ROGUE WAVES IN OCEANIC TURBULENCE
&
VORTICES IN AXIS-SYMMETRIC TURBULENT FLOW:

AN EXTREME VIEW

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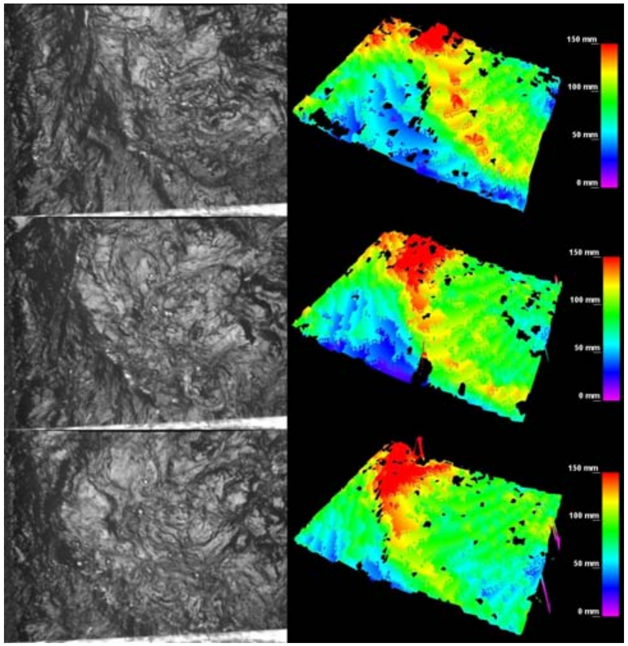
ATMOSPHERIC BOUNDARY LAYER & WIND-WAVE INTERACTION

STEREO-VIDEO IMAGERY & HOT-WIRE ANEMOMETRY EXPERIMENTS



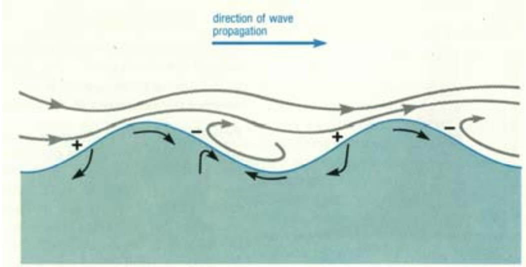


Figure 1.3 Jeffreys' 'sheltering' model of wave generation. Curved lines indicate air flow; short, straight arrows show water movement, which will be explained more fully in Section 1.2.1. The rear face of the wave against which the wind blows experiences a higher pressure than the front face, which is sheltered from the force of the wind. Air eddies are formed in front of each wave, leading to differences in air pressure. The excesses and deficiencies of pressure are shown by plus and minus signs respectively. The pressure difference pushes the wave along.

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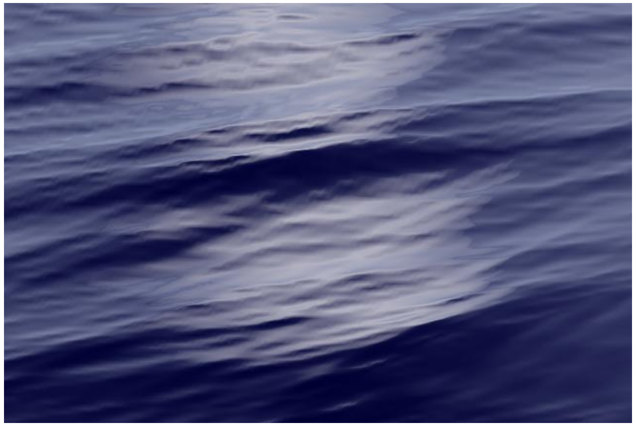
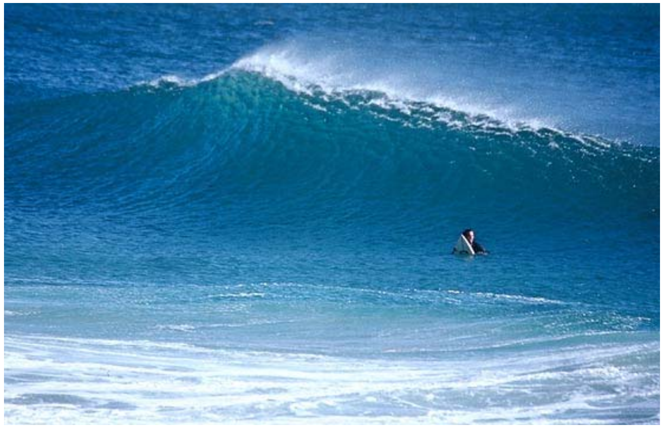
ROGUE WAVES , HURRICANE WAVES , GIANT WAVES, FREAK WAVES ....



A NATURAL BEAUTY !



Freak waves

Rogue waves

Giant waves

Extreme waves

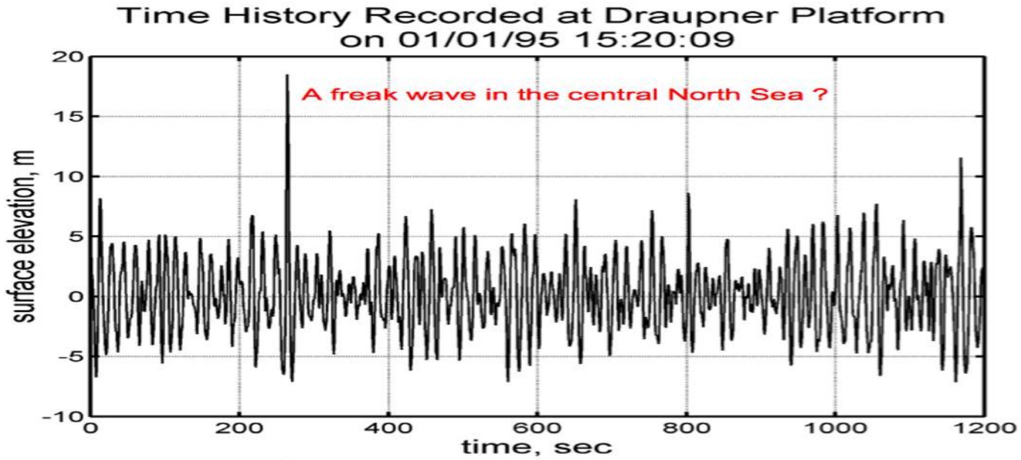
Rogue waves

Extreme waves

Giant waves

Freak waves

DRAUPNER EVENT JANUARY 1995



H_{max} = 25.6 m !

Extremely rare event
according to Gaussian model
Probability < 10^{-6} !!!

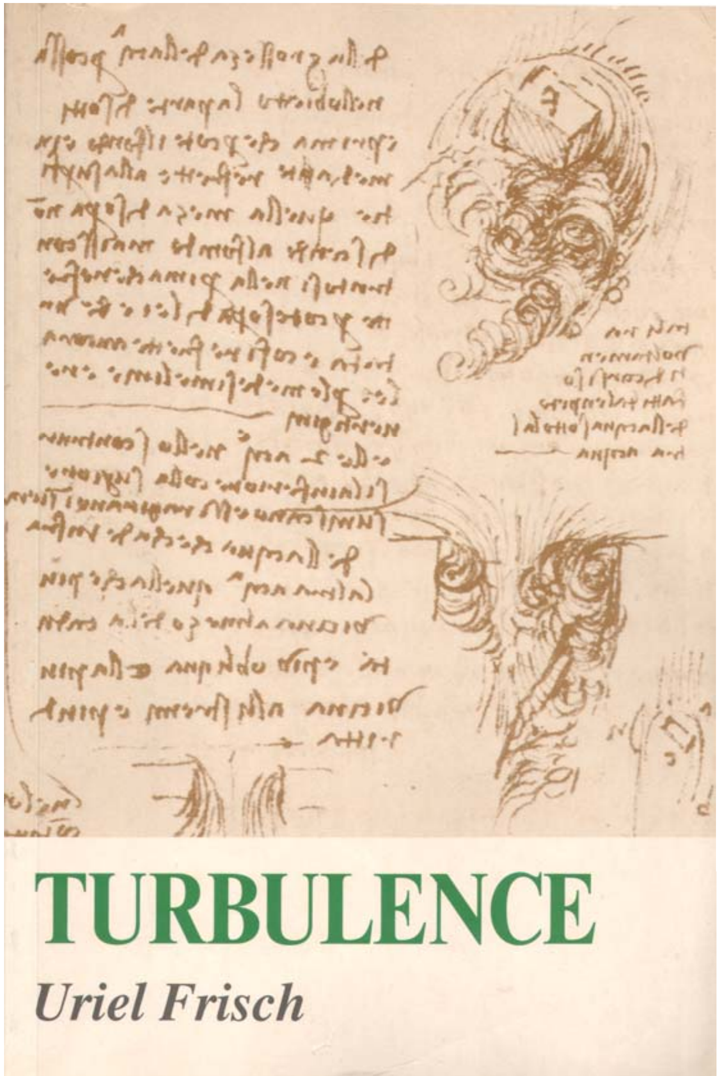
But they still occur in open
ocean !

ROGUE WAVES

Rare events of a normal population
or
typical events of a special population ?

➡ OCEANIC TURBULENCE OF ZAKHAROV
-weak wave turbulence –
- NLS turbulence –

➡ Concept of STOCHASTIC WAVE GROUP
(my contribution)



1.1 Turbulence and symmetries

In Chapter 41 of his *Lectures on Physics*, devoted to hydrodynamics and turbulence, Richard Feynman (1964) observes this:

Often, people in some unjustified fear of physics say you can't write an equation for life. Well, perhaps we can. As a matter of fact, we very possibly already have the equation to a sufficient approximation when we write the equation of quantum mechanics:

$$H\psi = -\frac{\hbar}{i}\frac{\partial \psi}{\partial t}. \tag{1.1}$$

Of course, if we only had this equation, without detailed observation of biological phenomena, we would be unable to reconstruct them. Feynman believes, and this author shares his viewpoint, that an analogous situation prevails in *turbulent* flow of an incompressible fluid. The equation, generally referred to as the Navier–Stokes equation, has been known since Navier (1823):

$$\partial_t \boldsymbol{v} + \boldsymbol{v}\cdot\nabla\boldsymbol{v} = -\nabla p + \nu\nabla^2\boldsymbol{v}, \tag{1.2}$$

$$\nabla\cdot\boldsymbol{v} = 0. \tag{1.3}$$

It must be supplemented by initial and boundary conditions (such as the vanishing of $\boldsymbol{v}$ at rigid walls). We shall come back later to the choice of notation.

Quantum version of the
The Nonlinear Schrödinger (NLS) equation
cousin
of
the Korteweg-de Vries Equation

$$i\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial^2 u}{\partial \xi^2} + k|u|^2 u = 0$$

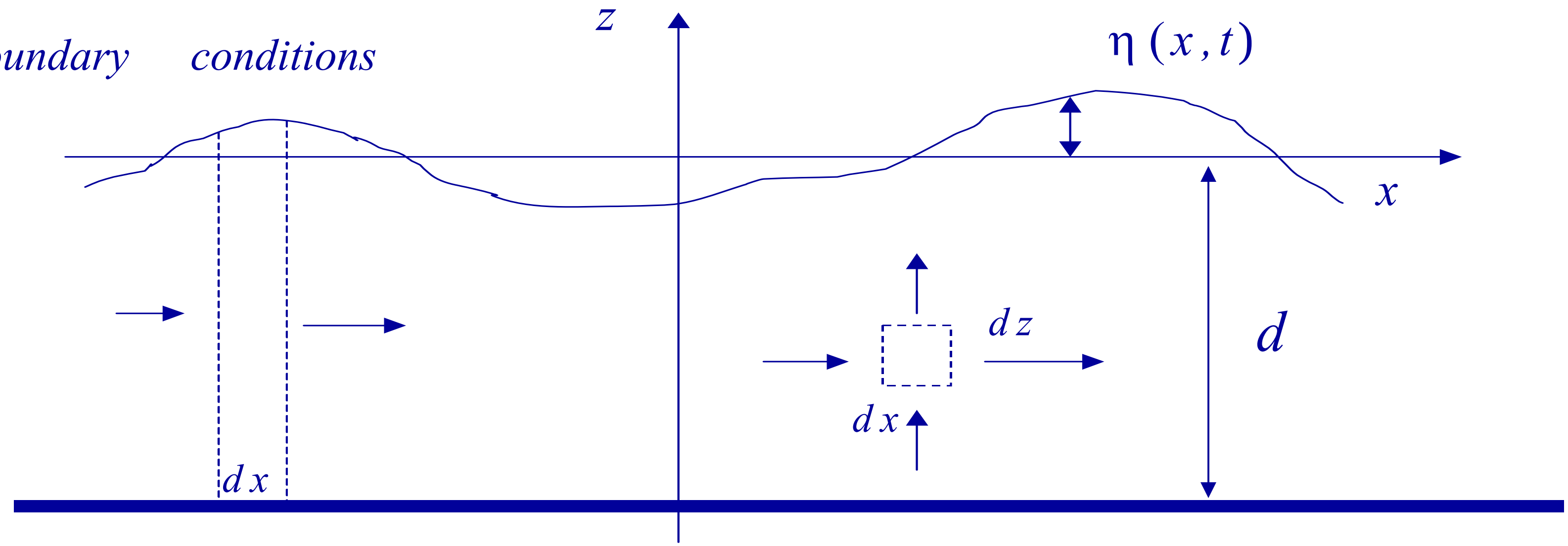
$$\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial \xi^3} + ku\frac{\partial u}{\partial \xi} = 0$$

...... START WITH NAVIER-STOKES EQUATIONS TO MODEL WAVE DYNAMICS ......

$$
\begin{cases}
\dfrac{\partial^2 \Phi}{\partial x^2} + \dfrac{\partial^2 \Phi}{\partial z^2} = 0 \\[2ex]
\left(\dfrac{\partial \Phi}{\partial z}\right)_{z=\eta} = \dfrac{\partial \eta}{\partial t} + \dfrac{\partial \eta}{\partial x}\left(\dfrac{\partial \Phi}{\partial x}\right)_{z=\eta} \\[2ex]
\left(\dfrac{\partial \Phi}{\partial t}\right)_{z=\eta} + \dfrac{1}{2}\left[\left(\dfrac{\partial \Phi}{\partial x}\right)^2 + \left(\dfrac{\partial \Phi}{\partial z}\right)^2\right]_{z=\eta} + g\eta = f(t) \\[2ex]
\textit{boundary} \quad \textit{conditions}
\end{cases}
$$

$$v_z = \frac{\partial \Phi}{\partial z} \qquad v_x = \frac{\partial \Phi}{\partial x}$$

Inviscid, irrotational

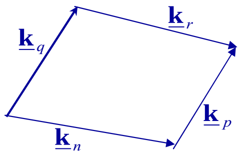


... and by multiple scale perturbation method you get Zakharov equation for WAVE TURBULENCE ....

Third order effects :
FOUR-WAVE RESONANCE
(*WAVE TURBULENCE*)

$$\eta(\underline{\mathbf{x}},t) = \frac{1}{\pi}\sum_n \sqrt{\frac{\omega_n}{2g}}\left|B_n(t)\right|\cos\left(\underline{\mathbf{k}}_n \cdot \underline{\mathbf{x}} + \left|\varphi_n(t)\right|\right)$$

$$\frac{dB_n}{dt} + i\omega_n B_n = -i\sum_{p,q,r} T_{npqr}\delta_{n+p-q-r}B_p^* B_q B_r$$



Quartet interaction

$$\underline{\mathbf{k}}_n + \underline{\mathbf{k}}_p = \underline{\mathbf{k}}_q + \underline{\mathbf{k}}_r$$

Conserved quantities :
Hamiltonian
Wave action
Wave momentum

$$\mathrm{H} = \sum_n \omega_n B_n(t) B_n^*(t) + \frac{1}{2}\sum_{n,p,q,r} T_{npqr}\delta_{n+p-q-r} B_n^*(t) B_p^*(t) B_q(t) B_r(t)$$

$$\mathbf{A} = \sum_n B_n(t) B_n^*(t) \qquad \mathbf{M} = \sum_n \mathbf{k}_n B_n(t) B_n^*(t)$$

Chaotic behavior of a sea of weakly dispersive nonlinear waves

... moreover for narrow-band waves the Zakharov equation reduces to...

$$i\frac{\partial u}{\partial t}+\frac{1}{2}\frac{\partial^2 u}{\partial \xi^2}+k|u|^2u=0$$

In deep water (NLS)

$$\frac{\partial u}{\partial t}+\frac{\partial^3 u}{\partial \xi^3}+ku\frac{\partial u}{\partial \xi}=0$$

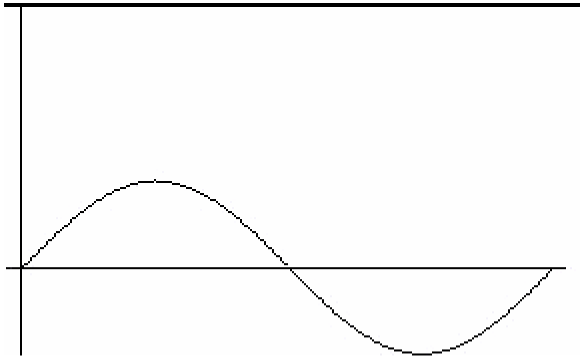
In shallow water (KdV)

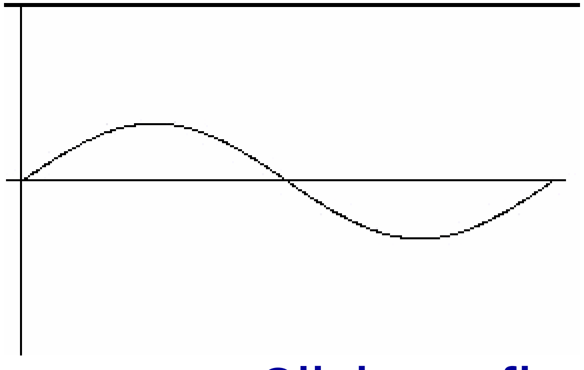
Exact analytical solutions via the Inverse Scattering Transform Technique !

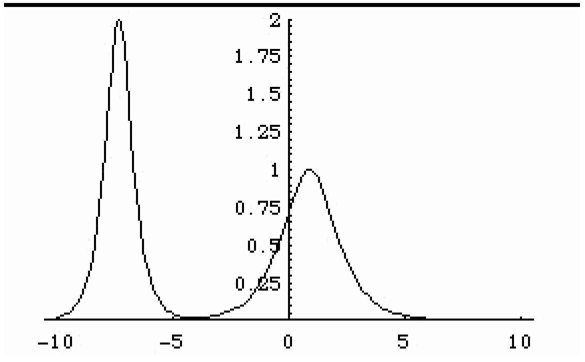
NLS solitons and KdV Cnoidal waves

chaotic behavior due to nonlinear interaction of waves and solitons

$$
\begin{aligned}
&\operatorname{sech}^2\left(\frac{\sqrt{b_1}\,(x-2\,t\,b_1)}{\sqrt{2}}\right)b_1- \\
&\left(2\,(b_2-b_3)\left(\left(2\,(b_3-b_1)\left(\operatorname{sech}^2\left(\frac{\sqrt{b_3}\,(x-2\,t\,b_3)}{\sqrt{2}}\right)b_3-\operatorname{sech}^2\left(\frac{\sqrt{b_1}\,(x-2\,t\,b_1)}{\sqrt{2}}\right)b_1\right)\right)\Big/\right.\right. \\
&\quad\left(\sqrt{2}\,\sqrt{b_3}\,\tanh\left(\frac{\sqrt{b_3}\,(x-2\,t\,b_3)}{\sqrt{2}}\right)-\sqrt{2}\,\sqrt{b_1}\,\tanh\left(\frac{\sqrt{b_1}\,(x-2\,t\,b_1)}{\sqrt{2}}\right)\right)^2- \\
&\quad\left(2\,(b_1-b_2)\left(b_2\operatorname{csch}^2\left(\frac{\sqrt{b_2}\,(x-2\,t\,b_2)}{\sqrt{2}}\right)+\operatorname{sech}^2\left(\frac{\sqrt{b_1}\,(x-2\,t\,b_1)}{\sqrt{2}}\right)b_1\right)\right)\Big/ \\
&\quad\left.\left.\left(\sqrt{2}\,\sqrt{b_1}\,\tanh\left(\frac{\sqrt{b_1}\,(x-2\,t\,b_1)}{\sqrt{2}}\right)-\sqrt{2}\,\coth\left(\frac{\sqrt{b_2}\,(x-2\,t\,b_2)}{\sqrt{2}}\right)\sqrt{b_2}\right)^2\right)\right)\Big/ \\
&\left((2\,(b_1-b_2))\Big/\left(\sqrt{2}\,\sqrt{b_1}\,\tanh\left(\frac{\sqrt{b_1}\,(x-2\,t\,b_1)}{\sqrt{2}}\right)-\sqrt{2}\,\coth\left(\frac{\sqrt{b_2}\,(x-2\,t\,b_2)}{\sqrt{2}}\right)\sqrt{b_2}\right)-\right. \\
&\quad\left.(2\,(b_3-b_1))\Big/\left(\sqrt{2}\,\sqrt{b_3}\,\tanh\left(\frac{\sqrt{b_3}\,(x-2\,t\,b_3)}{\sqrt{2}}\right)-\sqrt{2}\,\sqrt{b_1}\,\tanh\left(\frac{\sqrt{b_1}\,(x-2\,t\,b_1)}{\sqrt{2}}\right)\right)\right)^2
\end{aligned}
$$







Click on figures to see animations

More at ….

http://www.math.h.kyoto-u.ac.jp/~takasaki/soliton-lab/gallery/solitons/kdv-e.html

Classical (Linear) Fourier analysis

A_n stochastically independent
Linear interaction Harmonic waves

$$\Psi(x) = \sum A_n \exp(ik_n x)$$

Nonlinear Fourier analysis
(NLS , wave turbulence)

A_n dependent
nonlinear interaction
Of Solitons, wave groups

$$\Psi(x,t) = \sum A_n(t) \bullet (Soliton,\ wave\ Group)$$

Axis-symmetric pipe turbulence

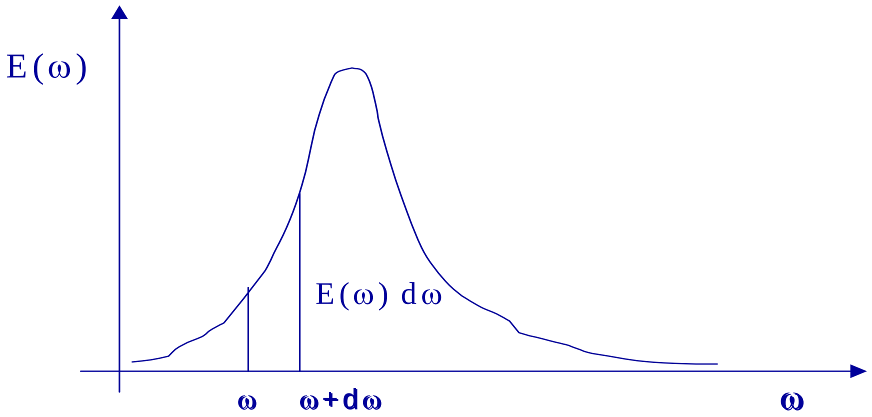
A_n dependent
nonlinear interaction
Of Cnoidal waves

$$\Psi(x,t) = \sum A_n(t) \bullet (Cnoidal\ \ waves)$$

'OPTIMAL' NONLINEAR BASES FOR GALERKIN PROJECTION

LINEAR WAVES : GAUSSIAN SEAS

$$\eta(x,t) = \sum_{j=1}^{N} a_j \cos(k_j x + \omega_j t + \varepsilon_j)$$

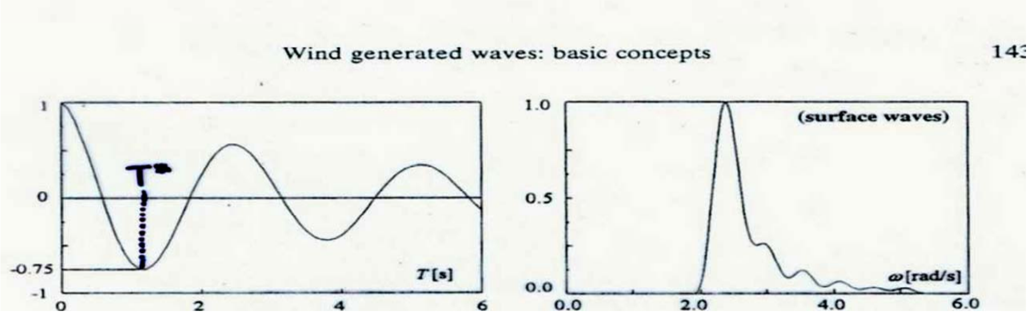


$$E(\omega)d\omega = \frac{1}{2}\sum_j a_j^2$$

$$\psi(T) = \langle \eta(x_0,t_0)\eta(x_0,t_0+T)\rangle = \int_0^\infty E(\omega)\cos\omega T\, d\omega$$

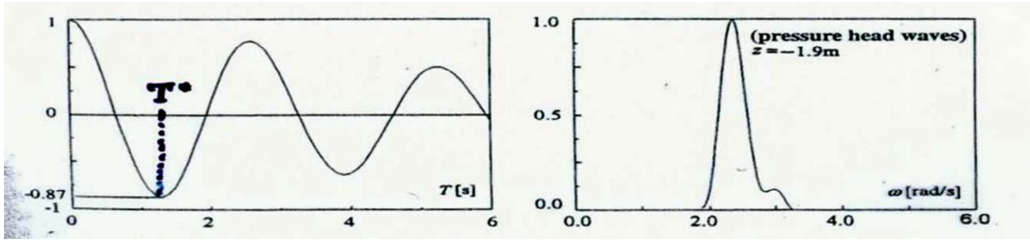
TYPICAL WAVE SPECTRA OF THE MEDITERRANEAN SEA*

Time covariance ***Spectrum***



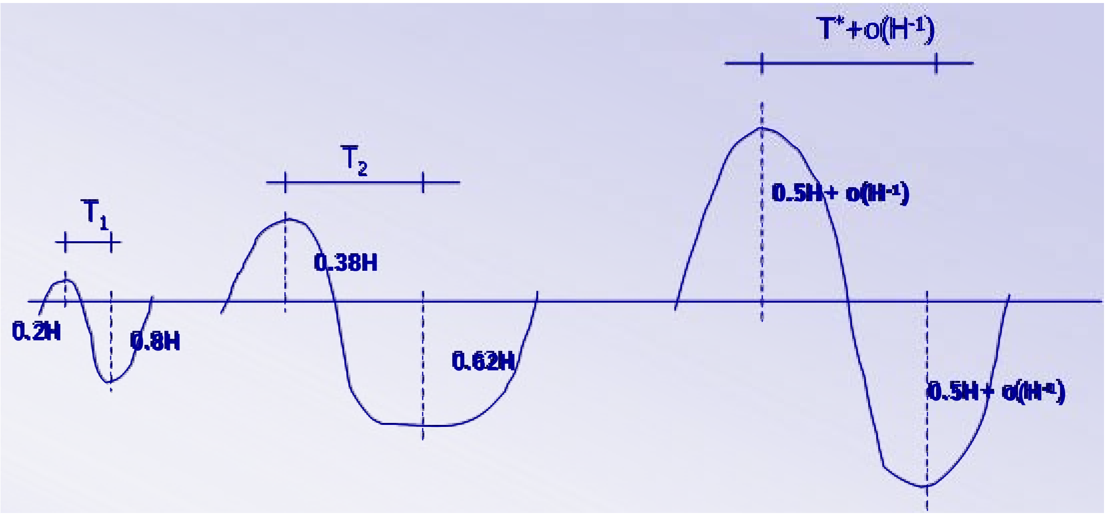
Broad-band spectra

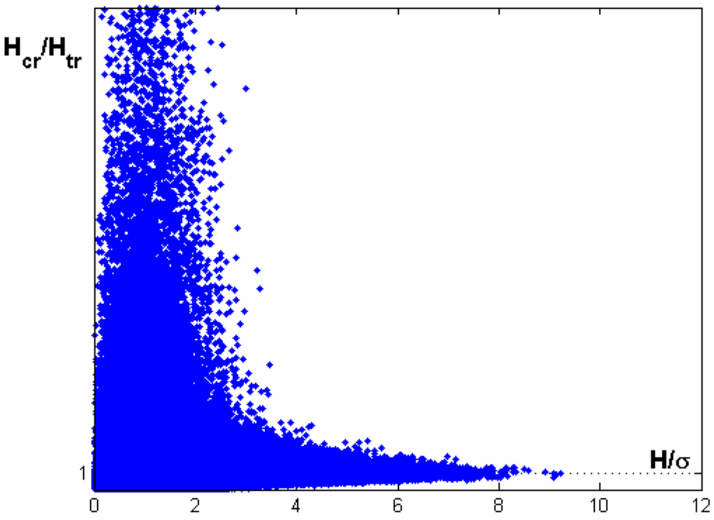
Narrow-band spectra

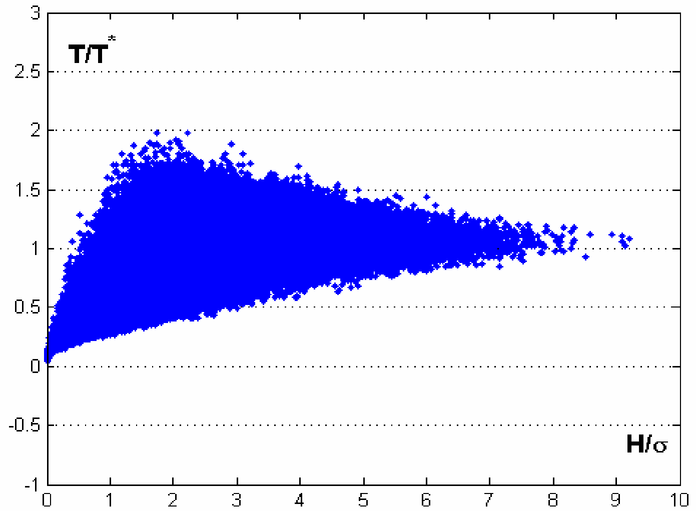


*from *Boccotti P.* **Wave Mechanics** *2000 Elsevier*

NECESSARY AND SUFFICIENT CONDITIONS FOR THE OCCURRENCE OF A HIGH WAVE IN TIME*







*Theory of quasi-determinism, *Boccotti P.* **Wave Mechanics** *2000 Elsevier*

What happens in the neighborhood of a point x_0 if a large crest followed by large trough are recorded in time at x_0 ?

What is the probability that

$$\eta(\mathrm{x}_0 + \mathrm{X}, t_0 + T) \in (u, u + du)$$

conditioned to

$$\eta(\mathrm{x}_0, t_0) = H/2, \;\; \eta(\mathrm{x}_0, t_0 + T_2^*) = -H/2 \;\; ?$$

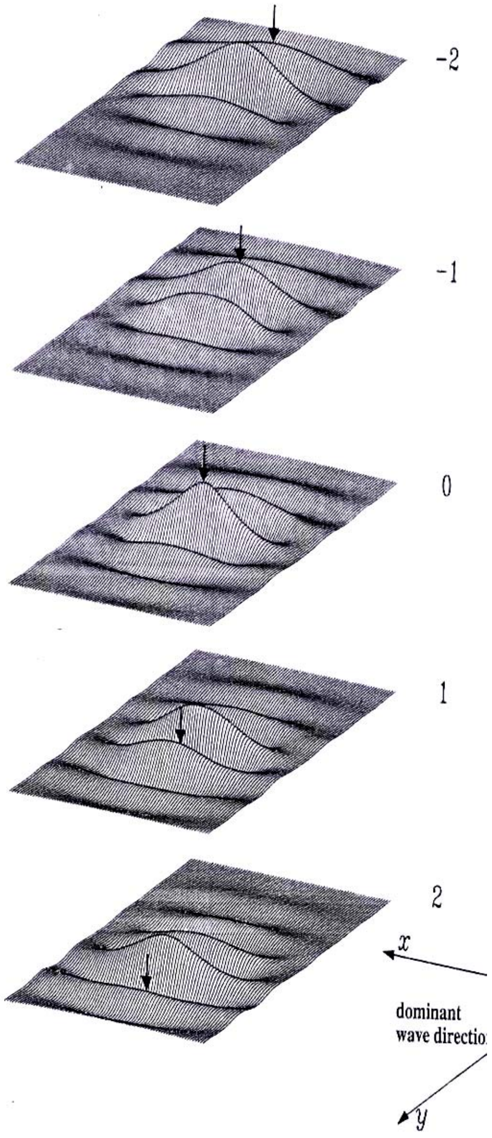
$$h = \frac{H}{\sigma} \rightarrow \infty$$

$$\{\eta | \eta(x_0, t_0) = h\} = h\Psi + \Delta$$

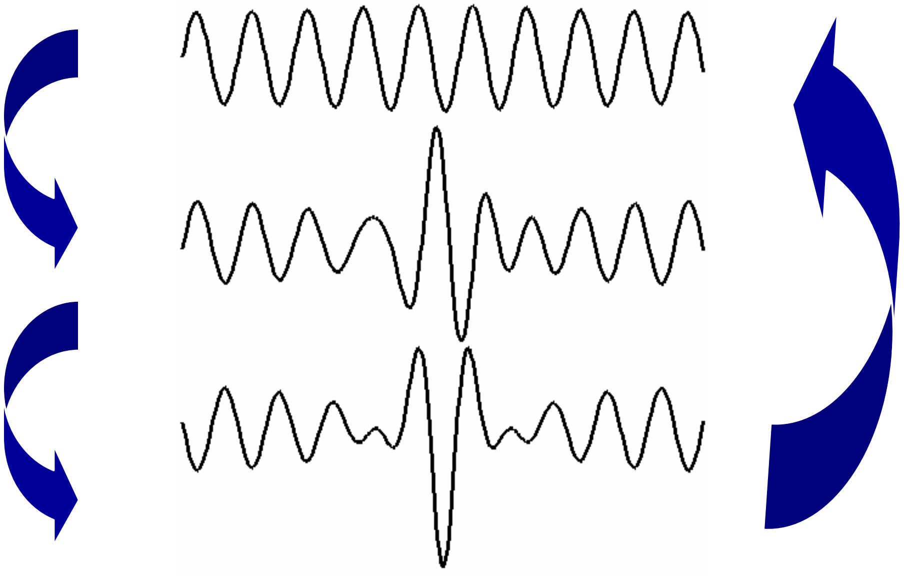
Ψ **SPACE-TIME covariance**

Δ random residual, h Rayleigh variable

stochastic wave group



NONLINEAR DYNAMICS: FOUR-WAVE RESONANCE



Crest-trough symmetry
kurtosis>3

Modulation instability

Effects on slow time scale >> wave period

DOMINANT ONLY IN
UNIDIRECTIONAL NARROW-BAND SEAS !

Weak turbulence

$$\eta = \eta_1 + f(\eta_1) \qquad f(\bullet)\ nonlinear$$

$O(\varepsilon)\uparrow \qquad O(\varepsilon^2)\uparrow$

Linear conditional process

(Gaussian group)

$$\{\eta_1 | \eta(x_0, t_0) = h_1\} = h_1 \Psi + \Delta$$

Nonlinear Conditional process

$$\{\eta | \eta(x_0, t_0) = h\} \ \ ?$$

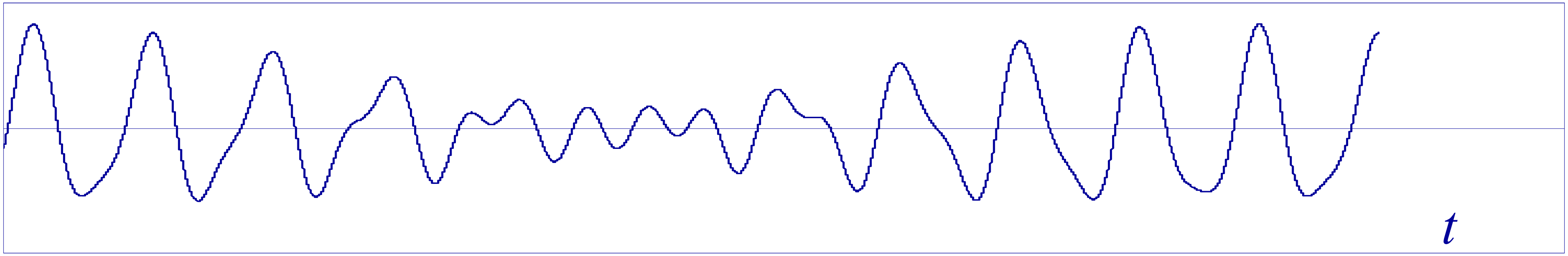
$$\{\eta | \eta(x_0, t_0) = h\} = \{\eta | \eta_1(x_0, t_0) = h_1\}$$

$$\Downarrow$$

Non-Gaussian group

$$\{\eta | \eta_1(x_0, t_0) = h_1\} = h_1 \Psi + \Delta + f(h_1 \Psi + \Delta)$$

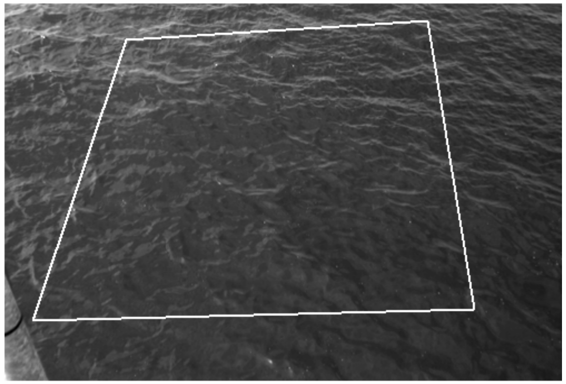
What is a probability of exceedance for crests ?

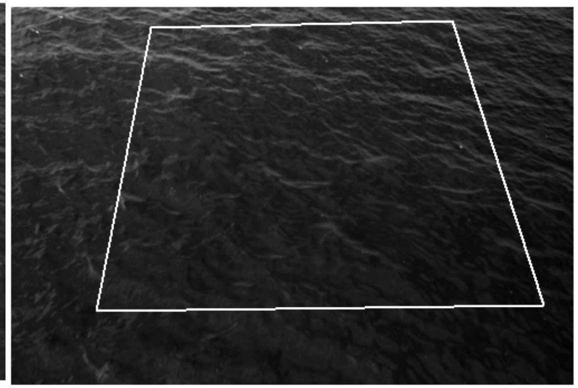


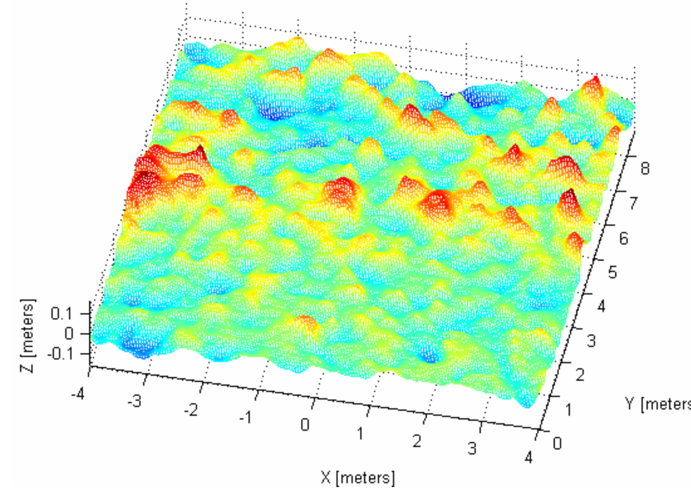
$$P[Z] = \frac{number\ of\ waves\ with\ crest\ greater\ than\ Z}{total\ number\ of\ waves}$$

$$\Pr(crest\ height > Z) = \exp[-\frac{1}{2\ \mu^{*2}}\ (-1 + \sqrt{1 + 2\,\mu^* Z}\ )^2\,]\ [1 + \frac{\Lambda}{64}\ (Z^4 - 8Z^2 + 8)\,]$$

VARIATIONAL WAVE ACQUISITION STEREO SYSTEM







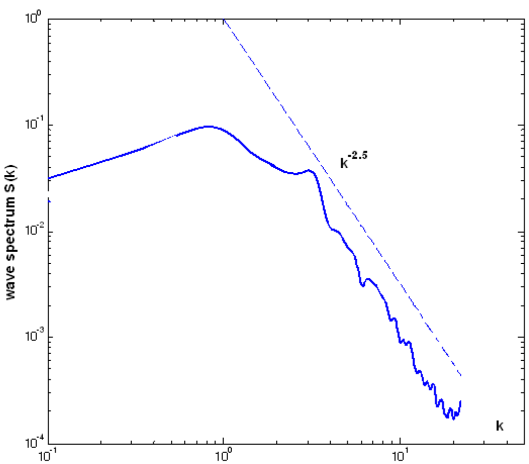


Figure 5. Wave spectrum *S(k)* as function of the wave number *k* computed from the reconstructed wave surface η in Figure 4. The spectrum tail decays as $k^{-2.5}$ in agreement with wave turbulence theory (Zakharov 1999, Socquet-Juglard et al. 2005).

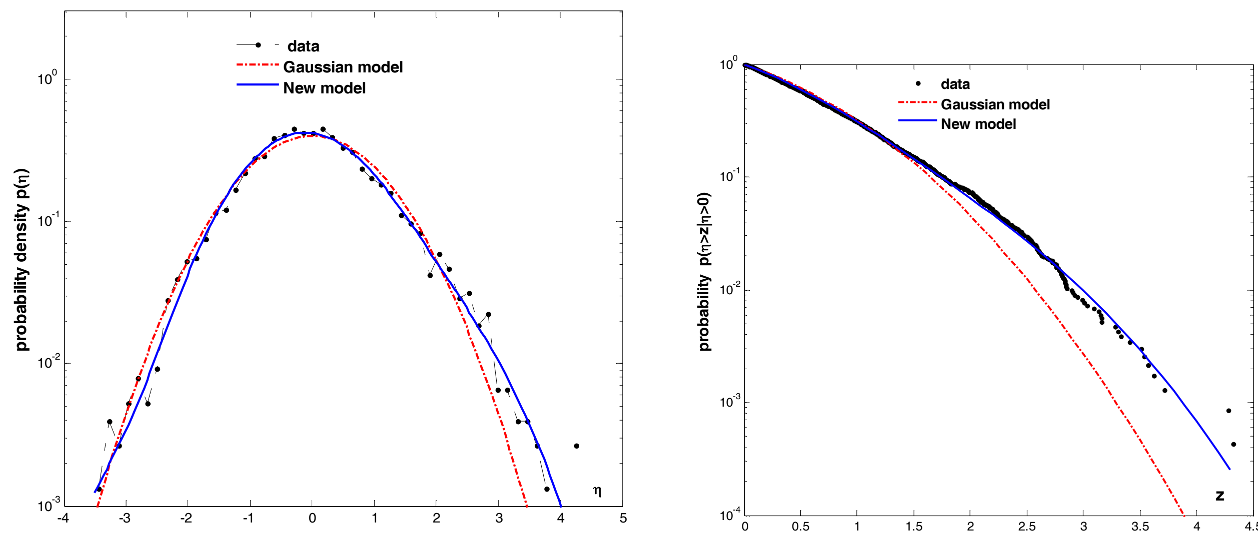


Figure 6. Probability density *p(η)* of the reconstructed wave surface η in Figure 4: comparisons with theoretical stochastic models for wave height probabilities (Tayfun & Fedele 2007, Fedele 2008).

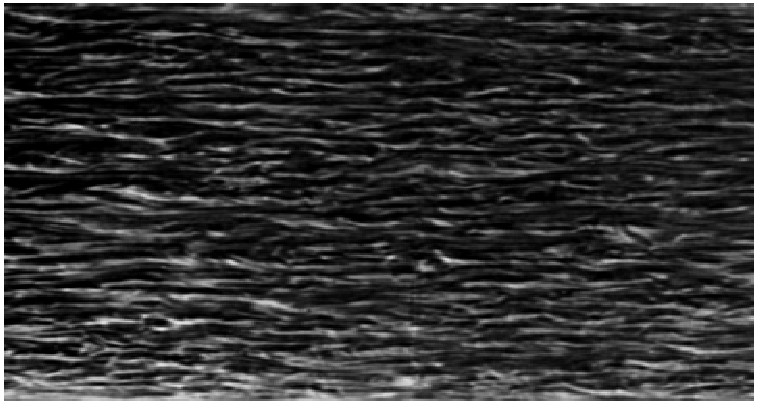
Key words: OCEANIC TURBULENCE, ROGUE WAVES, STOCHASTIC WAVE GROUP, NLS & KdV equations, Coherent structures

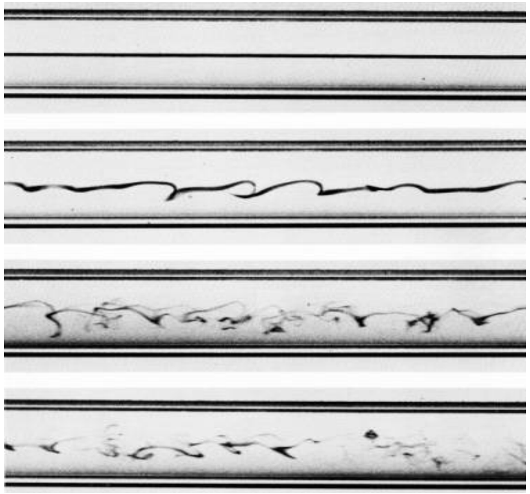
Chaotic behavior of a sea of weakly dispersive nonlinear waves

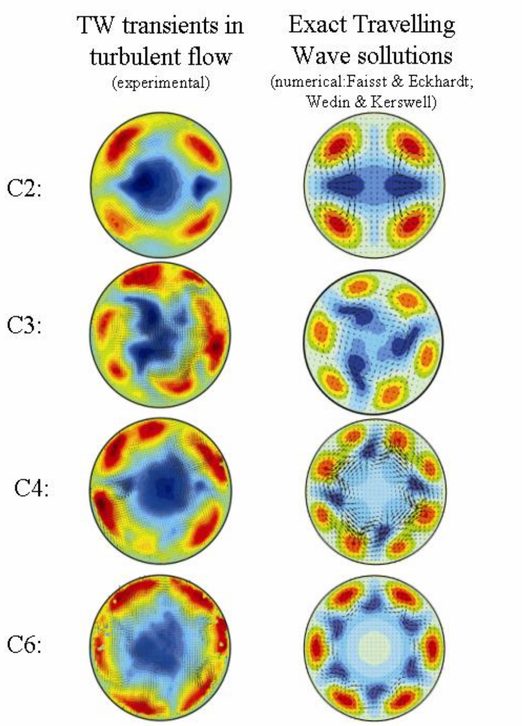
Rogue waves in oceanic turbulence occur due to the nonlinear dynamics of stochastic wave groups

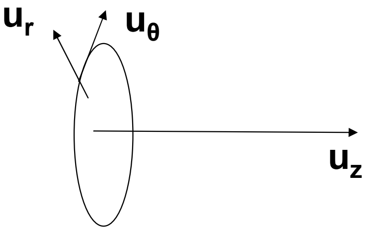
Can we extend these concepts to pipe turbulence ?

TURBULENCE IN PIPE FLOWS





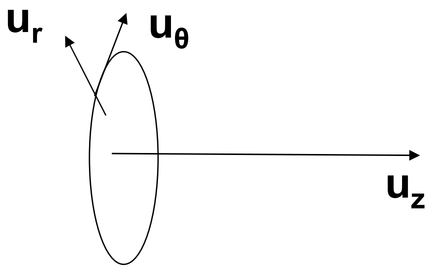




AXIS-SYMMETRIC TURBULENCE or "ACADEMIC TURBULENCE"

$$G\Psi_t + L\Psi + N\Psi = 0$$

$$G\Psi = \Psi_{rr} - \frac{\Psi_r}{r} - \Psi_{zz}$$



$$L\Psi = W_0\Psi_{zzz} + \left(W_0 G\Psi - \Psi G W_0\right)_z - R_e^{-1} G^2 \Psi$$

$$N\Psi = \Psi_r\left(1/r \cdot G\Psi\right)_z - \Psi_z\left(1/r \cdot G\Psi\right)_r$$

... expand as ....

$$\Psi = \sum A_n(z,t)\psi_n(r)$$

$$\lambda_n G\psi_n + L\psi_n = 0$$

$$A_j(z,t) = \varepsilon^2 a_j\left[\varepsilon\xi, \varepsilon^3 t\right] + ..., \qquad \varepsilon = \mathrm{Re}^{-1} \qquad \mathrm{Re} \to \infty$$

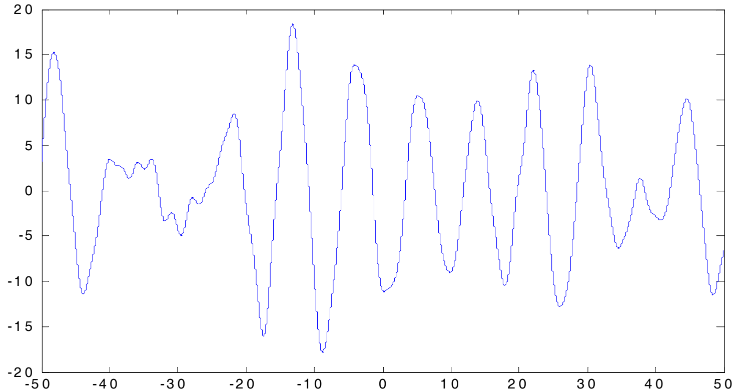
$$\xi = \left(z - c_j t\right) \qquad \text{moving frame streamwise direction}$$

KdV system

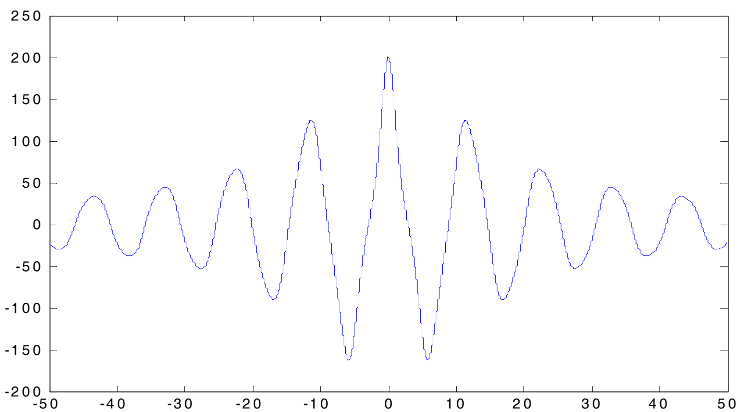
$$\frac{\partial a_j}{\partial t} + \beta_{jj}\frac{\partial^3 a_j}{\partial \xi^3} + \Gamma_{jj} a_j \frac{\partial a_j}{\partial \xi} + \lambda_j a_j = -\sum_m \left(\Gamma_{jm} a_j \frac{\partial a_m}{\partial \xi} + \beta_{jm}\frac{\partial^3 a_m}{\partial \xi^3} \right)$$

chaotic behavior due to nonlinear interactions of *Cnoidal waves*

Incoherence

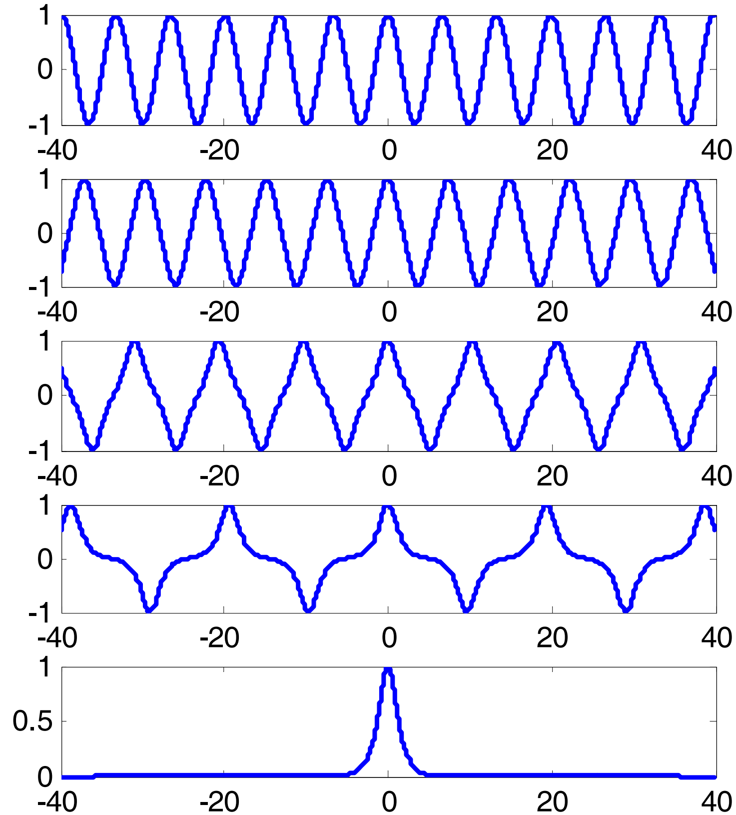


Coherence: Cnoidal wave group



streamwise direction

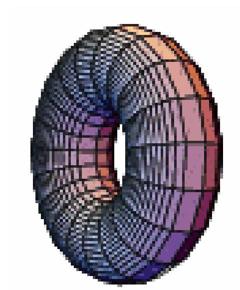
Fourier waves



Cnoidal waves

Cnoidal solitons

streamwise direction



Toroidal vortex tube
Modulated by Cnoidal waves in the streamwise direction

... more work to be done ...

QUESTIONS ?