Design under Uncertainty using a Combination of Evidence Theory and a Bayesian Approach

Jun Zhou and Zissimos P. Mourelatos
Mechanical Engineering Department
Oakland University, Rochester, MI 48309
email: mourelat@oakland.edu

Abstract: Early in the engineering design cycle, it is difficult to quantify product reliability due to insufficient data or information to model uncertainties. Probability theory can not be therefore, used. Design decisions are usually based on fuzzy information which is imprecise and incomplete. Various design methods such as Possibility-Based Design Optimization (PBDO) and Evidence-Based Design Optimization (EBDO) have been developed to systematically treat design with non-probabilistic uncertainties. In practical engineering applications, information regarding the uncertain variables and parameters may exist in the form of sample points, and uncertainties with sufficient and insufficient information may exist simultaneously. Most of the existing optimal design methods under uncertainty can not handle this form of incomplete information. They have to either discard some valuable information or postulate the existence of additional information. In this paper, a design optimization method is proposed based on evidence theory, which can handle a mixture of epistemic and random uncertainties. Instead of using "expert" opinions to form the basic probability assignment, a Bayesian approach is used with a limited number of sample points. A pressure vessel example demonstrates the merit of the proposed design optimization method. The results are compared with those from existing design methodologies under uncertainty.

1. INTRODUCTION

Engineering design under uncertainty has recently gained a lot of attention. Uncertainties are usually modeled using probability theory. In Reliability-Based Design Optimization (RBDO), variations are represented by standard deviations which are typically assumed constant, and a mean performance is optimized subject to probabilistic constraints [1-5]. In general, probability theory is very effective when sufficient data is available to quantify uncertainty using probability distributions. However, when sufficient data is not available or there is lack of information due to ignorance, the classical probability methodology may not be appropriate. For example, during the early stages of product development, quantification of the product's reliability or compliance to performance targets is practically very difficult due to insufficient data for modeling the uncertainties. A similar problem exists when the reliability of a complex system is assessed in the presence of incomplete information on the variability of certain design variables, parameters, operating conditions, boundary conditions etc.

Uncertainties can be classified in two general types; aleatory (stochastic or random) and epistemic (subjective) [6-10]. Aleatory or irreducible uncertainty is related to inherent variability and is efficiently modeled using probability theory. However, when data is scarce or there is lack of information, the probability theory is not useful because the needed probability distributions cannot be accurately constructed. In this case, epistemic uncertainty, which describes subjectivity, ignorance or lack of information, can be used. Epistemic uncertainty is also called reducible because it can be reduced with increased state of knowledge or collection of more data.

Formal theories to handle uncertainty have been proposed in the literature including evidence theory (or Dempster – Shafer theory) [9, 10], possibility theory [11] and interval analysis [12]. Two large classes of fuzzy measures, called belief and plausibility measures, respectively, characterize the mathematical theory of evidence. They are mutually dual in the sense that one of them can be uniquely determined from the other. Evidence theory uses plausibility and belief (upper and lower bounds of probability) to measure the likelihood of events. When the plausibility and belief measures are equal, the general evidence theory reduces to the classical probability theory. Therefore, the classical probability theory is a special case of evidence theory. Possibility theory handles epistemic uncertainty if there is no conflicting evidence among experts [9]. It uses a special subclass of dual plausibility and belief measures, called possibility and necessity measures, respectively. In possibility theory, a fuzzy set approach is common, where membership functions characterize the input uncertainty [13]. Even if a probability distribution is not available due to limited information, lower and upper bounds (intervals) on uncertain design variables are usually known. In this case, interval analysis [12, 14, 15] and fuzzy set theory [13] have been extensively used to characterize and propagate input uncertainty in order to calculate the interval of the uncertain output. An efficient method for reliability estimation with a combination of random and interval variables is presented in [16]. However, it is not implemented in a design optimization framework. A few design optimization studies have been also reported, where some or all of the uncertain design variables are in interval form [17-19].

Optimization with input ranges has also been studied under the term anti-optimization [20, 21]. Anti-optimization is used to describe the task of finding the "worst-case" scenario for a given problem. It solves a two-level (usually nested) optimization problem. The outer level performs the design optimization while the inner level performs the anti-optimization. The latter seeks the worst condition under the interval uncertainty [21]. A decoupled approach is suggested in [21] where the design optimization alternates with the anti-optimization rather than nesting the two. It was mentioned that this method takes longer to converge and may not even converge at all if there is strong coupling between the interval design variables and the rest of the design variables. A "worst-case" scenario approach using interval variables has also been considered in multidisciplinary systems design [19, 22].

Very recently, possibility-based design algorithms have been proposed [23-25] where a mean performance is optimized subject to possibilistic constraints. It was shown that more conservative results are obtained compared with the probability-based RBDO. A comprehensive comparison of probability and possibility theories is given in [26] for design under uncertainty.

Evidence theory is more general than probability and possibility theories, even though the methodologies of uncertainty propagation are completely different [27, 28]. It can be used in design under uncertainty if limited, and even conflicting, information is provided from experts. Furthermore, the basic axioms of evidence theory allow to combine aleatory (random) and epistemic uncertainty in a straightforward way without any assumptions [28]. Evidence theory however, has been barely explored in engineering design. One of the reasons may be its high computational cost due mainly to the discontinuous nature of uncertainty quantification. Evidence-based methods have been only recently used to propagate epistemic uncertainty [28, 29] in large-scale engineering systems. Although a computationally efficient method is proposed in [28, 29], the design issue is not addressed. We are aware of only one study which propagates epistemic uncertainty using evidence theory and also performs a design optimization [30]. The optimum design is calculated for multidisciplinary systems under uncertainty using a trust region sequential approximate optimization method with surrogate models representing the uncertain measures as continuous functions.

In engineering design, information regarding the uncertain quantities is usually available in the form of a set of finite samples, either from historical data or from actual measurements. These samples are not enough to infer a probability distribution. However, if we collapse them into intervals, we discard valuable information. Collecting more samples is often not possible due to the cost or time limitations. So RBDO, PBDO (Possibility-Based Design Optimization) [23, 24], and EBDO (Evidence-Based Design Optimization) [31] may not satisfactorily address the presence of incomplete information. We must utilize Bayesian inference to estimate design reliability with incomplete information.

Bayesian inference is an approach to statistics in which all forms of uncertainty are expressed in terms of probability. A Bayesian approach starts with the formulation of a model to describe the situation of interest. A prior distribution is formulated over the unknown parameters of the model, which is meant to capture the belief about the situation before seeing the data. Using available data, we apply Bayesian's rule to obtain a posterior distribution for these unknowns, which accounts for both the prior and the new data.

In this paper, a Bayesian approach is used to account for uncertainty in the design when limited information is provided by a limited number of sample points. A Bayesian approach is proposed using the extreme value distribution of the smallest value. The approach can handle both a mixture of epistemic and random uncertainties or pure epistemic uncertainties. The accuracy of predictions improves with the use of more sample points. Previous research, such as in [32-34] illustrate how to use a Bayesian approach in design utilizing the confidence percentile concept. In this paper, the available methodologies are improved by using the extreme value distribution of the smallest value instead of the conventional beta distribution. The extreme value distribution approach is necessary because we have only a small set of sample points which are different at each experiment. A Bayesian approach to design optimization (BADO) using the extreme value distribution is proposed. We show that the optimal design is conservative.

The proposed BADO approach can handle epistemic uncertainties or a mixture of aleatory and epistemic uncertainties. Also if only the number of sample points within a certain range is known instead of the exact distribution of the sample points, we propose a design methodology which combines the evidence theory and the Bayesian approach.

The difference between Possibility-Based Design Optimization (PBDO) and Bayesian-Based Design Optimization (BADO) is in the format of uncertain variables. Possibilistic variables are in the form of intervals and Bayesian uncertain variables are in the form of sample points. The latter provide more information compared with the possibilistic variables. Both of them are based on the confidence percentile concept. In PBDO, a membership function is constructed for each possibilistic variables. The PBDO approach provides a worst-case design because there is a minimal amount of information in the form of intervals. However, more information is available for the Bayesian uncertain variables in the form of sample points. For this reason, we will show that the BADO design is less conservative than the PBDO design.

The paper is organized as follows. Section 2 gives an introduction to the fundamentals of evidence theory. Section 3 presents an overview of an Evidence-Based Design Optimization (EBDO) algorithm. Section 4 presents the proposed Bayesian-Based Design Optimization (BADO) procedure and a methodology to estimate the BPA structure from limited available data using Bayesian statistics. The concepts in section 4 are demonstrated with a pressure vessel example. Comparisons among RBDO, EBDO, PBDO and BADO are also provided in order to demonstrate the value of added information in design. Finally, a summary and conclusions are given in section 5.

2. FUNDAMENTALS OF EVIDENCE THEORY

This section gives the fundamentals of evidence theory, how it can be used in design optimization and an introduction to fuzzy measures. Detailed information is provided in [8, 9, 11, 31, 35]. The role of fuzzy measures and the axiomatic definition of evidence theory are explained.

Evidence theory is based on the belief (*Bel)* and Plausibility (*Pl*) fuzzy measures. Fuzzy measures provide the foundation of fuzzy set theory. Before we introduce the basics of fuzzy measures, it is helpful to review the used notation on set representation. A universe X represents the entire collection of elements having the same characteristics. The individual elements in the universe X are denoted by x, and are usually called singletons. A set A is a collection of some elements of X. All possible sets of X constitute a special set called the power set $\wp(X)$.

A fuzzy measure is defined by a function g: $\wp(X) \rightarrow [0,1]$ which assigns to each crisp subset of X a number in the unit interval [0,1]. The assigned number in the unit interval for a subset $A \in \wp(X)$, denoted by g(A), represents the degree of available evidence or belief that a given element of X belongs to the subset A.

In order to qualify as a fuzzy measure, the function g must have certain properties. These properties are defined by axioms that are *weaker* than the probability theory axioms [8, 9]. Every fuzzy measure obeys the following three axioms:

Axiom 1 (boundary conditions): g($\varnothing$)=0 and g(X)=1.

Axiom 2 (monotonicity): For every $A, B \in \wp(X)$, if $A \subseteq B$, then $g(A) \leq g(B)$.

Axiom 3 (continuity): For every sequence ($A_i \in \wp(X)$, $i=1,2,\ldots$) of subsets of $\wp(X)$, if either $A_1 \subseteq A_i \subseteq \ldots$ or $A_1 \supseteq A_2 \supseteq \ldots$ (i.e., the sequence is monotonic), then $\lim_{i\to\infty} g(A_i) = g(\lim_{i\to\infty} A_i)$.

A belief measure is a function $Bel: \wp(X) \to [0,1]$ which satisfies the three axioms of fuzzy measures and the following additional axiom [9]:

$$Bel(A_1 \cup A_2) \geq Bel(A_1) + Bel(A_2) - Bel(A_1 \cap A_2) \quad . \tag{1}$$

The axiom (1) can be expanded for more than two sets. For $A \in \wp(X)$, $Bel(A)$ is interpreted as the degree of belief, based on available evidence, that a given element of X belongs to the set A.
A plausibility measure is a function

$$Pl : \wp(X) \Rightarrow [0,1] \tag{2}$$

which satisfies the three axioms of fuzzy measures and the following additional axiom [9]

$$Pl(A_1 \cap A_2) \leq Pl(A_1) + Pl(A_2) - Pl(A_1 \cup A_2) \tag{3}$$

Every belief measure and its dual plausibility measure can be expressed with respect to the non-negative function

$$m : \wp(X) \Rightarrow [0,1] \tag{4}$$

such that $m(\varnothing) = 0$ and

$$\sum_{A \in \wp(X)} m(A) = 1 \,. \tag{5}$$

The function m is called Basic Probability Assignment (BPA) due to the resemblance of Eq. (5) with a similar equation for probability distributions. The basic probability assignment $m(A)$ is interpreted either as the degree of evidence supporting the claim that a specific element of X belongs to the set A or as the degree to which we believe that such a claim is warranted. Every set $A \in \wp(X)$ for which $m(A)>0$ is called a focal element of m. Focal elements are subsets of X on which the available evidence focuses; i.e. available evidence exists.

Given a BPA m, a belief measure and a plausibility measure are uniquely determined by

$$Bel(A) = \sum_{B \subseteq A} m(B) \tag{6}$$

and

$$Pl(A) = \sum_{B \cap A \neq 0} m(B) \,. \tag{7}$$

which are applicable for all $A \in \wp(X)$.

In Eq. (6), $Bel(A)$ represents the total evidence or belief that the element belongs to A as well as to various subsets of A. The $Pl(A)$ in Eq. (7) represents not only the total evidence or belief that the element in question belongs to set *A* or to any of its subsets but also the additional evidence or belief associated with sets that overlap with *A*. Therefore,

$$Pl(A) \geq Bel(A). \tag{8}$$

It should be noted that belief and plausibility are complementary in the sense that one of them can be uniquely derived from the other.

Probability theory is a subcase of evidence theory. When the additional axiom of belief measures (see Eq. (1)) is replaced with the stronger axiom

$$Bel(A \cup B) = Bel(A) + Bel(B) \text{ where } A \cap B = \varnothing, \tag{9}$$

we obtain a special type of belief measures which are the classical probability measures. In this case, the right hand sides of Eq. (6) and (7) become equal and therefore,

$$Bel(A) = Pl(A) = \sum_{x \in A} m(x) = \sum_{x \in A} p(x) \tag{10}$$

for all $A \in \wp(X)$, where *p(x)* is the probability distribution function (PDF). Note that the BPA *m(x)* is equal to *p(x).* Therefore with evidence theory, we can simultaneously handle a mixture of input parameters. Some of the inputs can be described probabilistically (random uncertainty) and some can be described through expert opinions (epistemic uncertainty with incomplete data). In the first case, the range of each input parameter will be discretized using a finite number of intervals. The BPA value for each interval must be equal to the PDF area within the interval.

Evidence obtained from independent sources or experts must be combined. If the BPA's m_1 and m_2 express evidence from two experts, the combined evidence *m* can be calculated by the following Dempster's rule of combining [36]

$$m(A) = \frac{\sum_{B \cap C = A} m_1(B) m_2(C)}{1 - K} \quad \text{for} \quad A \neq 0 \tag{11}$$

where

$$K = \sum_{B \cap C = 0} m_1(B) m_2(C) \tag{12}$$

represents the *conflict* between the two independent experts. Dempster's rule filters out any conflict, or contradiction among the provided evidence, by normalizing with the complementary degree of conflict. It is usually appropriate for relatively small amounts of conflict where there is some consistency or sufficient agreement among the opinions of the experts. Yager [10] has proposed an alternative rule of combination where all degrees of contradiction are attributed to total ignorance. Other rules of combining can be found in [36].

2.1. ASSESSING BELIEF AND PLAUSIBILITY WITH DEMPSTER-SHAFER THEORY

The previous section described a methodology to quantify epistemic uncertainty, even when the experts provide conflicting evidence. This section shows how to propagate epistemic uncertainty through a given model (transfer function). We will illustrate that, using the following simple transfer function

$$y = f(a,b) \tag{13}$$

where $a \in A, b \in B$ are two independent input parameters and y is the output. The combined BPA's for both a and b are obtained from Dempster's rule of combining of Eq. (11) if multiple experts have provided evidence for either a or b. With combined information for each input parameter, we define a vector $c = \lfloor a_{ci}, b_{cj} \rfloor$, needed to calculate the output y as

$$C = A \times B = \{c = \lfloor a_{ci}, b_{cj} \rfloor, a_{ci} \in A, b_{cj} \in B\} \tag{14}$$

where subscript c stands for "combined" and i,j indicate focal elements.

Taking advantage of assumed parameter independency, the BPA for c is

$$m_c(h_{ij}) = m(a_{ci})m(b_{cj}) \tag{15}$$

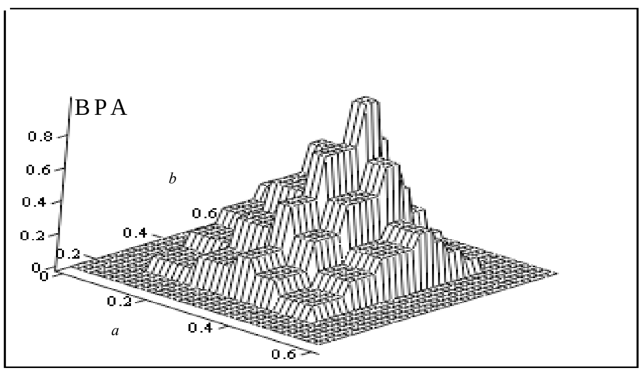

Figure 1. Representative BPA structure for two parameters *a* and *b*.

where $h_{ij} = \lfloor a_{ci}, b_{cj} \rfloor$ and a_{ci}, b_{cj} denote intervals such that $a \in a_{ci}$ and $b \in b_{cj}$. Eq. (15) can be used to calculate the combined BPA structure for the entire domain C. For every $(a,b) \in c \mid c \in C$, needed to evaluate the output y, the combined BPA m_c is used. A representative combined BPA structure is shown in Figure 1.

The Cartesian product C of Eq. (14) is also called frame of discernment (FD) in the literature. It consists of all focal elements (rectangles in Figure 1 with nonzero combined BPA) and can be viewed as the finite sample space in probability theory.

If a domain F is defined as

$$F = \{g : g = f(a,b) - y_0 > 0, (a,b) \in c, c = [a_c, b_c] \subset C\} \tag{16}$$

where y_0 is a specified value, *Bel(F)* and *Pl(F)* can be calculated from Eqs (6) and (7) where set F replaces set A. According to evidence theory, *Bel(F)* and *Pl(F)* bracket the true probability $p_f = P(g > 0)$ [9,27]; i.e.

$$Bel(F) \leq p_f \leq Pl(F). \tag{17}$$

The *Bel (F)* and *Pl (F)* are calculated using Eqs (6) and (7) where set A is equal to set F of Eq. (16) and B is a rectangular domain (focal element) such that $B \subseteq A$ for Eq. (6) and $B \cap A \neq 0$ for Eq. (7). In other words, $B \subseteq A$ means that the focal element must be entirely within the domain *g>0* and $B \cap A \neq 0$ means that the focal element must be entirely or partially within the domain *g>0* (see Fig. 2). In general, in order to identify if a focal element B satisfies $B \subseteq A$ or $B \cap A \neq 0$, the following minimum and maximum values of g must be calculated

$$[g_{\min}, g_{\max}] = \left[\min_{\mathbf{X}} g(\mathbf{X}), \max_{\mathbf{X}} g(\mathbf{X})\right] \tag{18}$$

for $\mathbf{X}^L \leq \mathbf{X} \leq \mathbf{X}^U$ where $(\mathbf{X}^L, \mathbf{X}^U)$ defines the focal element domain. For monotonic functions, the vertex method [37] can be used to calculate the minimum and maximum values in Eq. (18) by simply identifying the minimum and maximum values among all vertices of the focal element domain. If for a focal element, $g_{\min}$ and $g_{\max}$ are both positive, the focal element will contribute to the calculation of belief and plausibility according to Eqs (6) and (7). On the other hand, if $g_{\min}$ and $g_{\max}$ are both negative, the focal element will not contribute to the calculation of belief or plausibility. If however, $g_{\min}$ is negative and $g_{\max}$ is positive, the focal element will not contribute to the belief but it will contribute to the plausibility calculation. This is shown schematically in Figure 2.

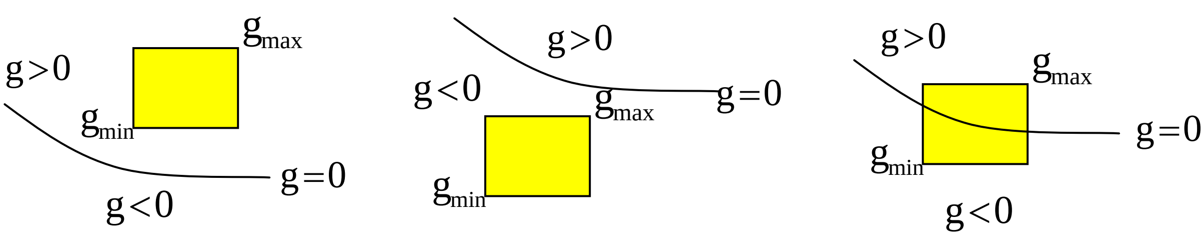

Figure 2. Schematic illustration of focal element contribution to belief and plausibility measures.

In summary the following tasks are performed in order to calculate the belief and plausibility of the failure region:

1) For each input parameter, combine the evidence from the experts by combining the individual BPA’s from each expert using Dempster’s rule of combining (Eq. (11)).

2) Construct the BPA structure for the *m*-dimensional frame of discernment, where *m* is the number of input parameters. Assuming independent input parameters, Eq. (15) is used.
3) Identify the failure region space (set *F* of Eq. (16)).
4) Use Eqs (6) and (7) to calculate the belief and plausibility measures of the failure region. The failure region must be identified only within the frame of discernment. The true probability of failure is bracketed according to Eq. (17).

3. EVIDENCE-BASED DESIGN OPTIMIZATION (EBDO)

In deterministic design optimization, an objective function is minimized subject to satisfying each constraint. In Reliability-Based Design Optimization (RBDO), where all design variables are characterized probabilistically, an objective function is usually minimized subject to the probability of satisfying each constraint, being greater than a specified high reliability level. In this section, a methodology is presented on how to use evidence theory in design. We will show that the evidence theory-based design is conservative compared with all RBDO designs obtained with different probability distributions.

If feasibility of a constraint g is expressed with the non-negative null form $g \geq 0$, we have shown in the previous section that $P(g \geq 0)$ is bracketed by the belief $Bel(g \geq 0)$ and plausibility $Pl(g \geq 0)$; i.e. $Bel(g \geq 0) \leq P(g \geq 0) \leq Pl(g \geq 0)$. Therefore,

$$P(g < 0) \leq p_f \text{ is satisfied if } Pl(g < 0) \leq p_f \tag{19}$$

where p_f is the probability of failure which is usually a small prescribed value. The above statement is equivalent to

$$P(g \geq 0) \geq R \text{ is satisfied if } Bel(g \geq 0) \geq R \tag{20}$$

where $R = 1 - p_f$ is the corresponding reliability level.

An evidence theory-based design optimization (EBDO) problem can be formulated as

$$\begin{aligned} &\min_{\mathbf{d},\mathbf{X}^N} f\left(\mathbf{d},\mathbf{X}^N,\mathbf{P}^N\right) \\ \text{s.t. } &Pl\left(g_i\left(\mathbf{d},\mathbf{X},\mathbf{P}\right) < 0\right) \leq p_{f_i},\ i = 1,\dots,n \\ &\mathbf{d}_L \leq \mathbf{d} \leq \mathbf{d}_U \\ &\mathbf{X}_L^N \leq \mathbf{X}^N \leq \mathbf{X}_U^N \end{aligned} \tag{21}$$

where $\mathbf{d} \in R^k$ is the vector of deterministic design variables, $\mathbf{X} \in R^m$ is the vector of uncertain design variables, $\mathbf{P} \in R^q$ is the vector of uncertain design parameters, $f(\ )$ is the objective function and *n, k, m* and *q* are the number of constraints, deterministic design variables, uncertain design variables and uncertain design parameters, respectively. According to the used notation, a bold letter indicates a vector, an upper case letter indicates an uncertain variable or parameter and a lower case letter indicates a realization of the uncertain variable. The superscript "N" in Problem (21) indicates nominal value of each uncertain design variable or design parameter. The uncertainty is provided by expert opinions.

It should be noted that the plausibility measure is used instead of the equivalent belief measure, in Problem (21). The reason is that at the optimum, the failure domain for each active constraint is usually much smaller than the safe domain over the frame of discernment (FD). As a result, the computation of the plausibility of failure is much more efficient than the computation of the belief of safe region.

3.1. IMPLEMENTATION OF THE EBDO ALGORITHM

This section describes a computationally efficient solution of Problem (21). As a geometrical interpretation of Problem (21), we can view the design point (**d**,**X**) moving within the feasible domain so that the objective f is minimized. If the entire FD is in the feasible domain, the constraints are satisfied and are inactive. A constraint becomes active if part of the FD is in the "failure" region so that the plausibility of constraint violation is equal to p_f. In general, Problem (21) represents movement of a hyper-cube (FD) within the feasible domain.

In order to save computational effort, the bulk of the FD movement, from the initial design point to the *vicinity* of the optimal point (point B of Figure 3), can be achieved by *moving a hyper-ellipse which contains the FD*. The center of the hyper-ellipse is the "approximate" design point and each axis is arbitrarily taken equal to three times the standard deviation of a hypothetical normal distribution. This assumes that each dimension of the FD hyper-cube is equal to six times the standard deviation of the hypothetical normal distribution. The hyper-ellipse can be easily moved in the design space by solving a Reliability-Based Design Optimization (RBDO) problem. The RBDO optimum (point B of Figure 3) is in the vicinity of the

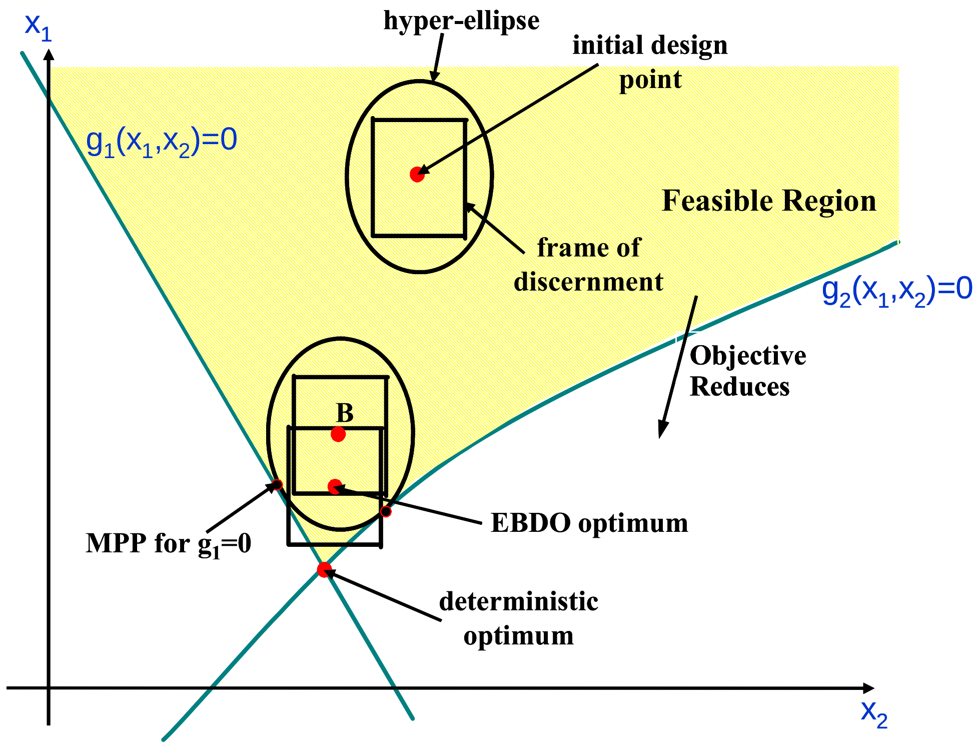

Figure 3. Geometrical interpretation of the EBDO algorithm.

solution of Problem (21) (EBDO optimum). The RBDO solution also identifies all active constraints and their corresponding most probable points (MPP's). The maximal possibility search algorithm [38] can also be used to move the FD hyper-cube in the feasible domain. It should be noted that the 3-sigma axes hyper-ellipse is arbitrary. The size of the hyper-ellipse is not however, crucial to the user because it is only used to calculate the initial point (point B of Figure 3) of the EBDO algorithm. The latter calculates the true EBDO optimum accurately. From our experience, 3 to 4-sigma size works fine.

At this point, we generate a *local* response surface of each active constraint around its MPP. In this work, the Cross-Validated Moving Least Squares (CVMLS) [39] method is used based on an Optimum Symmetric Latin Hypercube (OSLH) [40] "space-filling" sampling.

A derivative-free optimizer calculates the EBDO optimum. It uses as initial point the previously calculated RBDO optimum which is close to the EBDO optimum. Problem (21) is solved, considering only the identified active constraints. For the calculation of the plausibility of failure $Pl(g<0)$ of each active constraint, the algorithm of next section is used. The algorithm identifies all focal elements which contribute to the plausibility of failure. The computational effort is significantly reduced because accurate local response surfaces are used for the active constraints. The cost can be much higher if the optimization algorithm evaluates the actual active constraints instead of their efficient surrogates (response surfaces). It should be noted that a derivative-free optimizer is needed due to the discontinuous nature of the combined BPA structure. The DIRECT (DIvisions of RECTangles) derivative-free, global optimizer is used in this work. DIRECT is a modification of the standard Lipschitzian approach that eliminates the need to specify a Lipschitz constant [41].

3.1. CALCULATION OF PLAUSIBILITY OF FAILURE

In Problem (21), the plausibility of failure or equivalently the plausibility of constraint violation, $Pl(g<0)$, must be calculated every time the optimizer evaluates a constraint. The algorithm is given below.

Step 1. Initialize sets C = {FD} and F = {0} and counter m = 1

Step 2. Consider all sets $E_k : E_k \subset C$ or $E_k \subseteq C$

Initialize counter n = 0
Empty set C; i.e. C = {0}
For k = 1 to m
  Partition E_k into E_k^1 and E_k^2
  For j = 1 to 2
    Calculate $g_{\min}\left(E_k^j\right)$
    If $g_{\min}\left(E_k^j\right)<0$ then
      Calculate $g_{\max}\left(E_k^j\right)$
      If $g_{\max}\left(E_k^j\right)>0$ then $C=C\cup E_k^j$ and n = n+1
      If $g_{\max}\left(E_k^j\right)\leq 0$ then $F=F\cup E_k^j$
    End if (for the loop of $g_{\min}\left(E_k^j\right)<0$)
  End if (for the loop of j = 1 to 2)
End if (for the loop k = 1 to m)
Set counter m = n
If C can be partitioned, go to step 2.
If C can not be partitioned, stop and calculate plausibility of failure from Eq. (22)

$$Pl(g<0)=\sum_{B\in F} m(B)+\sum_{B\in C} m(B) \tag{22}$$

as the sum of BPA values of all focal elements B which belong to sets F and C.

A set C which is initially equal to the entire frame of discernment FD (see step 1) is partitioned into sets E^1 and E^2. The partitioning sequence is explained at the end of this section. The minimum and maximum values of g in the E^1 and E^2 domains are calculated; i.e.

$g_{\min}(E_i) = \min g(\mathbf{X}),\ \ \mathbf{X} \in E_i, i = 1,2$ and $g_{\max}(E_i) = \max g(\mathbf{X}),\ \ \mathbf{X} \in E_i, i = 1,2$ (see step 2). If $g_{\min}(E_i) < 0$ and $g_{\max}(E_i) > 0$, E^i is placed in set *C*. If $g_{\min}(E_i) < 0$ and $g_{\max}(E_i) < 0$, E^i is placed in set *F*. Otherwise, E^i is not considered further. For a subsequent iteration *k* in step 2, each set which has been placed in *C* (denoted by E_k) is further partitioned into sets E_k^1 and E_k^2, and the process continues. If all sets put in *C* represent focal elements and therefore, can not be partitioned further, the algorithm stops and Eq. (22) is used to calculate the plausibility of failure.

The above algorithm is demonstrated with a hypothetical example. Figure 4 shows the location of the FD relative to the limit state g=0 for a particular iteration. A hypothetical BPA structure is also shown. Each "rectangle" represents a focal element. In this case, we have 20 focal elements denoted by m_i, *i*=1,2,…,20. A set which is initially equal to FD, is partitioned into sets E^1 and

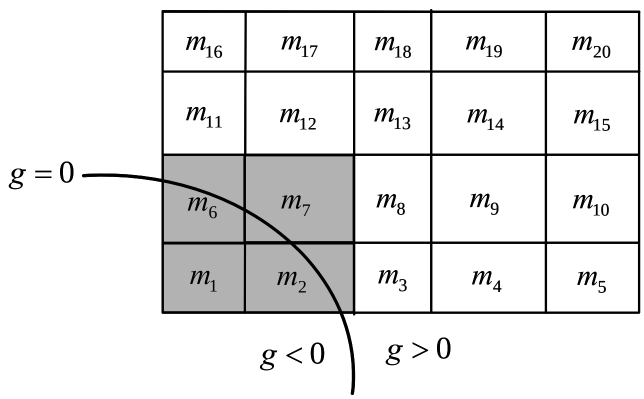

Figure 4. A hypothetical two-dimensional BPA structure.

E^2 such that $E^1 = \bigcup_i m_i \ \ , i = 1,2,6,7,11,12,16,17$ and $E^2 = \bigcup_i m_i \ \ , i = 3,4,5,8,9,10,13,14,15,18,19,20$. Subsequently, the minimum and maximum values of *g* in the E^1 and E^2 domains ($g_{\min}(E^i)$ and $g_{\max}(E^i)$, *i*=1,2) are calculated. Because $g_{\min}(E^1) < 0$ and $g_{\max}(E^1) > 0$, E^1 is placed in *C*. However, $g_{\min}(E^2) > 0$ and therefore, E^2 is not considered further. This is the end of the first iteration.

The second iteration starts by partitioning *C* which is equal to E^1 of the first iteration, into $E^{11} = \bigcup_i m_i \quad, i = 11,12,16,17$ and $E^{12} = \bigcup_i m_i \quad, i = 1,2,6,7$. Similarly to the first iteration, E^{11} is discarded and E^{12} is placed in *C* which is now composed of E^{12} only. Note that at the end of the second iteration, set *F* is empty. At the third iteration, *C* or equivalently E^{12}, is partitioned into $E^{121} = \bigcup_i m_i \quad, i = 1,6$ and $E^{122} = \bigcup_i m_i \quad, i = 2,7$ which are both placed in *C*. At the fourth iteration, E^{121} is partitioned into $E^{1211} = m_1$ and $E^{1212} = m_6$ and E^{122} is partitioned into $E^{1221} = m_2$ and $E^{1222} = m_7$. Now E^{1211} is placed in *F* and E^{1212}, E^{1221} and E^{1222} are placed in *C*. Because all previous sets consist of one focal element each, they can not be partitioned further. Therefore, the algorithm stops. Finally, $F = m_1$ and $C = \bigcup_i m_i \quad, i = 2,6,7$. Eq. (22) is used to calculate the plausibility of $g<0$ as the sum of BPA values of all focal elements in *F* and *C*; i.e.

$$Pl(g < 0) = \sum_{B \in F} m(B) + \sum_{B \in C} m(B).$$

The described algorithm uses the following partitioning scheme for an *n*-dimensional hyper-rectangle representing the FD which corresponds to *n* uncertain variables and parameters. For the k^{th} iteration ($k = 1,\ldots,n$), the hyper-rectangle is partitioned into two parts with an (n-1)-dimensional hyper-plane perpendicular to the k^{th} dimension. Each part has roughly the same number of focal elements. For iteration $k > n$, the (n-1)-dimensional hyper-plane is perpendicular to the $(k-n)^{th}$ dimension.

4. BAYESIAN RELIABILITY-BASED DESIGN OPTIMIZATION

It has been mentioned that if we only know the bounds within which an uncertain variable varies, interval analysis or possibility theory can be used to quantify and propagate uncertainty. If additional information is available in terms of expert opinions for example, the evidence theory can be used. It is common however, in engineering design, to know the bounds of the uncertain variables and also have additional information in the form of a discrete but limited number of sample points based on historic data or experiment data. In this case, we can not infer a probabilistic distribution because of the limited number of sample points. However, a Bayesian approach [32, 33] can be used to estimate the probability distribution. If more information is obtained later in the form of additional sample points, a more accurate estimation of the probability distribution can be obtained. The next subsections provide the basics of Bayesian approach as well as the introduction of the extreme value distribution in the Bayesian approach in order to account for the fact that we only have a small set of sample points which are different at each experiment.

4.1. BAYESIAN RELIABILITY ESTIMATION

Let us denote available data by D and the probability of success by θ. We wish to improve our knowledge about the unknown quantity θ by utilizing the known information in the available data D. To make inferences about θ, we build a conditional probability distribution $P(\theta|D)$ that describes how we believe θ is distributed considering the existence of data D. Using the Bayesian rule, it can be shown that

$$P(\theta \mid D) = \frac{\Gamma(N+\alpha+\beta)}{\Gamma(D+\alpha)\Gamma(N-D+\beta)}\theta^{D+\alpha-1}(1-\theta)^{N-D+\beta-1} = Beta\left(D+\alpha, N-D+\beta\right) \quad (23)$$

where,

$$Beta(\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1} \quad (24)$$

with α, and β being the *Beta* distribution parameters. As expected, the posterior distribution is also a *Beta* distribution because *Beta* is a conjugate family of distributions.

It should be noted that initially there is no prior information about θ and any values between 0 and 1 may be assumed equally probable. In this case, a uniform prior $P(\theta)=U(0,1)$ is used which is equivalent to *Beta(1,1)*.

If we have r_0 successes out of N_0 sample points, then the probability distribution $P(\theta \mid r_0)$ is proportional to Beta (r_0 +1, N_0 - r_0 +1). If additional N_1 data is obtained later, where r_1 is the number of successes, then the total number of success is $r_0 + r_1$ and the total number of failures is $N_0 + N_1 - r_0 - r_1$. In this case, the probability distribution $p(\theta \mid r_0 + r_1)$ is proportional to Beta ($r_0 + r_1$ +1, $N_0 + N_1 - r_0 - r_1$ +1). Note that if we use the constraint function g to divide the design space into feasible and infeasible domains, then a feasible realization of g is considered a success and an infeasible realization is considered a failure.

Let us define two vectors $\boldsymbol{R}$ = [$\boldsymbol{Y}$, $\boldsymbol{Z}$] and $\boldsymbol{U}$ = [$\boldsymbol{X}$, $\boldsymbol{P}$] where $\boldsymbol{Y}$ and $\boldsymbol{Z}$ denote the random variables and parameters whose PDFs are known and $\boldsymbol{X}$ and $\boldsymbol{P}$ denote the uncertain variables and parameters whose PDFs are not known. If $\boldsymbol{R}$ is not empty, each realization of $\boldsymbol{U}$ = [$\boldsymbol{X}$, $\boldsymbol{P}$] results in a distribution of g values. In this case, we can calculate the probability $Pr\,[g\,(\boldsymbol{Y}, \boldsymbol{Z}) > 0 \mid (\boldsymbol{X}, \boldsymbol{P})]$ that a sample point [$\boldsymbol{Y}$, $\boldsymbol{Z}$] will result in a feasible realization given the sample point [$\boldsymbol{X}$, $\boldsymbol{P}$]. This conditional probability is the expected feasible realization of one sample.

Using a limited number of sample points we obtain therefore, a probability distribution instead of a single probability value. Because there are few sample points and the samples are random, the *Beta* distribution may not be accurately representing the actual distribution. In order to increase our confidence of the predicted probability, we propose to use the extreme distribution of the smallest value using the *Beta* distribution as the basic distribution. If X is a Beta distributed uncertain variable and there are n available sample points of X, the CDF of the extreme minimum value Y_1 (i.e. $Y_1 = \min(X_1, X_2, ..., X_n)$) is given by

$$F_{Y_1}(y) = 1 - [1 - F_X(y)]^n \,. \tag{25}$$

4.2. CONSTRUCTION OF THE EXTREME SMALLEST VALUE DISTRIBUTION

Because we have a limited number of sample points [***X**, **P***] the probability *Pr* [*g (**Y**, **Z**) > 0* | *(**X**, **P**)*] is approximated by the *Beta* distribution. To increase our confidence of constraint satisfaction (reliability) exceeding a specified target reliability R, we express each probabilistic constraint in terms of a confidence percentile [42]. For the i^{th} constraint, this is expressed as

$$\nabla\left(P\left(g_i(\mathbf{d},\mathbf{Y},\mathbf{Z}) \geq 0 \,|(\mathbf{X},\mathbf{P})\right) \geq R_i\right) = \int_{p_f^t}^{1} P(\theta \,|\, D)\, d\theta \geq \sigma \,, \tag{26}$$

where σ is a specified confidence percentile, $p_f^t = -\Phi(1 - R_i)$ is the target probability of failure for the i^{th} constraint, and ∇ denotes the confidence percentile. The latter is calculated based on the extreme value distribution. It provides a conservative distribution of the probability of constraint satisfaction which is not a scalar. It should be noted that the extreme value distribution provides a much smaller confidence percentile compared with the *Beta* distribution for the same reliability. This means that it is much safer (or more conservative) to use the extreme value distribution in design optimization.

For a confidence percentile σ, let us denote by P_B and P_B' the probability corresponding to σ based on the extreme value and *Beta* distributions, respectively. Also, let us assume that the number of available sample points is *N*. For the extreme value distribution $1 - \sigma = 1 - [1 - F_X(P_B)]^N$, resulting in $P_B = F_X^{-1}[1 - \sqrt[N]{\sigma}]$ where *X* ~ Beta (*a, b*). Similarly for the *Beta* distribution $1 - \sigma = 1 - [1 - F_X(P_B')]$, or $P_B' = F_X^{-1}[1 - \sigma]$ where *X* ~ Beta (*a, b*). It is easy to see that if *N*=1, $P_B = P_B'$. However because $\sqrt[N]{\sigma} \geq \sigma$ or $1 - \sqrt[N]{\sigma} \leq 1 - \sigma$, P_B is less than P_B', if *N* is larger than 1. For this reason, the extreme value distribution based confidence percentile provides a more conservative (smaller) probability compared with the *Beta* distribution.

4.3. EVALUATION OF BAYESIAN TARGET RELIABILITY

In design optimization, the target reliability must be predefined. Because we do not have however, enough data, it is not practical to set the target reliability very high (e.g. $\beta = 3$). If the predefined target reliability is high, the confidence percentile will be low. In this section, we will calculate the maximum target reliability based on an existing sample size *N*.

If we have *N* sample points, the safest *Beta* distribution is Beta(N+1,1). The maximum Bayesian target reliability is therefore, equal to $P_B = F_X^{-1}[1 - \sqrt[N]{\sigma}]$ where *X* ~ Beta(N+1, 1), and σ is the confidence percentile. The larger the *N*, the higher the maximum target reliability is. However, the latter must be always lower than the allowable maximum reliability. For example, if we have 50 sample points, the maximum target reliability with confidence percentile 0.8 must be lower than 90%.

A Bayesian-based design optimization process entails the following steps:

1. Construct *Beta* distribution based on existing sample data.
2. Construct an extreme smallest value distribution using the above *Beta* distribution as the basic distribution.
3. Calculate the maximum target reliability for a specified confidence percentile.
4. Solve the design optimization problem using reliabilities which are based on the extreme smallest value distribution with a specified confidence percentile.

4.4. A BAYESIAN APPROACH TO DESIGN OPTIMIZATION

Reliability-based design optimization (RBDO) provides optimum designs in the presence of only random (or aleatory) uncertainty. A typical RBDO problem is formulated

$$\begin{aligned} &\min_{\mathbf{d},\boldsymbol{\mu}_{\mathbf{Y}}} f(\mathbf{d},\boldsymbol{\mu}_{\mathbf{Y}},\boldsymbol{\mu}_{\mathbf{Z}}) \\ \text{s.t.}\quad & P(g_i(\mathbf{d},\mathbf{Y},\mathbf{Z}) \geq 0) \geq R_i = 1 - p_{f_i}, \quad i = 1,\dots,n \\ & \mathbf{d}^{\mathrm{L}} \leq \mathbf{d} \leq \mathbf{d}^{\mathrm{U}}, \ \boldsymbol{\mu}_{\mathbf{Y}}^{\mathrm{L}} \leq \boldsymbol{\mu}_{\mathbf{Y}} \leq \boldsymbol{\mu}_{\mathbf{Y}}^{\mathrm{U}} \end{aligned} \tag{27}$$

where $\mathbf{Y} \in R^{\ell}$ is the vector of random design variables and $\mathbf{Z} \in R^{r}$ is the vector of random design parameters.

For a variety of practical applications, the uncertain information may be provided as a mixture of sample points and probability distributions. In this case, a Bayesian approach can be used based on the confidence percentile concept. A Bayesian Approach Design Optimization (BADO) problem with a combination of random and Bayesian uncertain variables can be formulated as

$$\begin{aligned} &\min_{\mathbf{d},\mathbf{x}^N,\boldsymbol{\mu}_Y} \mathrm{f}(\mathbf{d},\boldsymbol{\mu}_{\mathbf{Y}},\boldsymbol{\mu}_{\mathbf{Z}},\mathbf{x}^N,\mathbf{p}^N) \\ \text{s.t.}\quad & \nabla(P(g_i(\mathbf{d},\mathbf{X},\mathbf{P}) \geq 0) \geq R_i) \geq \sigma, \quad i = 1,\dots,n \\ & \mathbf{d}_L \leq \mathbf{d} \leq \mathbf{d}_U, \ \boldsymbol{\mu}_{\mathbf{Y}}^L \leq \boldsymbol{\mu}_{\mathbf{Y}} \leq \boldsymbol{\mu}_{\mathbf{Y}}^U \\ & \mathbf{x}_L \leq \mathbf{x}^N \leq \mathbf{x}_U \end{aligned} \tag{28}$$

where $\mathbf{d} \in R^k$ is the vector of deterministic design variables, $\mathbf{X} \in R^m$ is the vector of Bayesian uncertain design variables, $\mathbf{P} \in R^q$ is the vector of Bayesian uncertain design parameters, $\mathbf{Y} \in R^{\ell}$ is the vector of random design variables, $\mathbf{Z} \in R^r$ is the vector of random design parameters, R_i is the target reliability, σ is the confidence percentile factor, and ∇ is the confidence function.

All constraints in Problem (28) are expressed using a confidence percentile because the predicted probability is distributed based on the extreme value distribution instead of having a single value. We need the confidence percentile in order to calculate a single probability value. It should be noted that the described formulation represents a double-loop optimization sequence. The design optimization of the outer loop calls a series of Bayesian uncertain constraints. Each Bayesian uncertain constraint is in general, a global optimization problem.

It should be noted that the double-loop optimization structure of Problem (28) is different from the double-loop RBDO structure. In the outer loop, the deterministic variables **d**, the mean values $\boldsymbol{\mu}_{\mathbf{Y}}$ of random variables and the normal points $\mathbf{x}^N$ of Bayesian uncertain variables are used as design variables. In the inner loop, based on the distributions of some of the input design variables and the available sample points for the remaining design variables, an extreme value distribution is constructed using the Bayesian approach. Subsequently, we calculate the reliability of the constraint using the confidence percentile principle. Because the Bayesian uncertain variables are represented using discrete sample points, we can not use a gradient-based local optimizer to calculate the optima. Instead, we must use a global optimizer.

4.4.1. A PRESSURE VESSEL EXAMPLE

This example considers the design of a thin-walled pressure vessel [43] which has hemispherical ends as shown in Figure 5. The design objective is to calculate the radius *R*, mid-section length *L* and wall thickness *t* in order to maximize the volume while avoiding yielding of the material in both the circumferential and radial directions under an internal pressure *P*. Geometric constraints are also considered. The material yield strength is *Y*. A safety factor *SF* = 2 is used.

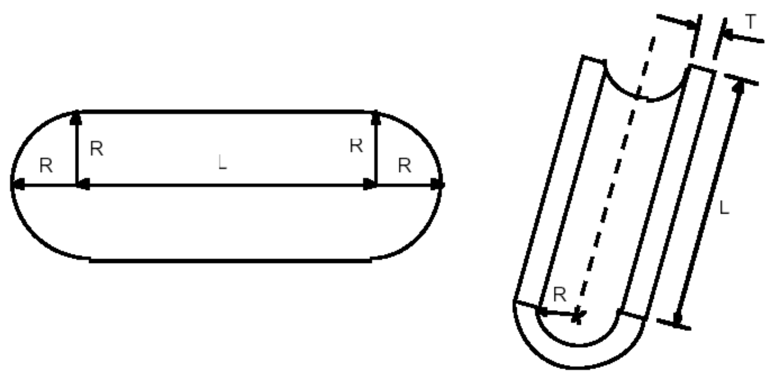

Figure 5. Thin-walled pressure vessel.

The BADO problem is stated as

$$\max_{R_N, L_N, t_N} \mathrm{f} = \frac{4}{3}\pi R_N^3 + \pi R_N^2 L_N \tag{29}$$

$$\text{s.t.} \quad \nabla\left(P\left(g_i(\mathbf{X}) \geq 0\right) \geq R_i\right) \geq \sigma,\ j = 1,\ldots,5$$

$$\mathrm{g}_1(\mathbf{X}) = 1.0 - \frac{P(R+0.5t)SF}{2tY}$$
$$\mathrm{g}_2(\mathbf{X}) = 1.0 - \frac{P(2R^2+2Rt+t^2)SF}{(2Rt+t^2)Y}$$
$$\mathrm{g}_3(\mathbf{X}) = 1.0 - \frac{L+2R+2t}{60}$$
$$\mathrm{g}_4(\mathbf{X}) = 1.0 - \frac{R+t}{12}$$
$$\mathrm{g}_5(\mathbf{X}) = 1.0 - \frac{5t}{R}$$
$$0.25 \le t_N \le 2.0$$
$$6.0 \le R_N \le 24 \quad .$$
$$10 \le L_N \le 48$$

There are three design variables (R, L, t) and two design parameters (P, Y) where P is the internal pressure and Y is the material yielding strength. The design variable R is considered a Bayesian uncertain variable. To compare results with RBDO, we sample 50 points based on the PDF of a normal distribution N (R^N ,1.5) The design variables L and t and the design parameters P and Y

Table 1. Comparison of BADO and PBDO optima for the pressure vessel example.

	Design Variables			**Objective**
	R^N	L^N	t^N	*f(X)*
Det. Opt.	11.75	36	0.25	22400
Reliability Optimum (*p*=0.85/*β*=1.036)	10.1926	34.7147	0.25	15757
Bayesian Uncertainty (*N*=50)				
σ=0.6, *p*=0.85	9.50	33.2099	0.4306	13100
σ=0.6, *p*=0.75	10.2778	34.4444	0.2639	15970
σ=0.8, *p*=0.85	9.50	31.4815	0.4853	12511
σ=0.8, *p*=0.75	10.50	31.4815	0.3750	15745
Possibilistic Uncertainty				
σ=0.6, *p*=0.85	8.9464	33.0912	0.25	11314
σ=0.6, *p*=0.75	8.9825	34.1069	0.25	11676
σ=0.8, *p*=0.85	8.0464	33.0912	0.25	8908
σ=0.8, *p*=0.75	8.0825	34.1069	0.25	9207

are normally distributed random variables with standard deviations equal to 3, 0.1, 50 and 13000, respectively. The mean values of *P* and *Y* are equal to 1,000 and 260,000.

For the vessel example with a combination of Bayesian and random variables, Table 1 gives the BADO results based on different target reliabilities and confidence percentiles. When the confidence percentile is σ=0.8, and the target probability is p=0.85, the BADO and RBDO results are 12511 and 15757, respectively. Because RBDO uses probabilistic distribution information, it utilizes more information compared with BADO which uses only a limited number of sample points. Thus, the BADO result should be more conservative. Because the objective is maximized in this example, the BADO result is less than the RBDO result. For the same confidence percentile of σ=0.8, if the target probability is 0.75, the BADO objective is 15745. If the target probability is 0.85, then the objective is equal to 12511. The higher the confidence percentile is, the lower the objective becomes. It should be noted that the uncertain variables in BADO are characterized only by a limited number of sample points, while only the bounds are known for the uncertain variables in PBDO. Therefore, the latter represent the least amount of information. For this reason, the PBDO design has the smallest objective value of 8908 which is obtained for a confidence percentile of σ=0.8 and a target probability of p= 0.85.

4.5. A COMBINED BAYESIAN AND EBDO APPROACH

For the above Bayesian approach, we know the range of the uncertain variables and parameters and also have a limited number of sample points. In actual engineering design however, assuming that this range is partitioned into a number of segments, we only know how many sample points are within a certain segment. In this case, we do not have an exact distribution of those sample points within the segment and we can not use therefore, the BADO methodology of section 4.4 to construct the probability distribution function of the constraint. Also, since the total number of sample points is limited, we can not assume that the probability of being within a segment is equal to the number of samples in the segment divided by the total number of samples in the whole range.

In this case, in order to utilize the existing information, we can use evidence theory to calculate the Basic Probability Assignment (BPA) for a segment of each Bayesian variable. In summary, the following tasks are performed in order to calculate the belief and plausibility of the failure region:

1) For each Bayesian input variable and parameter, construct a *Beta* distribution using the available data, and then form the extreme value distribution. Calculate the BPA structure for each variable and parameter using a predefined confidence percentile and the extreme value distribution.
2) Construct the BPA structure for the *m*-dimensional frame of discernment, where *m* is the number of input variables and parameters. Assume independent input variables and parameters.
3) Identify the failure region space based on the limit state functions (constraints).

4) Calculate the belief and plausibility measures of the failure region. The failure region must be identified only within the frame of discernment. The true probability of failure is bracketed by the belief and plausibility measures.
5) If more information becomes available, we can obtain a more accurate estimate of the BPA structure using an assumed confidence percentile.

This process is illustrated with an example in the following subsection.

4.5.1. THE PRESSURE VESSEL EXAMPLE

The same pressure vessel example of section 4.4.1 is considered here. We initially assume that we have only 100 sample points. Based on this limited available information, we only know the number of sample points within specified segments (bins) as is for example, indicated in Table 2 and shown in Figure 6 for R_N . However, we do not know the exact distribution of the sample points within each segment.

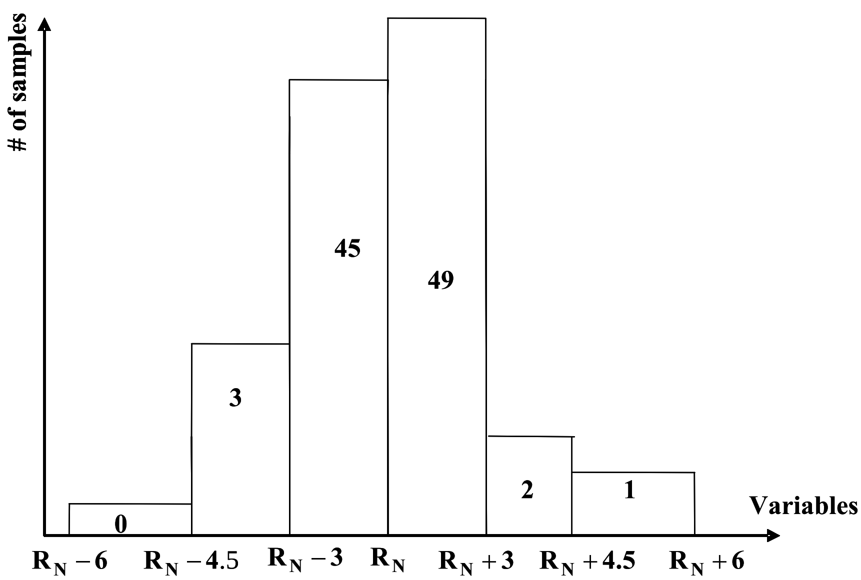

Figure 6. Histogram of sample points.

According to Figure 6, there are three sample points of R_N within the [R_N - 4.5, R_N - 3] segment. We cannot assume that the probability of having samples in that segment is equal to 3/100=0.03, because there are not enough sample points. However, we know that the extreme probability distribution for the smallest value will be $F_{Y_1}(p) = 1 - [1 - F_X(p)]^{100}$, where p denotes probability and X ~ Beta (3+1, 100-3+1) =Beta (4, 98). If we use a predefined confidence percentile of σ=0.8, then $Pr(R_N - 4.5 < R < R_N - 3 \mid \sigma=0.8) = p = F_X^{-1}[1-\sqrt[100]{0.8}]$ =0.0054, which is smaller than the real CDF (approximately equal to 3/50=0.06). Similarly, $Pr(R_N - 3.0 < R < R_N \mid \sigma=0.8)$ = 0.3155, which is also smaller than 45/100=0.45. Because of the existing uncertainty, if the confidence percentile is large enough, the values of the BPA structure calculated using the Bayesian approach of section 4.2 will be smaller than the actual values.

Table 2. BPA structure for Bayesian Variable *R* (100 sample points).

R	**# of Sample Points**	**BPA(Extreme Value)**
[R_N - 6.0 , R_N - 4.5]	0	2.2e-5
[R_N - 4.5 , R_N - 3.0]	3	0.0054
[R_N - 3.0 , R_N]	45	0.3155
[R_N , R_N + 3.0]	49	0.3523
[R_N + 3.0 , R_N + 4.5]	2	0.0025
[R_N + 4.5 , R_N + 6.0]	1	0.00068
[R_N - 6.0 , R_N + 6.0]	**---**	**0.3236**

If we have more sample points, the BPA structure can be estimated more accurately. In Tables 3 and 4, we utilize 300 and 1000 sample points, respectively. For the same confidence percentile of σ=0.8, for 300 samples, the estimated probability is $Pr(R_N - 3.0 < R < R_N \mid \sigma=0.8) = p = F_X^{-1}[1-\sqrt[300]{0.8}]$ =0.393. For 1000 samples, the estimated probability is equal to 0.4457, which is very close to the CDF of normal distribution 0.475. It should be noted that for 100 sample points, the same probability is equal to 0.3155. Using the calculated BPA structure, we can use steps 2 to 5 of section 4.5 to determine the optimal design using the EBDO algorithm. The more accurate the BPA structure is, the less conservative (smaller objective in this example) the optimum design will be. At the limit, the design approaches the RBDO design.

Table 3. BPA structure for Bayesian Variable *R* (300 sample points).

R	**# of Sample Points**	**BPA(Extreme Value)**
[R_N - 6.0 , R_N - 4.5]	2	0.00057
[R_N - 4.5 , R_N - 3.0]	6	0.0048
[R_N - 3.0 , R_N]	145	0.393
[R_N , R_N + 3.0]	138	0.371
[R_N + 3.0 , R_N + 4.5]	8	0.0078
[R_N + 4.5 , R_N + 6.0]	1	0.00013
[R_N - 6.0 , R_N + 6.0]	**---**	**0.2228**

Table 4. BPA structure for Bayesian Variable *R* (1000 Sample points).

R	**# of Sample Points**	**BPA(Extreme Value)**
[R_N - 6.0 , R_N - 4.5]	7	0.00157
[R_N - 4.5 , R_N - 3.0]	22	0.00985
[R_N - 3.0 , R_N]	501	0.4457
[R_N , R_N + 3.0]	445	0.3906
[R_N + 3.0 , R_N + 4.5]	16	0.0062
[R_N + 4.5 , R_N + 6.0]	9	0.00244
[R_N - 6.0 , R_N + 6.0]	---	**0.1436**

Based on the evidence theory the sum of all BPA values should be equal to one. However in Tables 2, 3 and 4, the sums of BPA are 0.6764, 0.7772 and 0.8564, respectively. The difference is due to unavailable information because of the limited number of sample points. It represents the uncertain belief of being somewhere between R_N - 6.0 and R_N + 6.0 without knowing the exact segment. Figure 7 shows the BPA values based on 100 samples. The uncertain belief is equal to 1-0.6764=0.3236 for the 100 sample point case, equal to 1-0.7772=0.2228 for the 300 sample point case, and equal to 1-0.8564=0.1436 for the 1000 sample point case. The uncertain belief will contribute to the belief measure (see section 3.1) if the range [R_N - 6.0, R_N + 6.0] is within the feasible area.

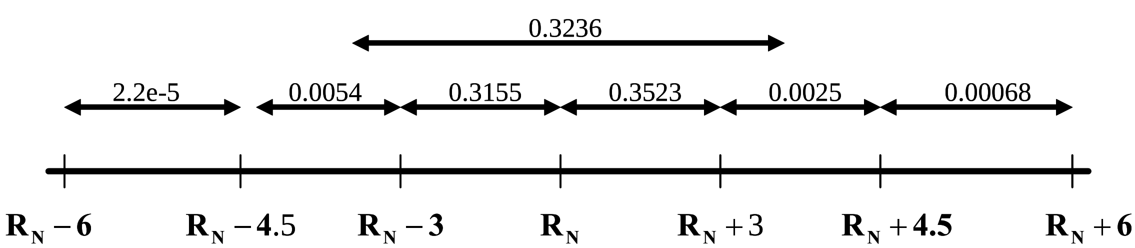

Figure 7. BPA of *R* for 100 sample points.

Considering the information from the above example, Table 5 compares the results between the Bayesian approach, EBDO and RBDO for a reliability index of $\beta = 0.385$ (p_f =0.35) and a confidence percentile of σ=0.8. The EBDO results are based on the assumption that a very large number of sample points is available from which the BPA structure is calculated. According to Table 5, the Bayesian approach (BADO of section 4.4) provides the most conservative result (smallest objective of 15098 for p_f =0.35) compared with the EBDO and RBDO optima of 16802 and 19610, respectively, because it utilizes the least amount of information among the three approaches. For comparison purposes the PBDO optimum of 7269 is also shown in Table 5 for the zero α -cut (worst-case design) as well as the Bayesian Evidence optimum of 9805. As expected, the Bayesian Evidence optimum is better than the PBDO optimum because it uses more information. However, the Bayesian Evidence optimum of 9805 is smaller than the Bayesian optimum of 15098 because the BPA structure of the former is more conservative than the extreme value distribution of the latter. It should also be noted that although the Bayesian Evidence approach is the most conservative compared with the RBDO, EBDO and BADO approaches, it is less conservative than the worst-case scenario of PBDO, as expected. Table 5 also compares results for p_f =0.45 with similar trends observed.

Table 5. Comparison of design optimization approaches.

	Reliability Optimum (RBDO)	**Bayesian Optimum (BADO)**		**Bayesian Evidence Optimum**		**Possibility Optimum (PBDO)**		**Evidence Optimum (EBDO)**	
Design Variables	p_f **=0.35, (** β **=0.385)**	p_f **=0.45**	p_f **=0.35**	p_f **=0.45**	p_f **=0.35**	***a*=0, p_f =0.45**	***a* =0, p_f =0.35**	p_f **=0.45**	p_f **=0.35**
R_N	11.153	10.574	10.166	8.654	8.481	7.346	7.211	10.778	10.555
L_N	35.330	33.539	32.963	34.032	32.098	34.905	34.905	33.703	33.950
t_N	0.264	0.300	0.291	0.254	0.254	0.25	0.25	0.263	0.263
Objective									
$f(R_N, L_N)$	19610	16725	15098	10718	9805	7574	7269	17535	16802

5. SUMMARY AND CONCLUSIONS

If only the bounds are available within which an uncertain variable varies, interval analysis or possibility theory can be used to quantify and propagate uncertainty. If additional information is known in terms of expert opinions for example, the evidence theory can be used. If in addition to the bounds of the uncertain variables, there is information in the form of a discrete but limited number of sample points, we can not infer a probabilistic distribution because of the limited number of sample points. However, a Bayesian approach can be used to estimate the probability distribution which can be subsequently, utilized in a Reliability-Based Design Optimization algorithm.

This paper has presented a method called Bayesian Approach Design Optimization (BADO) to solve design problems with uncertain variables in the form of both finite sample points and probability distributions. Also, a Bayesian approach was proposed to estimate the Basic Probability Assignment (BPA) for a specified confidence percentile, using only the number of available sample points within ranges. Subsequently, the evidence theory was used to obtain the optimal design.

A pressure vessel example was used to demonstrate the proposed Bayesian approach in design optimization and compare the results with known design methods such as reliability-based, possibility-based and evidence-based (RBDO, PBDO and EBDO) design optimization. It was clearly demonstrated that reducing the amount of available information in quantifying uncertainty, results in a more conservative design. We showed that the proposed Bayesian approach as well as the existing RBDO, PBDO and EBDO methods can quantify the tradeoff between available information and less optimal design (loss of optimality).

ACKNOWLEDGEMENTS

This study was performed with funding from the General Motors Research and Development Center and the Automotive Research Center (ARC), a U.S. Army Center of Excellence in Modeling and Simulation of Ground Vehicles at the University of Michigan. The support is gratefully acknowledged. Such support does not however, constitute an endorsement by the funding agencies of the opinions expressed in the paper.

REFERENCES

1. Tu, J., Choi, K. K. and Park, Y. H., "A New Study on Reliability-Based Design Optimization", ASME Journal of Mechanical Design, 121, 557-564, 1999.
2. Liang, J., Mourelatos, Z. P., and Tu, J., "A Single-Loop Method for Reliability-Based Design Optimization," Proceedings of ASME Design Engineering Technical Conferences, Paper# DETC2004/ DAC-57255, 2004.
3. Wu, Y.-T., Shin, Y., Sues, R. and Cesare, M., "Safety – Factor Based Approach for Probabilistic – Based Design Optimization," 42nd AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics and Materials Conference, Seattle, WA, 2001.

4. Lee, J. O., Yang, Y. O. and Ruy, W. S., “A Comparative Study on Reliability Index and Target Performance Based Probabilistic Structural Design Optimization,” Computers and Structures, 80, 257-269, 2002.
5. Youn, B. D., Choi, K. K. and Park, Y. H. “Hybrid Analysis Method for Reliability-Based Design Optimization,” ASME Journal of Mechanical Design, 125(2), 221-232, 2001.
6. Oberkampf, W., Helton, J. and Sentz, K., “Mathematical Representations of Uncertainty,” AIAA Non-Deterministic Approaches Forum, AIAA 2001-1645, Seattle, WA, April 16-19, 2001.
7. Sentz, K. and Ferson, S., “Combination of Evidence in Dempster – Shafer Theory,” Sandia National Laboratories Report SAND2002-0835, April 2002.
8. Klir, G. J. and Yuan, B., Fuzzy Sets and Fuzzy Logic: Theory and Applications, Prentice Hall, 1995.
9. Klir, G. J. and Filger, T. A., Fuzzy Sets, Uncertainty, and Information, Prentice Hall, 1988.
10. Yager, R. R., Fedrizzi, M. and Kacprzyk, J. (Editors), Advances in the Dempster – Shafer Theory of Evidence, John Wiley & Sons, Inc., 1994.
11. Dubois, D. and Prade, H., Possibility Theory, Plenum Press, New York, 1988.
12. Moore, R. E., Interval Analysis, Prentice-Hall, 1966.
13. Zadeh, L. A., “Fuzzy Sets,” Information and Control, 8, 338-353, 1965.
14. Muhanna, R. L. and Mullen, R. L., “Uncertainty in Mechanics Problems – Interval-Based Approach,” Journal of Engineering Mechanics, 127(6), 557-566, 2001.
15. Mullen, R. L. and Muhanna, R. L., “Bounds of Structural Response for all Possible Loadings,” ASCE Journal of Structural Engineering, 125(1), 98-106, 1999.
16. Penmetsa, R. C. and Grandhi, R. V., “Efficient Estimation of Structural Reliability for Problems with Uncertain Intervals,” Computers and Structures, 80, 1103-1112, 2002.
17. Du, X. and Sudjianto, A., “Reliability-Based Design with a Mixture of Random and Interval Variables,” Proceedings of ASME Design Engineering Technical Conferences, Paper# DETC2003/ DAC-48709, 2003.
18. Rao, S. S. and Cao, L., “Optimum Design of Mechanical Systems Involving Interval Parameters,” ASME Journal of Mechanical Design, 124, 465-472, 2002.
19. Gu, X., Renaud, J. E. and Batill, S. M., “An Investigation of Multidisciplinary Design Subject to Uncertainties,” 7th AIAA/USAF/NASA/ISSMO Multidisciplinary Analysis & Optimization Symposium, St. Louis, Missouri, 1998.
20. Elishakoff, I. E., Haftka, R. T. and Fang, J. “Structural Design under Bounded Uncertainty – Optimization with Anti-Optimization,” Computers and Structures, 53, 1401-1405, 1994.
21. Lombardi, M. and Haftka, R. T., “Anti-Optimization Technique for Structural Design under Load Uncertainties,” Computer Methods in Applied Mechanics and Engineering, 157, 19-31, 1998.
22. Du, X. and Chen, W., “An Integrated Methodology for Uncertainty Propagation and Management in Simulation-Based Systems Design,” AIAA Journal, 38(8), 1471-1478, 2000.
23. Mourelatos, Z. P. and Zhou, J., “Reliability Estimation with Insufficient Data Based on Possibility theory”, AIAA Journal, 43(8), 1696-1705, 2005.

24. Zhou, J. and Mourelatos, Z. P., "A Sequential Algorithm for Possibility-Based Design Optimization," ASME Journal of Mechanical Design, 130(1), 2008.
25. Choi, K. K., Du, L. and Youn, B. D., "A New Fuzzy Analysis Method for Possibility-Based Design Optimization," 10th AIAA/ISSMO Multidisciplinary Analysis and Optimization Conference, AIAA 2004-4585, Albany, NY, 2004.
26. Nikolaidis, E., Chen, S., Cudney, H., Haftka, R. T. and Rosca, R., "Comparison of Probability and Possibility for Design Against Catastrophic Failure Under Uncertainty," ASME Journal of Mechanical Design, 126, 2004.
27. Oberkampf, W. L. and Helton, J. C., "Investigation of Evidence Theory for Engineering Applications," AIAA Non-Deterministic Approaches Forum, AIAA 2002-1569, Denver, CO, April, 2002.
28. Bae, H-R, Grandhi, R. V. and Canfield, R. A., "An Approximation Approach for Uncertainty Quantification Using Evidence Theory," Reliability Engineering and System Safety, 86, 215-225, 2004.
29. Bae, H-R, Grandhi, R. V. and Canfield, R. A., "Epistemic Uncertainty Quantification Techniques Including Evidence Theory for Large-Scale Structures," Computers and Structures, 82, 1101-1112, 2004.
30. Agarwal, H., Renaud, J. E., Preston, E. L. and Padmanabhan, D., "Uncertainty Quantification Using Evidence Theory in Multidisciplinary Design Optimization," Reliability Engineering and System Safety, 85, 281-294, 2004.
31. Mourelatos, Z. P. and Zhou, J., "A Design Optimization Method using Evidence Theory," ASME Journal of Mechanical Design, 128(4), 901-908, 2006.
32. Box, G. and Tiao, G., Bayesian Inference in Statistical Analysis, John Wiley & Sons, New York, 1992.
33. Dempster, A. P., "A Generalization of Bayesian Inferences based on a Sample from a Finite Univariate Population," Biometrics, 54(2-3), 515-528, 1967.
34. Siu, N. O. and Kelly, D. L., "Bayesian Parameter Estimation in Probabilistic Risk Assessment," Reliability Engineering System Safety, 62(1-2), 89-116, 1998.
35. Ross, T. J., Fuzzy Logic with Engineering Applications, McGraw Hill, 1995.
36. Sentz, K. and Ferson, S., "Combination of Evidence in Dempster – Shafer Theory," Sandia National Laboratories Report SAND2002-0835, April 2002.
37. Dong, W. M. and Shah, H. C., "Vertex Method for Computing Functions of Fuzzy Variables," Fuzzy Sets and Systems, 24, 65-78, 1987.
38. Choi, K. K., Du, L. and Youn, B. D., "A New Fuzzy Analysis Method for Possibility-Based Design Optimization," 10^{th} AIAA/ISSMO Multidisciplinary Analysis and Optimization Conference, AIAA 2004-4585, Albany, NY, 2004.
39. Tu, J. and Jones, D. R., "Variable Screening in Metamodel Design by Cross-Validated Moving Least Squares Method", Proceedings 44^{th} AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics and Materials Conference, AIAA-2003-1669, Norfolk, VA, April 7-10, 2003.

40. Ye, K. Q., Li, W. and A. Sudjianto, "Algorithmic Construction of Optimal Symmetric Latin Hypercube Designs", Journal of Statistical Planning and Inference, 90, 145-159, 2000.
41. Jones, D. R., Perttunen, C. D. and Stuckman, B. E., “Lipschitzian Optimization without the Lipschitz Constant,” Journal of Optimization Theory and Applications, 73(1), 157-181, 1993.
42. Subroto, G. and Papalambros, P.Y., “A Bayesian Approach to Reliability-Based Optimization with Incomplete Information,” ASME Journal of Mechanical Design, 128(4), 909-918, 2006.
43. Lewis, K. and Mistree, F., “Collaborative, Sequential and Isolated Decisions in Design,” Proceedings of ASME Design Engineering Technical Conferences, Paper# DETC1997/ DTM-3883, 1997.