Stochastic wave groups in weakly nonlinear random waves

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Abstract. A stochastic model of wave groups is presented to explain the occurrence of large waves in nonlinear random seas. The model leads to the description of the non-Gaussian statistics of oceanic waves and to a new asymptotic distribution of crest heights over large waves in a form that generalizes the Tayfun model. Comparisons based on a first wave data set collected at the Tern platform in the northern North Sea during an extreme storm, and a second set collected in the southern North Sea (WACSIS) show good agreement with the new theoretical wave distributions. In particular, for broad band seas, the Tayfun model seems to fit the data, and thus it can be regarded suitable for describing crest statistics for engineering applications.

Keywords: crest height; stochastic wave group; second order effects; probability of exceedance; Gaussian sea; quasi-determinism, Slepian model.

1. INTRODUCTION

To the leading order of approximation, the free surface displacement $\eta(t)$ is a Gaussian process of time. Lindgren (1970,1972) showed that locally near a very high crest, the surface displacement tends to assume the same shape as the covariance function $\psi(T) = \langle \eta(t)\eta(t+T)\rangle$. This is the Slepian model (Kac & Slepian 1959) whose time-domain formulation was used by Tromans et al. (1991) to analyze wave measurements.

An alternative view of the Slepian model was offered in the eighties by Boccotti (1989,2000). His theory of quasi determinism revealed the mechanics of three dimensional wave groups and their relation to the occurrence of extreme waves in a Gaussian sea and confirmed with field experiments (Boccotti et al., 1993a,1993b, Phillips et al. 1993a, 1993b).

In Gaussian sea waves, both crest and trough distributions follow the same Rayleigh law for narrow-band spectra (Longuet-Higgins, 1952). In the more general case of Gaussian waves with finite-band spectra, the Rayleigh distribution serves as an upper bound for the exceedance probability of crest heights.

In reality, water waves are nonlinear, and the probability density function of the surface displacement tends to deviate from the Gaussian form. In particular, due to second order nonlinearities the water surface presents sharper crests and shallower rounded troughs. Thus, the skewness λ_3 of surface elevations is not zero (Longuet-Higgins 1963). The exact theoretical form of the corresponding distribution of nonlinear wave crests is not known under general conditions. A series expansion based on the Edgeworth's form the Gram-Charlier distribution was proposed by Longuet-Higgins (1963), but can lead to expressions that violate the non-negativity condition on probability densities. Crest heights of large waves can be over predicted unrealistically in steep storm seas in deep or transitional water depths. Convenient and simple narrow-band approximation for deep-water waves was given by Tayfun (1980, 1986a, 2006) in the early eighties based upon weakly second order wave theory. As a corollary, Tayfun (1980) also derived an analytical distribution for the crest statistics and a least-upper-bound (*lub*) distribution of crest heights (Tayfun and Al-Humoud, 2002). Comparisons of such models with various deep and shallow water second-order simulations have been carried out by Forristall (2000) and Prevosto & Forristall (2002).

The recent experimental results of Onorato et al. (2006) and the numerical simulations of Socquet-Juglard et al. (2005) both show that for the case of multidirectional random waves, the nonlinear effects are due dominantly to bound waves and the Tayfun distribution explains very well the crest statistics. Deviations from the Tayfun distribution may occur only in long-crested narrow-band waves due to third order nonlinear effects, such as the Benjamin- Feir type modulation instability (Zakharov 1999, Janssen 2003) as shown by Onorato et al. (2006) and Socquet-Juglard et al. (2005). Thus, for practical engineering applications where realistic oceanic conditions are characterized by multidirectional spectra, the second order Stokes theory, and thus the Tayfun model, still offers a valid theoretical framework for the wave statistics.

In this paper, we propose an alternative view of second order wave theory and a generalization of the Tayfun model. We first present an extension of the theory of quasi-determinism of Boccotti (1989,2000), defining a stochastic wave group that describes the dynamics of the wave surface around a randomly chosen very large crest (Lindgren 1970,1972). The stochastic wave group can be thought as a first order regression approximation according to Rychlik (1987) and Lindgren & Rychlik (1991).

In the second part of the paper, we shall study the nonlinear evolution of the stochastic wave group in the context of second order Stokes waves. This analysis will reveal the expected shape of large nonlinear crests and their statistics. In particular, we prove that the distribution of second order extreme crests is uniquely defined by the skewness λ_3 of the nonlinear surface displacement. This result is in perfect agreement with the narrow-band model of Tayfun (1980,1986a,2006), and it is valid for waves at deep and transitional water depths in a manner free of any constraints on their directionality or spectral bandwidth in agreement with the analytical results of Fedele & Arena (2005). In addition, a generalization of the Tayfun model (1980, 1986a) is proposed. Both the models are free of any bandwidth constraints and depends only on the global properties of the spectrum available from wave hindcasts. We also consider the Weibull model of Forristall (2000), and an exact closed form solution of the crest distribution based on the asymptotics for the h-upcrossings in Gaussian multivariate processes derived by Breitung and Richter (1996) which yields to the First Order Reliability Method (FORM).

Comparisons based on a first wave data set collected at the Tern platform in the northern North Sea during an extreme storm, and a second set collected in the southern North Sea (WACSIS) are presented. In particular, for broad band seas, the new theoretical models do not improve upon the Tayfun distribution (Tayfun 1980,1986, 2006), which thus can be regarded suitable for describing crest statistics for engineering applications.

2. Second order random waves

Consider weakly nonlinear random waves propagating in water of uniform depth d. The second order sea surface displacement ζ from the mean sea level at a fixed point $\mathbf{x}$ is given by

$$\zeta(\mathbf{x},t)=\zeta_1(\mathbf{x},t)+\zeta_2(\mathbf{x},t) \tag{1}$$

where the first order linear Guassian component ζ_1 is of the form

$$\zeta_1(\mathbf{x},t)=\sum_{i=1}^{N} z_i\cos(\boldsymbol{\theta}_i) \tag{2}$$

and the second order correction ζ_2 is given by

$$\zeta_2(\mathbf{x},t)=\frac{1}{4}\sum_{i,j=1}^{N} z_i z_j\left[A_{ij}^{+}\cos(\boldsymbol{\theta}_i+\boldsymbol{\theta}_j)+A_{ij}^{-}\cos(\boldsymbol{\theta}_i-\boldsymbol{\theta}_j)\right], \tag{3}$$

with

$$\boldsymbol{\theta}_i=\mathbf{k}_i\cdot\mathbf{x}-\omega_i t+\varepsilon_i=k_i x\cos\phi_i+k_i y\sin\phi_i-\omega_i t+\varepsilon_i.$$

Here, A_{ij}^{+} and A_{ij}^{+} are second order interaction coefficients (see e.g. Sharma & Dean 1979, Forristall 2000), $\mathbf{k}_i$ are horizontal wave-number vectors, with $k_i=|\mathbf{k}_i|$, the directional angles ϕ_i refer to the x axis, $\mathbf{x}=(x,y)$ is the horizontal spatial vector coincident with the mean water surface, ω_i is the wave frequency related to $\mathbf{k}_i$ through the dispersion relation $k_i\tanh k_i d=\omega_i^2/g$. We assume that frequencies ω_i are different from each other, the number N is infinitely large and that the phase angles ε_i are independent and uniformly distributed in $[0,2\pi]$. The linear wave amplitudes z_i are related to the wave spectral density $S(\mathbf{k})$ as

$$S(\mathbf{k})d\mathbf{k}{=}S(k,\phi)k\ \delta k\ \delta\phi=\sum_i\frac{z_i^2}{2},$$

where the sum is over i's for which $(k_i,\phi_i)\in([k,k+\delta k],[\phi,\phi+\delta\phi])$.

2.1. Basic definitions and assumptions

The jth order moment of the linear spectrum is

$$m_j=\int_0^\infty\omega^j S(\mathbf{k})\,d\mathbf{k}.$$

The validity of the form assumed for ζ is measured by the smallness of the *rms* surface gradient (Tayfun 1993)

$$\mu_1=\sqrt{\left\langle|\nabla\zeta_1|^2\right\rangle}=m_4/g^2<<1 \tag{4}$$

where $\langle\cdot\rangle$ means time average. The spectral mean frequency ω_m, the mean zero-upcrossing frequency ω_0 of the underlying linear process ζ_1 and the bandwidth ν of the spectral density $S(\mathbf{k})$ are defined respectively as

$$\omega_m = \frac{m_1}{m_0}, \quad \omega_0 = \sqrt{\frac{m_2}{m_0}}, \quad \nu = \sqrt{\frac{m_0 m_2}{m_1^2} - 1}. \tag{5}$$

Moreover, $EX_+ = \omega_0/2\pi$ is the expected number per unit time of zero up-crossings of ζ, correct to $O(\mu_1)$. To the same order, the space-time covariance $\Psi(\mathbf{X},T)$ of ζ is given by

$$\Psi(\mathbf{X},T) = \langle\zeta_1(\mathbf{X},t)\,\zeta_1(\mathbf{X},t+T)\rangle = \int S(\mathbf{k})\cos(\mathbf{k}\cdot\mathbf{X} - \omega T)d\mathbf{k}$$

where $\mathbf{X} = (X,Y)$ and $\psi(T) = \Psi(\mathbf{0},T)$ for brevity. Hereafter, the first absolute minimum of $\psi(T)$ occurs at time $T = T^*$ and that $\psi(T)$ decreases monotonically between $T = 0$ (when the absolute maximum is attained) and $T = T^*$.

The first moment $\langle\zeta\rangle = 0$, and the higher order moments $\langle\zeta^p\rangle$ with $p = 2,..4$ are given, correct to $O(\mu_1)$, by

$$\begin{aligned}
\left\langle\zeta^2\right\rangle &= m_0 + O\left(\mu_1^2\right), \\
\left\langle\zeta^3\right\rangle &= \frac{3}{2}\int S(\mathbf{k}_1)\,S(\mathbf{k}_2)\left[A^+(\mathbf{k}_1,\mathbf{k}_2) + A^-(\mathbf{k}_1,\mathbf{k}_2)\right]d\mathbf{k}_1 d\mathbf{k}_2 + O\left(\mu_1^2\right), \\
\left\langle\zeta^4\right\rangle &= 3m_0^2 + O\left(\mu_1^2\right),
\end{aligned} \tag{6}$$

where $A^{\pm}(\mathbf{k}_i,\mathbf{k}_j) = A_{ij}^{\pm}$. The spectral mean frequency and the mean zero-upcrossing frequency of the nonlinear process ζ are given by ω_m and ω_0 in Eq. (5) and they are correct to $O(\mu_1)$.

3. Large crests in Gaussian seas

Assume for the moment that a large wave crest of amplitude h is observed at $\mathbf{x} = \mathbf{x}_0 = (x_0, y_0)$ and $t = t_0$. Boccotti (2000) and Fedele (2006b) showed that as $h/\sigma \to \infty$, with probability approaching 1, a well defined wave group passes through the point $\mathbf{x} = \mathbf{x}_0$, with the apex of its development stage occurring at time $t = t_0$. As $h/\sigma \to \infty$, the surface displacement ζ_c around $\mathbf{x} = \mathbf{x}_0$ and $t = t_0$ is asymptotically described by the sum of a deterministic part $\zeta_{\rm det}$ of $O(h)$ and a residual random process R_ζ of $O(1)$, viz.

$$\zeta_c(\mathbf{X},T) = \zeta_{\rm det}(\mathbf{X},T) + R_\zeta(\mathbf{X},T), \tag{7}$$

where

$$\zeta_{\rm det}(\mathbf{X},T) = \langle\zeta_1(\mathbf{X},T)\,|\zeta_1(\mathbf{0},0) = h\rangle = \frac{h}{\sigma^2}\Psi(\mathbf{X},T). \tag{8}$$

Thus, ζ_c represents the conditional process $\zeta_1(\mathbf{X},T)\,|\zeta_1(\mathbf{0},0) = h$ and $\zeta_{\rm det}$ is its conditional expectation. As $h/\sigma \to \infty$ in (7), the residual R_ζ becomes negligible relative to the first term, leading to

$$\zeta_c(\mathbf{X},T) = \zeta_{\rm det}(\mathbf{X},T) + O(h^0). \tag{9}$$

Thus, a high local maximum also corresponds to a local wave crest since ζ_{det} attains its absolute maximum at $(T = 0, \mathbf{X} = \mathbf{0})$. Moreover, ζ_c can be also interpreted as the wave surface around a randomly chosen large crest (Lindgren 1970,1972; Boccotti 2000) if h is assumed to be a random variable described by the Rayleigh probability density

$$p_R(h) = \frac{EX(h)}{EX_+} = \exp\left(-\frac{h^2}{2\sigma^2}\right)\frac{h}{\sigma^2}, \qquad (10)$$

where $EX(h)dh$ represents the expected number per unit time of local maxima of the surface displacement recorded at $\mathbf{X} = 0$ and $T = 0$, whose amplitudes lie between h and $h + dh$. This model is the first order regression approximation of the wave process locally near a randomly chosen large crest (Rychlik 1987, Lindgren & Rychlik 1991). The random process (9) represents a family of wave groups which evolves in space and time attaining the largest crest at $\mathbf{X} = 0$ and $T = 0$. Thus, $\zeta_c \approx \zeta_{\text{det}}$ is asymptotically correct to $O(h)$, and it either represents the wave field locally to a given crest height h, or it defines the conditional process for the dynamics in space-time around a randomly chosen crest if h is interpreted as a Rayleigh distributed random variable.

Our principal interest is in two-dimensional crests of the surface displacement, viz. the largest maxima of a surface time series recorded at a fixed point. Therefore, h is Rayleigh distributed. In general though, (9) can be also interpreted as a snapshot of the wave surface locally around a three-dimensional crest at a particular instant of time. In this case, the variable h is not distributed according to the Rayleigh law. In fact, in Gaussian processes the crest height follows the Rayleigh distribution by virtue of the one-to-one correspondence between each h-upcrossing point and a maximum of amplitudes greater than a large threshold h. In multi-dimensional Gaussian fields, this one-to-one correspondence is lost since h-upcrossings are level curves. In this case, an appropriate definition of a h-upcrossing is necessary, yielding an asymptotic form of the crest distribution different from the Rayleigh law (Adler 1981, Adler & Hasofer 1976, Wilson & Adler 1982, Piterbarg 2003).

4. Stochastic wave groups

We now extend and generalize some results of Boccotti (1989) to wave groups with large crests. Boccotti considers, as $H/\sigma \to \infty$, the conditional process

$$\zeta_b(\mathbf{X}, T) = (\zeta_1(\mathbf{X}, T)\,|\zeta_1(\mathbf{X}, 0) = H/2,\ \zeta_1(\mathbf{X}, T_w) = -H/2)$$

where H represents the largest wave height in the group, and $T_w = T^* + O(H^{-1})$ is the time-lag between the crest of the wave and the following trough. In particular, Boccotti derives the asymptotic form of the statistical distribution of H (Boccotti 1989, 2000, see also Tayfun & Fedele 2007b). Boccotti (2000) and later Fedele (2007b) both show that largest wave heights occur not as waves reach the apex of a group, but just after they pass it. In the present case, we draw upon Boccotti's concepts but consider the largest crest which occurs at the apex of a wave group. Specifically, we examine the conditional process $\zeta_c(\mathbf{X}, T)$ around a large crest, and analyse its $O(h^0)$-random residual R_ζ and thus devise a new formulation of wave groups in Gaussian seas. First, the

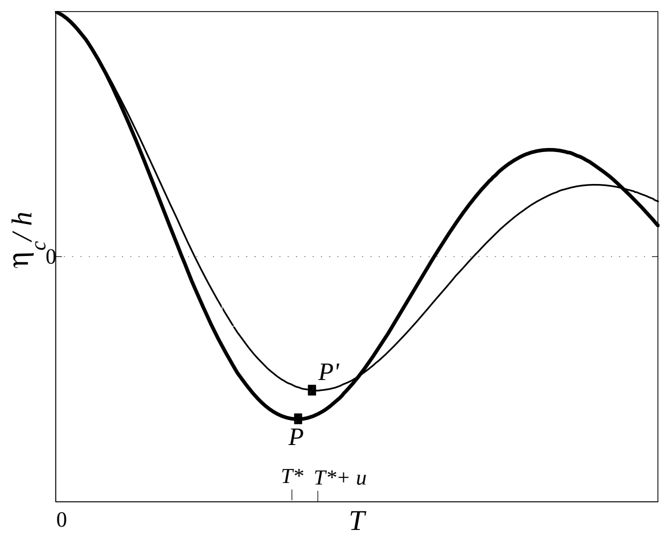


Figure 1.

wave profile $\eta_c(T)$ at $\mathbf{X} = \mathbf{0}$ is expressed in terms of an $O(h)$ contribution $\eta_{\rm det}(T) = \zeta_{\rm det}(\mathbf{0}, T)$ and the random residual $r(T) = R_\zeta(\mathbf{0}, T)$ of $O(h^0)$ as

$$\eta_c(T) = \eta_{\rm det}(T) + r(T) \tag{11}$$

where

$$\eta_{\rm det}(T) = \zeta_{\rm det}(\mathbf{0}, T) = h\frac{\psi(T)}{\sigma^2}.$$

We can now determine the effects of the residual $r(T)$ on η_c. Specifically, as $h/\sigma \to \infty$, with probability approaching 1, the surface profile locally near a large crest tends to assume the shape given by $\eta_{\rm det}(T)$ (see Lindgren 1972, Boccotti 2000). The latter represents a wave profile with a crest of amplitude h at time $T = 0$ followed by a local minimum of amplitude $\eta_{\rm det}(T^*)$ at $T = T^*$, with T^* being the abscissa of the first local minimum of $\psi(T)$ (point P in figure 1). Further, when the absolute minimum of $\psi(T)$ occurs at $T = T^*$, then $\eta_{\rm det}(T)$ represents a large wave with period $T_h \approx 2T^*$ and a crest-to-trough amplitude H given by

$$H = h\left(1 - \frac{\psi(T^*)}{\sigma^2}\right).$$

For large h, the wave trough of the profile $\eta_c(T)$ following the crest of amplitude h shall now occur at time $T = T^* + u$, shown as point $P\prime$ in figure 1, with u being random. To obtain an explicit expression for u, we set the time derivative of the profile η_c equal to zero at $T = T^* + u$ and use the expansion

$$\dot{\eta}_c(T^* + u) = \ddot{\eta}_{\rm det}(T^*)u + \dot{r}(T^*) + O(u^2) = 0.$$

Thus,

$$u=-\frac{\dot{r}(T^*)}{\ddot{\eta}_{\rm det}(T^*)}+O(u^2h^{-1}). \tag{12}$$

Note that u is of $O(h^{-1})$ because the residual process $\dot{r}(T^*)$ is of $O(h^0)$ and $\ddot{\eta}_{\rm det}(T^*)$ is of $O(h)$. Thus, the residual terms in (12) are of $O(h^{-3})$ and negligible. By expansion, the value of the surface displacement $\eta_c(T)$ at T^*+u is then given by

$$\eta_c(T^*+u)=\eta_{\rm det}(T^*)+\frac{1}{2}\ddot{\eta}_{\rm det}(T^*)u^2+r(T^*)+O(h^{-2}). \tag{13}$$

Because u is of $O(h^{-1})$, it follows that

$$\eta_c(T^*+u)=\eta_{\rm det}(T^*)+\Delta+O\left(h^{-1}\right),$$

where $\Delta=r(T^*)$ is the residual at T^* of $O(h^0)$. Correct to the same order, $\eta_c(T^*)=\eta_c(T^*+u)$. Thus, as $h/\sigma\to\infty$, a crest of amplitude h that occurs at $T=0$, is followed after a time lag T^*+u by a trough, and $\eta_c(T)$ and its first time derivative $\dot{\eta}_c(T)$ at $T=T^*$ attain values given, correct to $O\left(h^0\right)$, by

$$\begin{aligned}\eta_c(T^*) &= \eta_{\rm det}(T^*)+\Delta+O\left(h^{-1}\right), \\ \dot{\eta}_c(T^*) &= -\ddot{\eta}_{\rm det}(T^*)u+O\left(h^{-1}\right).\end{aligned} \tag{14}$$

Conversely, if the conditions in (14) hold, then a crest of amplitude h at time $T=0$ is followed by a trough at time $T=T^*+u$.

Next, we describe $\eta_c(T)$ locally near a randomly chosen crest, using a regression approximation (Rychlik 1987, Lindgren & Rychlik 1991). In particular, such an approximation must satisfy the conditions in (14), viz. it must have a local maximum of amplitude h at time $T=0$ followed by a trough of amplitude $\eta^*_{\rm det}+\Delta$ at $T=T^*+u$. For linear Gaussian functions, an approximation to $\eta_c(T)$ satisfying both conditions exactly is given by

$$\eta_c(T)=A\psi(T)+B\psi(T-T^*-u), \tag{15}$$

where

$$A=\frac{\psi(0)h-\psi(T^*+u)\cdot(\psi(T^*)h+\Delta)}{\psi^2(0)-\psi^2(T^*+u)},\qquad B=\frac{\psi(0)\cdot[\psi(T^*)h+\Delta]-\psi(T^*+u)h}{\psi^2(0)-\psi^2(T^*+u)}.$$

To $O(h^0)$, u drops out, and $\eta_c(T)$ becomes

$$\eta_c(T)=\eta_{\rm det}(T)+\frac{\Delta}{\sigma^2}\frac{-\psi^*\psi(T)+\psi(T-T^*)}{1-\psi^{*2}}+O(h^{-1}), \tag{16}$$

ignoring terms of $O(h^{-1})$, and $\psi^*\equiv\psi(T^*)/\psi(0)$. With the random residual r of $O(1)$ explicitly determined now, it can be differentiated from $\eta_{\rm det}(T)$ of $O(h)$ in (11).

The necessary conditions for the existence of a local maximum at $T = 0$, i.e. $\ddot{\eta}_c(0) < 0$, and a local minimum at $T = T^*$, i.e. $\ddot{\eta}_c(T^* + u) > 0$, yield the following inequality constraint:

$$h > \Delta \ \min\left(\frac{\psi^* + \ddot{\psi}^*}{1-\psi^{*2}}, \frac{\psi^* + 1/\ddot{\psi}^*}{1-\psi^{*2}}\right), \tag{17}$$

where $\ddot{\psi}^* \equiv \ddot{\psi}(T^*)/\left|\ddot{\psi}(0)\right|$. If the surface spectral density is defined over a compact support in the frequency domain, then the moments m_j for $j > 3$ are finite, and $\eta(t)$ is differentiable at least twice. Thus, the terms appearing in (17) are bounded, and since Δ is of $O(1)$, h can be chosen sufficiently large to satisfy the above inequality, viz. $\Delta/h \sim O(h^{-1})$.

It is straightforward to extend the above time formulation to the space-time domain obtaining a new approximation of the stochastic wave group ζ_c in (7) in the form

$$\zeta_c(\mathbf{X}, T) = \zeta_{\text{det}}(\mathbf{X}, T) + \frac{\Delta}{\sigma^2}\frac{-\psi^* \ \Psi(\mathbf{X}, T) + \Psi(\mathbf{X}, T - T^*)}{1-\psi^{*2}} + O(h^{-1}). \tag{18}$$

Evidently, this is an improved expression of the wave surface locally around a large crest correct to $O(h^0)$, where the random residual R_ζ in (7) is explicitly determined as $\Delta/h \to 0$, and terms of $O(h^{-1})$ have been neglected.

For a given h, ζ_c is the conditional processes locally around a given crest, i.e. $\zeta_1(\mathbf{X}, T)\,|\zeta_1(\mathbf{0}, 0) = h$. If we instead interpret h and Δ as random, then ζ_c identifies a *stochastic wave group*, describing the dynamics locally around a randomly chosen crest.

The joint pdf of the random variables h, Δ and u, as $h/\sigma \to \infty$, is given by (Boccotti 1989)

$$p(h, \Delta, u) = h\frac{\exp\left(-\frac{h^2}{2\sigma^2} - \frac{\Delta^2}{2\sigma^2(1-\psi^{*2})} - \frac{h^2u^2\left|\ddot{\psi}(0)\right|}{2\sigma^4\gamma^2}\right)}{\sigma^2 2\pi\sqrt{\sigma^2(1-\psi^{*2})\frac{\sigma^4\gamma^2}{h^2\left|\ddot{\psi}(0)\right|}}}. \tag{19}$$

The probability $p(h, \Delta, u)dhd\Delta du$ can be interpreted as the fraction of realizations of linear ζ_1 with a large crest of amplitude h occurring at some t_0, proceeded by a trough of amplitude $\eta^*_{\text{det}} + \Delta$ at $T^* + u$. As $h/\sigma \to \infty$, each realization of ζ_1 resembles a wave group evolving in accordance with (18).

The joint probability density of h and Δ follows from (19), with $\xi \to \infty$, as

$$p_{\xi,\tilde{\Delta}}(\xi, \tilde{\Delta}) = \int_{-\infty}^{\infty} p(\xi, \tilde{\Delta}, u)du = p_\xi(\xi)p_{\tilde{\Delta}}(\tilde{\Delta}), \tag{20}$$

where

$$p_\xi(\xi) = \xi \exp\left(-\frac{\xi^2}{2}\right), \qquad p_{\tilde{\Delta}}(\tilde{\Delta}) = \frac{\exp\left(-\frac{\tilde{\Delta}^2}{2(1-\psi^{*2})}\right)}{\sqrt{2\pi(1-\psi^{*2})}}, \tag{21}$$

and $\xi = h/\sigma$ and $\tilde{\Delta} = \Delta/\sigma$ are dimensionless variables. Thus, ξ and $\tilde{\Delta}$ are independent. Note that with h given in (18), averaging over $\tilde{\Delta}$ yields the conditional mean

$$\langle\zeta_1(\mathbf{X}, T)\,|\zeta_1(\mathbf{0}, 0) = h\rangle = \langle\zeta_c(\mathbf{X}, T)\rangle_{\tilde{\Delta}} = \zeta_{\text{det}}(\mathbf{X}, T),$$

as expected.

5. Nonlinear stochastic groups and large crests

Herein, we examine the nonlinear evolution of the stochastic wave group $\zeta_c(\mathbf{X},T)$ in second order random seas, explaining how its linear structure is distorted by nonlinearities. We argue that, prior to focussing, the nonlinear wave group tends to reflect the characteristics of a well defined Gaussian group that can be defined by (18). Due to nonlinearities, the Gaussian group will nonlinearly evolve forming an extreme crest with a different amplitude $h_{nl} > h$, h being the linear crest height. The relationship between h and h_{nl} is given by the nonlinear conditional process $\zeta_{nc} = (\zeta\,|\zeta_1 = \zeta_c)$. For large waves, ζ_{nc} is equivalent to $\zeta(\mathbf{X},T)\,|\zeta_1(\mathbf{0},0) = h$, drawing upon Fedele & Arena (2005). The nonlinear mapping $f(\zeta_1)$ between ζ_1 and ζ is known from (1),(2) and (3), and it yields

$$\zeta_{nc} = (\zeta(\mathbf{X},T)\,|\zeta_1(\mathbf{0},0) = h) = f(\zeta_c). \tag{22}$$

We recall that h and $\tilde{\Delta}$ are random variables with the joint pdf (20), and $\zeta_{nc} = f(\zeta_c)$ is the nonlinear stochastic group which describes the wave dynamics locally around a randomly chosen crest. To compute $f(\zeta_c)$, we note that (1) along with (2) and (3) *not only* defines weakly nonlinear random waves *but also* the general analytical solution for the second order surface displacement, if the amplitudes c_i and the phases $\boldsymbol{\theta}_i$ are regarded as deterministic variables. Thus, if we set in (1) the linear component ζ_1 of the surface ζ equal to ζ_c in (18), it follows that

$$\zeta_{nc} = f(\zeta_c) = \zeta_c + \frac{h^2}{4\sigma^4}\mathcal{F} + \frac{h\Delta}{2\sigma^4}\frac{-\psi^*\mathcal{F}+\mathcal{G}}{1-\psi^{*2}} + O\left(\Delta^2\right), \tag{23}$$

where

$$\mathcal{F}(\mathbf{X},T) = \int S_1 S_2\left(A_{12}^{+}\cos\left(\beta_{12}^{+}\right) + A_{12}^{-}\cos\left(\beta_{12}^{-}\right)\right) d\mathbf{k}_1 d\mathbf{k}_2, \tag{24}$$

$$\mathcal{G}(\mathbf{X},T) = \int S_1 S_2\left[A_{12}^{+}\cos\left(\beta_{12}^{+} + \omega_1 T^*\right) - A_{12}^{-}\cos\left(\beta_{12}^{-} + \omega_1 T^*\right)\right] d\mathbf{k}_1 d\mathbf{k}_2,$$

with the abbreviated notation $S_j = S\left(\mathbf{k}_j\right)$, $j = 1,2$, and

$$A_{12}^{\pm} = A^{\pm}(\mathbf{k}_1,\mathbf{k}_2), \qquad \beta_{12}^{\pm} = (\mathbf{k}_1 \pm \mathbf{k}_2)\cdot\mathbf{X} - (\omega_1 \pm \omega_2)\,T.$$

6. Crest Statistics from nonlinear groups

The highest crest of the nonlinear stochastic wave group ζ_{nc} also occurs at $\mathbf{X} = 0$ and $T = 0$ correct to $O(\mu_1)$, with a dimensionless amplitude $\xi_{\max} = h_{nl}/\sigma$ given by

$$\xi_{\max} = \xi + \frac{\mu}{2}\xi^2 + \frac{\mu K}{2}\tilde{\Delta}\xi, \tag{25}$$

where

$$\mu = \frac{\lambda_3}{3} = \frac{\langle \zeta(t)^3 \rangle}{3\sigma^3}, \qquad K = 2\frac{-\psi^* + \kappa_1}{1 - \psi^{*2}} \tag{26}$$

with

$$\kappa_1 = \frac{\mathcal{G}(\mathbf{0},0)}{\mathcal{F}(\mathbf{0},0)} = \frac{\langle \zeta_1(\mathbf{0},t)\zeta_2(\mathbf{0},t)\ \zeta_1(\mathbf{0},t+T^*) \rangle}{2\mu}, \tag{27}$$

and λ_3 stands for the skewness coefficient of surface elevations correct to $O\left(\mu_1\right)$.

6.1. Recovering the Tayfun model

As $\xi \to \infty$, and ignoring terms of $O(\tilde{\Delta})$ in (25) we obtain

$$\xi_{\max} = \xi + \frac{\mu}{2}\xi^2. \tag{28}$$

Thus, the probability of exceedance for the nonlinear wave crest height $\xi_{\max}$ readily follows from the Rayleigh distribution of ξ as

$$\Pr\left\{\xi_{\max} > \lambda\right\} = \exp\left(-\frac{\xi(\lambda)^2}{2}\right), \tag{29}$$

where ξ follows from (28) with $\xi_{\max} = \lambda$. The result stated in (28) is valid for directional waves in waters of finite depth irrespective of the spectral bandwidth. It also agrees with the original narrow-band model of Tayfun (1980) appropriate to long-crested deep-water waves. In fact, Tayfun proposed the same expression for the crest height ξ_c, replacing μ with

$$\mu_m = m_0^{1/2}\frac{\omega_m^2}{g}. \tag{30}$$

This parameter is also a measure of steepness for unidirectional short-crested waves in deep water if one neglects the frequency-difference contributions. If the latter are included, then the parameter μ in (26) can be expressed explicitly for various theoretical spectra in the form

$$\mu = \mu_m(1 - \gamma\nu + \nu^2), \tag{31}$$

where, for example, $\gamma = 2/\sqrt{3} = 1.1547$ for rectangular spectra, and $\gamma = 2/\sqrt{\pi} = 1.1284$ for Gaussian spectra. For oceanic applications we shall assume that $\gamma = 1$ and define for the deep-water case

$$\mu_a = \mu_m(1 - \nu + \nu^2) \tag{32}$$

both for unidirectional waves and as an approximate upper bound for directional waves. As an alternative, Tayfun (2006) estimates μ from Forristall's Weibull model (Forristall 2000) as

$$\mu_{Fj} = 16\frac{\alpha_j^3}{\beta_j}\Gamma\left(\frac{3}{\beta_j}\right) - \frac{1}{4}\sqrt{\frac{\pi}{2}}, \tag{33}$$

where α_j and β_j represent the parameters of the Weibull distribution

$$\Pr\{\xi_{\max} > x\} = \exp\left[-\left(\frac{x}{4\alpha_j}\right)^{\beta_j}\right] \tag{34}$$

used by Forristall to fit (34) to simulations of second order random seas, and $j = 2$ or 3 corresponding to unidirectional (2D) or directional (3D) waves, respectively. Thus, not only for narrow-band waves, but also for high crest amplitudes, i.e. as $h/\sigma \to \infty$, crest heights are described by (28), with μ defined as $\lambda_3/3$ under the most general conditions. Moreover, all crest-height statistics depend clearly on a few integral properties such as m_0 (or σ), ω_m, ν and/or λ_3. These are easily estimated from a surface time series.

6.2. Generalizing the Tayfun model

As $\xi \to \infty$, and when we retain all the terms in (25), then

$$\Pr(\xi_{\max} > \lambda) = \int_{-\infty}^{\infty} \Pr\left\{\xi > \xi(\lambda, w) \middle| \tilde{\Delta} = w\right\} \; p_{\tilde{\Delta}}(\tilde{\Delta} = w) dw,$$

where $\xi(\lambda, w)$ follows from (25) with $\xi_{\max} = \lambda$ and

$$\Pr\left\{\xi > \xi(\lambda, w) \middle| \tilde{\Delta} = w\right\} = \exp\left[-\frac{\xi(\lambda, w)^2}{2}\right].$$

As $\lambda \to \infty$, an asymptotic solution to the preceding integral can be obtained, if we set

$$\xi\left(\lambda, \tilde{\Delta}\right) = \xi_0(\lambda) + a(\lambda)\,\tilde{\Delta} + O\left(\tilde{\Delta}^2\right) \tag{35}$$

where $\lambda = \xi_0 + \frac{\mu}{2}\xi_0^2$ and

$$a(\lambda) = -\frac{K}{2}\frac{\mu\,\xi_0}{1+\mu\,\xi_0}. \tag{36}$$

Because ξ and $\tilde{\Delta}$ are statistically independent and by neglecting $O(\tilde{\Delta}^3)$, it follows after some algebra that

$$\Pr\{\xi_{\max} > \lambda\} = \frac{\exp\left[-\frac{1-\beta(\lambda)}{2}\xi_0^2\right]}{\sqrt{1+\left(1-\psi^{*2}\right)a(\lambda)^2}}, \tag{37}$$

where $\lambda >> 1$ and

$$\beta(\lambda) = \frac{\left(1-\psi^{*2}\right)a(\lambda)^2}{1+\left(1-\psi^{*2}\right)a(\lambda)^2}.$$

We shall refer to this asymptotic result as the generalized Tayfun distribution. Evidently, it is not normalized to unity at the origin since its intended range of validity is over large waves. In the narrow-band limit as $\nu \to 0$, $K \to 0$, and the Tayfun distribution is recovered. An exact expression for K in terms of spectral parameters can be obtained because the frequency-difference terms have been ignored. Under this condition, we recall that α vanishes and K takes the form

$$K = K^{+} = -\frac{\ddot{\psi}^{*} + \psi^{*}}{1 - \psi^{*2}} \tag{38}$$

since $\kappa_1 = \left(\psi^{*} - \ddot{\psi}^{*}\right)/2$. Note further that in general, $\psi^{*} \to -1 + O(\nu)$ and $\ddot{\psi}^{*} \to 1 - O(\nu)$. Thus, if we include the frequency-difference terms, then $|K| \leq |K^{+}|$, and as $\nu \to 0$, $K^{+} \to K \to 0$.

7. Crest statistics from Breitung's asymptotics

Recently, Baxevani et al. (2005) improved the asymptotic formula of h-upcrossings in Gaussian multivariate processes derived by Breitung and Richter (1996). They presented a rigorous view of the FORM (first order feliability method) and SORM (second order reliability method) used in applications to compute crest exceedances. We restrict our attention to FORM, and consider the hypersurface in the Euclidean space R^{2N} defined by the second order surface displacement of Eq.(1) written in terms of the column vectors $\mathbf{p} = (p_1, p_2, ..., p_N)$ and $\mathbf{q} = (q_1, q_2, ..., q_N)$, where $\{p_n\}$ and $\{q_n\}$ represent the sets of the spectral components of the linear surface displacement ζ_1 and its Hilbert transform respectively (see Baxevani et al. 2005 for details), that is

$$\lambda = \zeta_1(\mathbf{p}, \mathbf{q}) + \zeta_2(\mathbf{p}, \mathbf{q}), \tag{39}$$

with λ being a fixed threshold. Moreover the components of the vectors $\mathbf{p}, \mathbf{q}$ are independent Gaussian variables with zero mean and unit variance. Then the crest exceedance in FORM is given by

$$\Pr\{\xi_{\max} > \lambda\} = \exp\left[-\frac{g(\lambda)^2}{2}\right] \tag{40}$$

where $g(\lambda) = \|\mathbf{z}_{\min}\|$ is the minimal distance between the origin and the point $P_{\min} \in \mathbb{R}^{2N}$ identified by the column vector $\mathbf{z}_{\min} = [\tilde{\mathbf{p}}, \tilde{\mathbf{q}}]$ on the hypersurface Γ defined by (39). Here, $\|\mathbf{z}_{\min}\| = \sqrt{\tilde{\mathbf{p}}^T \tilde{\mathbf{p}} + \tilde{\mathbf{q}}^T \tilde{\mathbf{q}}}$ is the classical Euclidean norm of the vector $\mathbf{d} \in \mathbb{R}^{2N}$, and T signifies the transpose. In this case, the solution for $\mathbf{z}_{\min}$ can be obtained numerically by using standard optimization techniques (Tromans and Vanderschuren, 2004). If we compare the crest exceedance distribution of (29) with the FORM distribution of (40), it is seen that the vector entries $(\tilde{\mathbf{p}}, \tilde{\mathbf{q}})$ of the optimal vector $\mathbf{z}_{\min}$ for very large N, can be written, with a little abuse of notation, as

$$\tilde{\mathbf{p}} = \left[\xi_0(\lambda)\frac{\sqrt{2S(\mathbf{k}_1)d\mathbf{k}}}{\sigma}, ..., \xi_0(\lambda)\frac{\sqrt{2S(\mathbf{k}_N)d\mathbf{k}}}{\sigma}\right], \qquad \tilde{\mathbf{q}} = \mathbf{0} \tag{41}$$

where $\lambda = \xi_0 + \frac{\mu}{2}\xi_0^2$. The Lagrange multiplier method and some algebra will show that $P_{\min} \in \mathbb{R}^{2N}$ pointed by the vector $\tilde{\mathbf{d}}$ is indeed the point on the hypersurface (39) at minimal distance from the origin, correct to $O(\mu\xi_0)$.

For simplicity, we shall prove the above statement for narrow-band waves only. In this case, the wave surface is given by (Tayfun 1980)

$$\zeta = \zeta_1 + \frac{\mu}{2}\left(\zeta_1^2 - \hat{\zeta}_1^2\right)$$

where $\hat{\zeta}_1$ is the Hilbert transform respect to time of ζ_1. Thus, from (39)

$$\zeta_1(\mathbf{p},\mathbf{q}) = \mathbf{z}^T\mathbf{p}, \qquad \zeta_2(\mathbf{p},\mathbf{q}) = \frac{\mu}{2}\left(\mathbf{p}^T\mathbf{z}\mathbf{z}^T\mathbf{p} - \mathbf{q}^T\mathbf{z}\mathbf{z}^T\mathbf{q}\right)$$

where μ is the steepness of the waves, and the column vector $\mathbf{z}$ has entries given by the spectral components $(\mathbf{z})_j = \sqrt{2S(\mathbf{k}_j)d\mathbf{k}}\,/\sigma$ such that $\mathbf{z}^T\mathbf{z} = 1$. Consider now the Lagrangian function

$$\mathcal{L} = \frac{1}{2}\left(\mathbf{p}^T\mathbf{p} + \mathbf{q}^T\mathbf{q}\right) + \chi\left(x - \mathbf{z}^T\mathbf{p} - \frac{\mu}{2}\left(\mathbf{p}^T\mathbf{z}\mathbf{z}^T\mathbf{p} - \mathbf{q}^T\mathbf{z}\mathbf{z}^T\mathbf{q}\right)\right)$$

where the Lagrange multiplier χ is introduced in order to minimize over the hypersurface Γ in (39). Some nontrivial algebra shows that the gradients $\frac{\partial\mathcal{L}}{\partial\mathbf{p}}$ and $\frac{\partial\mathcal{L}}{\partial\mathbf{q}}$ vanish for the critical vectors $(\tilde{\mathbf{p}},\tilde{\mathbf{q}})$ given by

$$\tilde{\mathbf{p}} = \rho\xi_c\mathbf{z}, \qquad \tilde{\mathbf{q}} = \mathbf{0}. \tag{42}$$

where

$$\rho = \frac{1}{1+\frac{\mu\xi_0}{2}} + \left(1+\frac{\mu\xi_0}{2}\right)\frac{\mu\xi_0}{2} = 1 + \frac{1}{2}\mu^2\xi_0^2 + O\left(\mu^3\xi_0^3\right).$$

Thus, the crest exceedance distribution is then given by

$$\Pr\left\{\xi_{\max} > \lambda\right\} = \exp\left[-\frac{\tilde{\mathbf{p}}^T\tilde{\mathbf{p}} + \tilde{\mathbf{q}}^T\tilde{\mathbf{q}}}{2}\right] = \exp\left[-\frac{\xi_0^2}{2}\rho^2\right]. \tag{43}$$

Also, one can show that the critical point $(\tilde{\mathbf{p}},\tilde{\mathbf{q}})$ on the hypersurface Γ of Eq. (42) is at minimal distance $g_{\min}(\lambda) = \rho\xi_0$ from the origin. Note finally that the Breitung distribution (43) coincides with the Tayfun distribution (29) correct to $O\left(\mu\xi_0\right)$.

8. Data Comparisons

In the following we shall present results of the analysis of two data sets. The first set comprises 9 hours of measurements gathered during a severe storm in January, 1993 with a Marex radar from the Tern platform located in the northern North Sea in 167 m water depth. The second set represents nearly 9 hours of measurements gathered in January, 1998 with a Baylor wave staff from Meetpost Noordwijk in 18 m average water depth in the southern North Sea. Forristall elaborates the nature of the first data, hereafter simply referred to as Tern. The second set is from Wave Crest Sensor Intercomparison Study and we shall call it as WACSIS for brevity (Forristall et al. 2002). The spectral properties of Tern are characterized by $\sigma = 3.02\ m$, $\nu = 0.629$ and $\lambda_3 = 0.174$ observed, and for WACSIS by $\sigma = 0.981\ m$, $\nu = 0.490$ and observed $\lambda_3 = 0.231$.

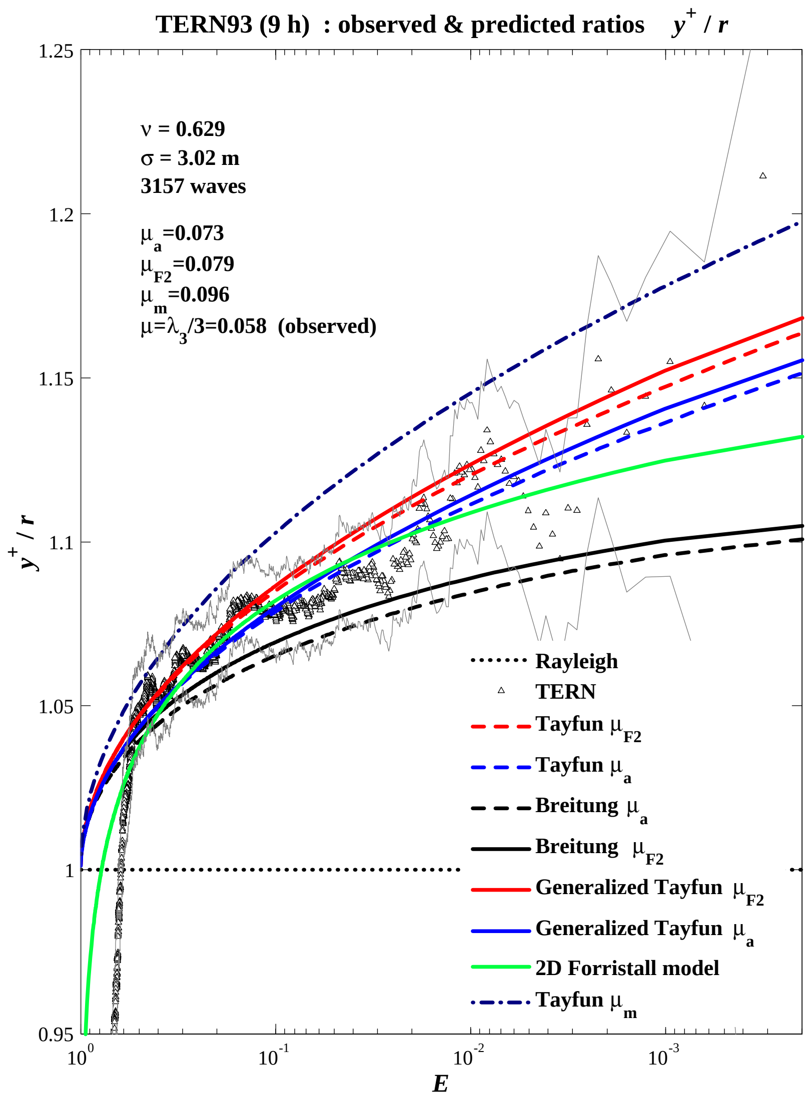
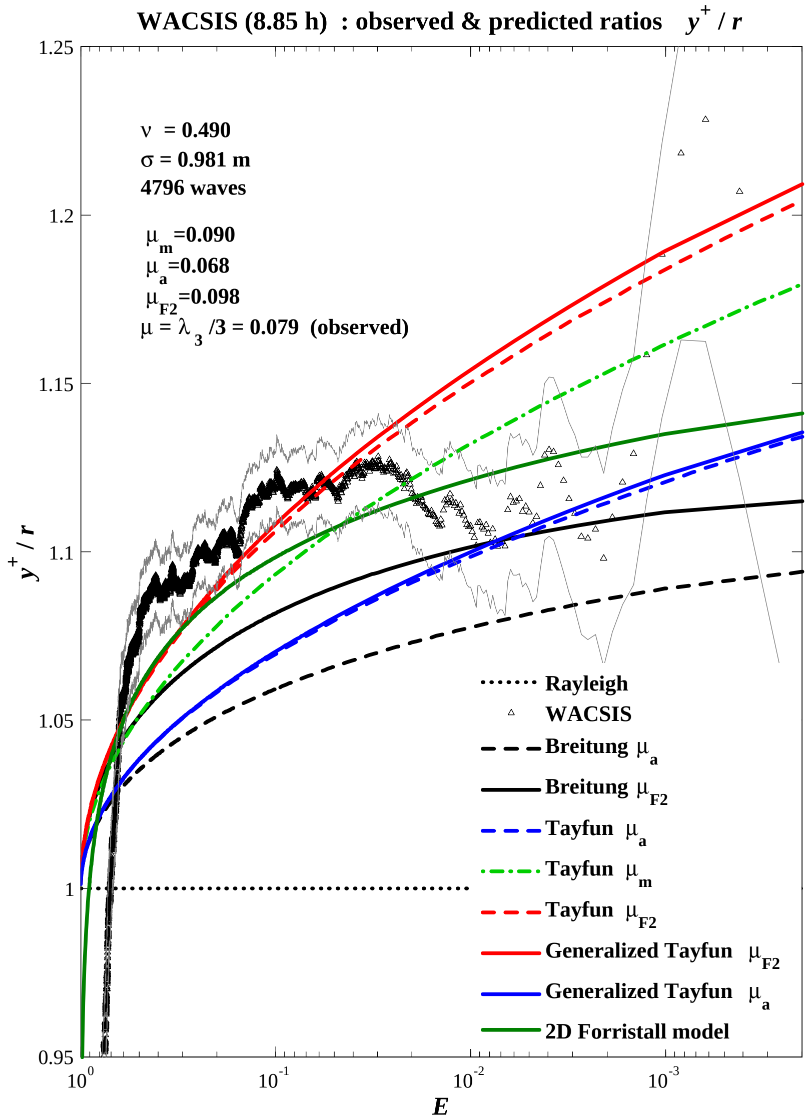

TERN93 (9 h) : observed & predicted ratios y+ / r
ν = 0.629
σ = 3.02 m
3157 waves
μa=0.073
μF2=0.079
μm=0.096
μ=λ3/3=0.058 (observed)
Rayleigh
TERN
Tayfun μF2
Tayfun μa
Breitung μa
Breitung μF2
Generalized Tayfun μF2
Generalized Tayfun μa
2D Forristall model
Tayfun μm
y+ / r
E
1.25
1.2
1.15
1.1
1.05
1
0.95
10^0
10^-1
10^-2
10^-3

Figure 2.



WACSIS (8.85 h) : observed & predicted ratios y+ / r
ν = 0.490
σ = 0.981 m
4796 waves
μm=0.090
μa=0.068
μF2=0.098
μ = λ3 /3 = 0.079 (observed)
Rayleigh
WACSIS
Breitung μa
Breitung μF2
Tayfun μa
Tayfun μm
Tayfun μF2
Generalized Tayfun μF2
Generalized Tayfun μa
2D Forristall model
y+ / r
E

Figure 3.

In figure 2, the ratio y^+/r of nonlinear crests y^+ to the corresponding linear Rayleigh-distributed crests defined as $r = \sigma\sqrt{-2\ln P}$, is plotted for Tern and compared against the original Tayfun model ($\mu \simeq \mu_m = 0.096$ from (30)), the approximate model ($\beta = 1$ in (31) and $\mu \simeq \mu_a = 0.073$), the 2D Tayfun-Forristall model ($\mu \simeq \mu_{F_2} = 0.079$, see (33)), the 2D Weibull model of Forristall (see 34, $\alpha_2 = 0.3715$, $\beta_2 = 1.8683$), the generalized Tayfun models ($K = 0.394$) from (37) based on the estimates μ_{F_2} and μ_a, respectively, and finally the Breitung's approximation of (43). It is evident that the 2D Tayfun-Forristall model describes the observed data extremely well, whereas the original Tayfun model overestimates the observed crest heights, but it also serves as a somewhat conservative upper bound to the distribution of crest heights over high waves. Evidently, the improvement of the new distributions (Breitung and generalized Tayfun models) is essentially negligible. Similar results also hold for WACSIS, as shown in figure 3.

9. CONCLUSIONS

We have presented a complete theory for second order random waves and their statistics based on the concept of stochastic wave group. This theory provides a framework for predicting the expected shape of large waves and the statistics of large wave crests quite accurately within the context of second-order random wave theory and it can be extended to analyze the properties of third order nonlinear random waves (Fedele 2006a,2006c). We have proposed a generalization of the Tayfun model valid under general conditions in transitional or deep water depths, and that depends upon spectral parameters easily estimated from wave hindcasts. Furthermore, we derive an exact closed form solution for the crest distribution of FORM based on the Breitung's asymptotics (Breitung and Richter, 1996).

The generalized Tayfun model and the FORM model although compare well with oceanic measurements gathered from the Tern platform in the northern North Sea (Tern) and with a Baylor wave staff in the southern North Sea (WACSIS), do not really improve upon the original model of Tayfun, which thus can be regarded suitable for describing crest statistics for engineering applications.

10. Acknowledgements

The authors thank George Forristall for providing both the Tern and WACSIS data.

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