Worst case bounds in the presence of correlated uncertainty

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Extended Abstract

This paper presents a method for computing rigorous bounds on the solution of linear systems whose coefficients have large, correlated uncertainties, with a computable overestimation factor that is frequently quite small.

Linear systems of equations are among the most frequently used tools in applied mathematics. In realistic applications, the data entering the coefficients of these equations are generally uncertain. Since linear equations become nonlinear when coefficients are uncertain and become variable, traditional sensitivity analysis remains valid only for sufficiently small errors. Unfortunately, it is usually unclear when the errors are sufficiently small for its validity: For errors larger than some unknown, problem-dependent margin, sensitivity analysis may be severely biased, since it does not account for the nonlinearities in the problem.

In problems where safety is an issue, worst case results are needed. For example, current safety regulation laws in civil engineering require a worst case analysis, and hence interval techniques, although current practice is still Monte Carlo with its deficiencies.

Recently, Neumaier & Pownuk (2) showed that using interval analysis, it is possible to do quantitative worst case sensitivity analysis even in high dimensions. The techniques presented there provide good and valid enclosures of all quantities of interest, and thus enables engineers to obtain guarantees whether the worst case satisfies all safety requirements.

However, previous worst case methods (including monotonicity methods which work only under additional assumptions) only compute the worst case when all uncertainties vary independently. This is frequently an unrealistic assumption. In the past, correlated uncertainties could be handled only with Monte Carlo methods which always underestimate the worst case, and sometimes drastically.

In this paper we develop a method for the worst case analysis of solutions of linear systems of the form

$$(K + A^T DA)u = a + Fb,$$

where D is diagonal, with correlated uncertainties in D and b, and no uncertainty in K, A, a, and F. This includes the case of linear systems arising in truss modeling.

The new results are obtained by generalizing those of NEUMAIER & POWNUK (2) to the case where the uncertainties are bounded by ellipsoids rather than boxes, thus reflecting the known correlations. A basic tool used is the following optimal bound for linear combinations of numbers ranging in an ellipsoid.

Proposition. Let $C \in \mathbb{R}^{n\times n}$ be symmetrtic and positive definite. If

$$x^T C^{-1} x \le q$$

then

$$|a^T x| \le \sqrt{q a^T C a}, \tag{1}$$

with equality iff $x = \lambda C a$ with $|\lambda| = \sqrt{q/a^T C a}$.

Proof. Any symmetrtic and positive definite matrix C has a Cholesky factorization $C = LL^T$ with nonsingular L. Using this, we have

$$\|L^{-1}x\|_2 = \sqrt{(L^{-1}x)^T L^{-1} x} = \sqrt{x^T C^{-1} x} \le \sqrt{q},$$

$$\|L^T a\|_2 = \sqrt{(L^T a)^T L^T a} = \sqrt{a^T C a}.$$

Now the Cauchy-Schwarz inequality implies

$$|a^T x| = |(L^{-1}x)^T L^T a| \le \|L^{-1}x\|_2 \|L^T a\|_2 \le \sqrt{q a^T C a},$$

giving (1). Equality can only hold if equality holds in the Cauchy-Schwarz inequality, hence if $L^{-1}x$ and $L^T a$ are parallel. This requires $x = \lambda C a$, and by substituting this into the equality case of (1), we find that $|\lambda| = \sqrt{q/a^T C a}$. $\square$

In addition, we employ ellipsoid arithmetic (cf. NEUMAIER (1)) to enclose the intermediate expressions in the calculations.

As an application, we show that it is feasible to compute worst case error bounds for the displacements of truss structures with uncertain stiffness coefficient, in the important case when the uncertainties are correlated. This includes the discussion of an appropriate deterministic model for uncertainty correlation.

Examples of numerical computations will be given for large truss structures with correlated uncertainty in the stiffness.

References

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