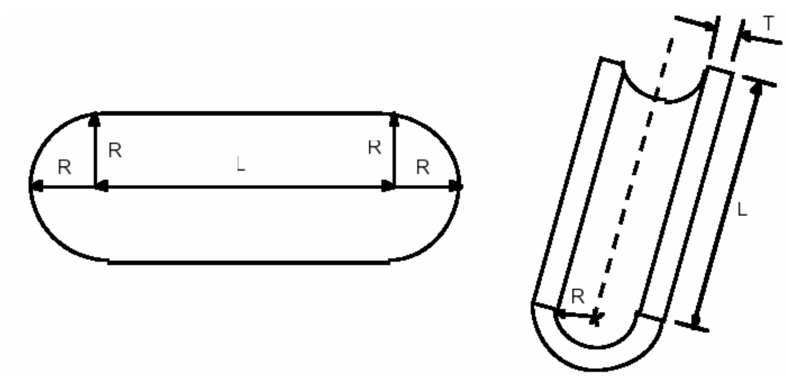
The EBDO problem formulation is the same with Problem (40) but with different constraints. The new constraints are $Pl(g_i < 0) \leq p_f$, i=1,2. The uncertain parameters **P**=[Y,Z,y,E] have the

BPA structure of Table 3. The BPA for each interval of an uncertain parameter is assumed to be equal to the area under the PDF used in RBDO, in order to compare the EBDO design with the corresponding RBDO design. This is not how the BPA is obtained in general. As it has been mentioned, expert opinions are used to construct the BPA structure. If however, a random variable or parameter is described probabilistically, equivalent BPA values within specified intervals are calculated as equal to the area under the PDF. In doing so, the evidence theory can be used to handle a mixture of probabilistic and non-probabilistic variables.

The last two columns of Table 2 show the EBDO results for p_f = 0.1 and 0.0013 ($\beta = 3$). As expected, the deterministic optimum of 7.6679 is less than the RBDO optimum of 9.5212 which in turn, is less than the EBDO optimum of 9.9868 at p_f =0.0013 ($\beta = 3$). For $p_f = 0.1$, the EBDO optimum reduces. Furthermore, the EBDO optimum of 9.9868 at p_f =0.0013 is better than the worst case PBDO optimum of 10.901 (*a*=0). Although only the first constraint is active for the RBDO and PBDO optima, both constraints are active for the EBDO optima, similarly to the deterministic case.

6.2. A Pressure Vessel Example

This example considers the design of a thin-walled pressure vessel (Lewis and Mistree, 1997) which has hemispherical ends as shown in Fig. 9. The design objective is to calculate the radius *R*, mid-section length *L* and wall thickness *t* in order to maximize the volume while avoiding yielding of the material in both the circumferential and radial directions under an internal pressure *P*. Geometric constraints are also considered. The material yield strength is *Y*. A safety



factor *SF* = 2 is use

Figure 9. Thin-walled pressure vessel.

The PBDO problem is stated as

$$\max_{R_N, L_N, t_N} f = \frac{4}{3}\pi R_N^3 + \pi R_N^2 L_N$$

$$\text{s.t.} \quad g^{\alpha}_{j_{\min}} \geq 0 \qquad j = 1,...,5$$

where,
$$g_1(\mathbf{X}) = 1.0 - \frac{P(R+0.5t)SF}{2tY}$$
$$g_2(\mathbf{X}) = 1.0 - \frac{P(2R^2+2Rt+t^2)SF}{(2Rt+t^2)Y}$$
$$g_3(\mathbf{X}) = 1.0 - \frac{L+2R+2t}{60}$$
$$g_4(\mathbf{X}) = 1.0 - \frac{R+t}{12}$$
$$g_5(\mathbf{X}) = 1.0 - \frac{5t}{R}$$
$$0.25 \le t_N \le 2.0$$
$$6.0 \le R_N \le 24$$
$$10 \le L_N \le 48$$

Table 4. BPA structure for *R*, *L*, *t*, *P* and *Y*

R	**L**	**t**	**BPA**
[R_N - 6.0 R_N - 4.5]	[L_N - 12 L_N - 9]	[t_N - 0.4 t_N - 0.3]	0.13%
[R_N - 4.5 R_N - 3.0]	[L_N - 9 L_N - 6]	[t_N - 0.3 t_N - 0.2]	2.15%
[R_N - 3.0 R_N]	[L_N - 6 L_N]	[t_N -0.2 t_N]	47.72%
[R_N R_N + 3.0]	[L_N L_N + 6]	[t_N t_N +0.2]	47.72%
[R_N + 3.0 R_N + 4.5]	[L_N + 6 L_N + 9]	[t_N + 0.2 t_N + 0.3]	2.15%
[R_N + 4.5 R_N + 6.0]	[L_N + 9 L_N + 12]	[t_N + 0.3 t_N + 0.4]	0.13%

P	**Y**	**BPA**
[800 850]	[208000 221000]	0.13%
[850 900]	[221000 234000]	2.15%
[900 1000]	[234000 260000]	47.72%
[1000 1100]	[260000 286000]	47.72%
[1100 1150]	[286000 299000]	2.15%
[1150 1200]	[299000 312000]	0.13%

The EBDO problem formulation is the same but with constraints $Pl(g_j(\mathbf{X})<0) \le p_f \quad j=1,...,5$. For the EBDO case, the uncertainty in design variables *R*, *L*, and *t* and design parameters *P* and *Y* are represented with the combined BPA structure of Table 4. To compare results with RBDO, the BPA values of *R*, *L*, *t*, *P* and *Y* are taken equal to the area under the PDF of a normal distribution for the intervals shown in Table 4. The normal distributions for *R*, *L*, *t*, *P* and *Y* have standard deviations equal to 1.5, 3, 0.1, 50 and 13000, respectively. The mean values for parameters *P* and *Y* are taken equal to 1000 and 260000. The intervals for *R*, *L*, *t*, *P* and *Y* extend four standard deviations from each side of the normal point, in an attempt to use a similar variation with the RBDO study. Finally, EBDO and PBDO use the same frame of discernment.

Table 5 compares the deterministic optimization, RBDO, PBDO and EBDO results. Similar conclusions with the previous example are drawn. A reliability index $\beta = 2.0$ (p_f = 0.0228) has been used in the RBDO study for all constraints. As expected, the deterministic maximum volume of 22400 is higher than the RBDO volume of 10791 which in turn, is higher than the EBDO volume of 7644. Also, the PBDO optimum of 6132 (*a*=0) which represents the worst case, is the lowest. For comparison purposes, the PBDO and EBDO results are also presented for a=0.2 and p_f =0.0228, respectively. It is noted that the constraint activity changes among the deterministic, RBDO, PBDO and EBDO optima. Only the third and fourth constraints are active for the deterministic case. However, the second, third and fourth constraints become active at the RBDO and PBDO optima. At the EBDO optimum all constraints are active except the fifth one.

Table 5. Comparison of deterministic, RBDO, PBDO and EBDO optima for vessel example

	Determ. Optimum	**Reliability Optimum**	**Possibility Optimum**		**Evidence Optimum**	
Design Variables			***a*=0.2**	***a* =0**	**p_f =0.2**	**p_f =0.0228**
R_N	11.750	8.7244	7.9107	7.0107	8.333	8.1111
L_N	36.000	33.5186	30.3867	30.3867	30.407	26.1852
t_N	0.250	0.269	0.2893	0.2893	0.347	0.3472
Objective						
$-f(R_N, L_N)$	22400	10791	8044	6132	9053	7644
Constraints						
$g_1(\mathbf{x})$	0.8173	0.5003	0.5	0.5	0	0
$g_2(\mathbf{x})$	0.6346	0	0	0	0.0137	0
$g_3(\mathbf{x})$	0	0	0	0	0	0.0183
$g_4(\mathbf{x})$	0	0	0	0	0	0.0118
$g_5(\mathbf{x})$	0.8936	0.6891	0.4325	0.0256	0.9994	0.1038

7. SUMMARY AND CONCLUSIONS

In this paper, the possibility and evidence theories were used to assess design reliability with incomplete information. The possibility theory was viewed as a variant of fuzzy set theory. The different types of uncertainty and formal uncertainty theories were first introduced using the fundamentals of fuzzy measures. Subsequently, the commonly used vertex and discretization methods which are used for propagating non-probabilistic uncertainty were reviewed and compared with a hybrid (global-local) optimization method. It was showed that the hybrid optimization method is very efficient and has the same accuracy with the "brute force" discretization method.

The possibility theory was also used in design. A possibility-based design optimization method was proposed where all design constraints are expressed possibilistically. It was shown that the method gives a conservative solution compared with all conventional reliability-based designs obtained with different probability distributions. A general possibility-based design optimization method was also presented which handles a combination of random and possibilistic design variables.

Furthermore, a computationally efficient design optimization method was described, which can handle a mixture of epistemic and random uncertainties. A mean performance is optimized subject to the plausibility of constraint violation being small. Uncertainty is quantified using "expert" opinions. Two examples demonstrated the proposed possibility-based and evidence-based design optimization methods. It was shown that both the PBDO and EBDO designs are more conservative compared with the RBDO design. However, the EBDO design is usually less conservative compared with the PBDO design.

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