

Worst Case Bounds on the Point-wise Discretization Error in Boundary Element Method for Elasticity Problem

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Abstract: In this work, local discretization error is bounded via interval approach for the elasticity problem using the interval boundary element formulation. The formulation allows for computation of the worst case bounds on the errors in the solution of elasticity problem. From these bounds the worst case bounds on the discretization error of any point in the domain of the boundary can be computed. Examples are presented to demonstrate the effectiveness of the treatment of local discretization error in elasticity problem via interval methods.

Keywords: interval boundary element method, interval analysis, discretization error, elastostatics

1. Introduction

Most of the problems in engineering mechanics are governed by partial differential equations, to which solutions, in general, cannot be obtained exactly due to complexities in the geometry of the system for which the applied boundary conditions must be satisfied. Therefore, numerical methods have been developed to approximate the true solution by a polynomial interpolation between discrete values. The foremost method is the finite element method (FEM), in which the domain of the system is discretized into elements consisting of polynomial interpolation functions between discrete values which are to be computed. Another numerical method used to approximate the solutions to partial differential equations is the boundary element method (BEM). In boundary element analysis (BEA), the domain variables are transformed to the boundary variables, thus decreasing the dimension of the problem by one. This allows, in general, decreasing the time necessary for mesh generation or mesh refinement. The domain transformation is performed by the use of fundamental solutions to the linear partial differential equations, thus restricting classical BEM to problems for which the fundamental solution is known. The boundary integral equations, resulting from weighted residual formulation, are solved using point collocation methods, in which the residual is set to zero in the domain and exists only on the boundary of the system. To achieve such residual, the weighted residual function in a weak formulation of the partial differential equation, takes the form of the fundamental solution. The transformed boundary integral equations are then solved by approximating the true solution over discrete boundaries, thus introducing the discretization error. Although discretization error estimates have been made for BEM (Rencis and Jong 1989) the worst case bounds on the local

discretization error have been computed only for the Laplace problem (Zalewski and Mullen 2007).

In this work the point-wise discretization error is studied for the elasticity problem. The boundary integral equations are bounded by interval boundary integral equations, eventually resulting in interval linear system of equations. A parametric solver is reviewed that enables the computation of non-naive bounds. Example problems are presented to illustrate the behavior of the discretization error bounds.

2. Boundary Element Analysis of Elasticity Problem

2.1. BEA FORMULATION FOR ELASTICITY PROBLEM

The boundary element formulation for the behavior of an isotropic and homogeneous body is discussed in the literature (Brebbia 1992, Hartmann 1889, Pilkey and Wunderlich 1994). The following section reviews the two dimensional boundary element formulation for the elasticity problem. The elasticity problem is:

$$\left. \begin{aligned} \sigma_{ij,j} + b_i &= 0 \quad \text{in } \Omega \\ u_i &= \hat{u}_i \quad \text{on } \Gamma_1, \quad t_i = \hat{t}_i \quad \text{on } \Gamma_2 \\ \bigcup_{i=1}^2 \Gamma_i &= \Gamma \quad \text{and} \quad \bigcap_{i=1}^2 \Gamma_i = \emptyset \end{aligned} \right\} \quad (1)$$

where Ω is the domain of the system, Γ is the boundary of the system, σ_{ij} is the stress tensor, b_i is the vector of body force, u_i is the displacement vector with a forced boundary condition \hat{u}_i on Γ_1 , and t_i is the traction vector with a natural boundary condition \hat{t}_i on Γ_2 . The first step in approximating the solution to Eq. (1) is to express it in a weighted residual form or a weak form:

$$\int_{\Omega} (\sigma_{ij,j} + b_i) u_i^* d\Omega = \int_{\Gamma_2} (t_i - \hat{t}_i) u_i^* d\Gamma_2 - \int_{\Gamma_1} (u_i - \hat{u}_i) t_i^* d\Gamma_1 \quad (2)$$

where u_i^* and t_i^* are the weighted residual functions. In the following steps Betti's reciprocal theorem is reviewed and used to formulate boundary integral equations. Expanding the left side of Eq. (2) results in:

$$\int_{\Omega} (\sigma_{ij,j} + b_i) u_i^* d\Omega = \int_{\Omega} \sigma_{ij,j} u_i^* d\Omega + \int_{\Omega} b_i u_i^* d\Omega = 0 \quad (3)$$

Applying the chain rule to the first integral on the right side of Eq. (3) yields:

$$\int_{\Omega} \sigma_{ij,j} u_i^* d\Omega = \int_{\Omega} (\sigma_{ij} u_i^*)_{,j} d\Omega - \int_{\Omega} \sigma_{ij} u_{i,j}^* d\Omega \quad (4)$$

Substituting $\varepsilon_{ij}^* = u_{i,j}^*$ in Eq. (4) results in:

$$\int_{\Omega} \sigma_{ij,j} u_i^* d\Omega = \int_{\Omega} (\sigma_{ij} u_i^*)_{,j} d\Omega - \int_{\Omega} \sigma_{ij} \varepsilon_{ij}^* d\Omega \quad (5)$$

where ε_{ij} is the linear strain tensor. Applying Gauss integral theorem to the first integral on the right side of Eq. (5):

$$\int_{\Omega} (\sigma_{ij} u_i^*)_{,j} d\Omega = \int_{\Gamma} \sigma_{ij} u_i^* n_j d\Gamma = \int_{\Gamma} \sigma_{ij} n_j u_i^* d\Gamma = \int_{\Gamma} t_i u_i^* d\Gamma \quad (6)$$

Substituting the result of Eq. (6) into Eq. (5) and rearranging terms yields:

$$\int_{\Omega} \sigma_{ij} \varepsilon_{ij}^* d\Omega + \int_{\Omega} \sigma_{ij,j} u_i^* d\Omega = \int_{\Gamma} t_i u_i^* d\Gamma \quad (7)$$

The equilibrium condition, $\sigma_{ij,j} = -b_i$, is substituted into Eq. (7) to obtain:

$$\int_{\Omega} \sigma_{ij} \varepsilon_{ij}^* d\Omega - \int_{\Gamma} b_i u_i^* d\Gamma = \int_{\Gamma} t_i u_i^* d\Gamma \quad (8)$$

Following the same procedure, Eq. (3) through Eq. (8), the following equation can be obtained:

$$\int_{\Omega} \sigma_{ij}^* \varepsilon_{ij}^* d\Omega - \int_{\Gamma} b_i^* u_i^* d\Gamma = \int_{\Gamma} t_i^* u_i^* d\Gamma \quad (9)$$

It is then considered that the body follows the linear elastic constitutive model:

$$\sigma_{ij} = E_{ijkl} \varepsilon_{kl} \quad (10)$$

where E_{ijkl} is the fourth order linear elasticity tensor. Eq. (10) can also be written as:

$$\sigma_{ij} = \frac{E}{1+\nu} \varepsilon_{ij} + \frac{\nu E}{(1+\nu)(1-2\nu)} \delta_{ij} \varepsilon_{kk} \quad (11)$$

Also by expansion of σ_{ij} tensor and symmetry of E_{ijkl} tensor with respect to i, j and k, l indices:

$$\sigma_{ij} \varepsilon_{ij}^* = E_{ijkl} \varepsilon_{kl} \varepsilon_{ij}^* = E_{klij} \varepsilon_{ij} \varepsilon_{kl}^* = E_{klji} \varepsilon_{kl}^* \varepsilon_{ij} = E_{ijkl} \varepsilon_{kl}^* \varepsilon_{ij} = \sigma_{ij}^* \varepsilon_{ij} \quad (12)$$

By equating the first integral terms in Eq. (8) and Eq. (9) due to Eq. (12), Betti's reciprocal theorem can be obtained:

$$\int_{\Gamma} t_i u_i^* d\Gamma + \int_{\Gamma} b_i u_i^* d\Gamma = \int_{\Gamma} t_i^* u_i d\Gamma + \int_{\Gamma} b_i^* u_i d\Gamma \quad (13)$$

Eq. (13) is the starting point of the boundary element formulation for the elasticity problem. Equilibrium equation $\sigma_{ij,j}^* = -b_i^*$ is substituted into Eq. (13) resulting in:

$$-\int_{\Gamma} \sigma_{ij,j}^* u_i d\Gamma + \int_{\Gamma} t_i^* u_i d\Gamma = \int_{\Gamma} u_i^* b_i d\Gamma + \int_{\Gamma} u_i^* t_i d\Gamma \quad (14)$$

In order to decrease the dimension of the integral equation, Eq. (14), the weighted residual function is set to be the Green's function, which is obtained by applying a point load in direction a_i . This can be expressed as:

$$\sigma_{ij,j}^* = -\delta(x - \xi) a_i \quad (15)$$

where ξ is a source point at which a concentrated force is applied, x is a field point at which a response to the concentrated force is observed, and $\delta(x - \xi)$ is the Dirac delta function. The resulting fundamental solution is:

$$u_i^* = u_{ji}^* a_j \quad (16)$$

$$t_i^* = t_{ji}^* a_j \quad (17)$$

where u_{ji}^* and t_{ji}^* are i components of the displacements and tractions, respectively, due to a concentrated force in the j direction, and a_j is a unit vector in the direction of the applied concentrated force. The kernel functions u_{ji}^* and t_{ji}^* are given as:

$$u_{ij}^* = \frac{1}{8\pi(1-\nu)G} \left[(4\nu - 3)\ln(r)\delta_{ij} + \frac{(\bar{x} - \bar{\xi}) \bullet \bar{i}}{r} \cdot \frac{(\bar{x} - \bar{\xi}) \bullet \bar{j}}{r} \right] \quad (18)$$

$$q_{ij}^* = \frac{-1}{4\pi(1-\nu)r} \left\{ \begin{array}{l} \left[(1-2\nu)\delta_{ij} + 2 \frac{(\bar{x} - \bar{\xi}) \bullet \bar{i}}{r} \cdot \frac{(\bar{x} - \bar{\xi}) \bullet \bar{j}}{r} \right] \cdot \frac{(\bar{x} - \bar{\xi}) \bullet \bar{n}}{r} \\ -(1-2\nu) \left[\frac{(\bar{x} - \bar{\xi}) \bullet \bar{i}}{r} n_y - \frac{(\bar{x} - \bar{\xi}) \bullet \bar{j}}{r} n_x \right] \end{array} \right\} \quad (19)$$

Substituting Eq. (15), Eq. (16), and Eq. (17) into Eq. (14) yields:

$$u_i(\xi)a_i + \int_{\Gamma} t_{ji}^* a_j u_i d\Gamma = \int_{\Gamma} u_{ji}^* a_j b_i d\Gamma + \int_{\Gamma} u_{ji}^* a_j t_i d\Gamma, \quad \xi \in \Omega \quad (20)$$

The indices are exchanged in all the integral terms in Eq. (20) as:

$$u_i(\xi)a_i + \int_{\Gamma} t_{ij}^* a_i u_j d\Gamma = \int_{\Gamma} u_{ij}^* a_i b_j d\Gamma + \int_{\Gamma} u_{ij}^* a_i t_j d\Gamma, \quad \xi \in \Omega \quad (21)$$

The a_i coefficients are constant and can be canceled out from Eq. (21):

$$u_i(\xi) + \int_{\Gamma} t_{ij}^* u_j d\Gamma = \int_{\Gamma} u_{ij}^* b_j d\Gamma + \int_{\Gamma} u_{ij}^* t_j d\Gamma, \quad \xi \in \Omega \quad (22)$$

Assuming that the body force is zero, Eq. (22) can be simplified to:

$$u_i(\xi) + \int_{\Gamma} t_{ij}^* u_j d\Gamma = \int_{\Gamma} u_{ij}^* t_j d\Gamma, \quad \xi \in \Omega \quad (23)$$

Eq. (23) is integrated such that the source point, ξ , is included on the circular boundary of radius ε , as $\varepsilon \rightarrow 0$. This results in the right side integral vanishing. For constant elements the left side integral results in $-1/2u_i(\xi)$. Thus on the boundary of the system, Eq. (23) can be rewritten as:

$$\frac{1}{2}u_i(\xi) + \int_{\Gamma} t_{ij}^*(x, \xi)u_j(x)d\Gamma = \int_{\Gamma} u_{ij}^*(x, \xi)t_j(x)d\Gamma, \quad \xi \in \Gamma \quad (24)$$

In most cases, the exact solution to Eq. (24) cannot be found. Therefore Eq. (24) can be approximately solved using numerical methods such as BEM.

2.2. BOUNDARY DISCRETIZATION USING CONSTANT ELEMENT

In general, boundary integral equations, such as Eq. (24), cannot be solved analytically. To obtain approximate solutions, the boundary integral equation is discretized into boundary elements for which the true solution is approximated by a polynomial interpolation between known values of either u or t . In this work, only boundary elements with constant shape functions are used to generate significant discretization errors. Higher order approximation is assumed to approximate the true solutions better thus decreasing the discretization error. Constant elements contain one node per element, leading to the following discretization:

$$u(x) = \Phi u_i \quad (25)$$

$$t(x) = \Phi t_i \quad (26)$$

where u_i and t_i are the vectors of nodal values of u or t , respectively, at node i , and Φ is the vector of constant shape functions. The discretized Eq. (24) can be written as:

$$\frac{1}{2}u_i + \sum_{\text{Elements } \Gamma_x} \int t_{ij}^*(x, \xi)\Phi d\Gamma_x u_j = \sum_{\text{Elements } \Gamma_x} \int u_{ij}^*(x, \xi)\Phi d\Gamma_x t_j \quad (27)$$

Eq. (27) can be written in a matrix form:

$$Hu = Gt \quad (28)$$

where matrix H is singular and therefore satisfies the rigid body motion. To obtain a unique solution to Eq. (28) at least one boundary condition in each direction of the problem must be specified for the displacement. Eq. (28) is then rearranged according to the appropriate boundary conditions and solved as a linear algebra problem:

$$Ax = f \quad (29)$$

The terms of H and G matrices can either be determined explicitly or are computed numerically using numerical integration schemes. The effects of the integration error and truncation error have been studied (Zalewski et al. 2007) and can be implemented to enclose the true solution of Eq. (29). In this work the impact of the discretization error on the solution to Eq. (24) is studied, following the boundary element formulation, using interval methods.

3. Interval Analysis

In this work, the discretization error in BEM is treated using an interval approach. The following is a review of interval analysis (Moore 1966, Neumaier 1990). An interval number $\tilde{x} = [a, b]$ is a set of real numbers such that:

$$[a, b] = \{x \mid a \leq x \leq b\} \quad (30)$$

where $(a, b) \in \mathfrak{R}$. Interval variables $\tilde{x} = [a, b]$ and $\tilde{y} = [c, d]$ behave according to the following operations:

Addition:

$$\tilde{x} + \tilde{y} = [a + c, b + d] \quad (31)$$

Subtraction:

$$\tilde{x} - \tilde{y} = [a - d, b - c] \quad (32)$$

Multiplication:

$$\tilde{x} \cdot \tilde{y} = [\min\{ac, ad, bc, bd\}, \max\{ac, ad, bc, bd\}] \quad (33)$$

Division:

$$\frac{\tilde{x}}{\tilde{y}} = [a, b] \cdot \left[\frac{1}{d}, \frac{1}{c} \right], \quad 0 \notin \tilde{y} \quad (34)$$

Integration of interval-valued function $f(x, \tilde{\xi})$, which is a class of all possible functions bounded by a given interval is performed as:

$$\int_{\Gamma} f(x, \tilde{\xi}) d\Gamma = \left[\int_{\Gamma} \underline{f}(x, \tilde{\xi}) d\Gamma, \int_{\Gamma} \bar{f}(x, \tilde{\xi}) d\Gamma \right], \quad \xi \in [\underline{\xi}, \bar{\xi}] \quad (35)$$

Subdistributive property:

$$\tilde{x} \cdot (\tilde{y} + \tilde{z}) \subseteq \tilde{x} \cdot \tilde{y} + \tilde{x} \cdot \tilde{z} \quad (36)$$

One of the major sources of overestimation or underestimation in interval solutions is the subdistributive property of interval numbers. Great emphasis should be made to the correct order of operations in interval analysis. If the correct representation is given by the left term in Eq. (36), expressing the operation by the right term may cause overestimation. If the correct representation is expressed as the right term in Eq. (36), expressing it as the left term may result in inner bounds and the enclosure of the solution may not be guaranteed. This issue will be farther referred to in considering interval kernel functions.

Another source of overestimation occurs due to the dependency of interval numbers, either linear or nonlinear. Linear dependency of interval numbers for $\tilde{x} = [-1,1]$ and $\tilde{y} = [-1,1]$ can be illustrated as:

$$\tilde{x} \cdot \tilde{y} = [-1,1] \quad (37)$$

$$\tilde{x} \cdot \tilde{x} = [0,1] \quad (38)$$

Eq. (37) considers the two sets to be independent; therefore, the operation must enclose all possible values. Eq. (38) takes into account that the same set is multiplied by itself; therefore, every number in set \tilde{x} is multiplied by itself. For engineering problems interval dependency occurs mostly due to the physics of the problem and needs to be considered for sharp solutions. Naive interval application may results in wide and unrealistic bounds. Considering an example:

$$\tilde{y} = 6 \cdot \tilde{x} \cdot \tilde{x} + 3 \cdot \tilde{x}, \quad \tilde{x} = [-1,1]$$

direct interval operation results in naive bounds for the solution, $\tilde{y} = [-9,9]$. However, considering interval dependency, the bounds on the solution result in exact bounds, $\tilde{y} = [-0.375,9]$.

Another source of overestimation is the order of operations in interval linear algebra. To obtain sharp results, interval operations should be performed last to reduce the overestimation due to the dependency in interval matrix coefficients. The following example demonstrates this consideration.

$$\tilde{y}_1 = A \cdot (B \cdot \tilde{x}), \quad \tilde{y}_2 = (A \cdot B) \cdot \tilde{x}, \text{ where}$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}, \quad \tilde{x} = \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}.$$

$$\tilde{y}_1 = \begin{bmatrix} a_{11}(b_{11}\tilde{x}_1 + b_{12}\tilde{x}_2) + a_{12}(b_{21}\tilde{x}_1 + b_{22}\tilde{x}_2) \\ a_{21}(b_{11}\tilde{x}_1 + b_{12}\tilde{x}_2) + a_{22}(b_{21}\tilde{x}_1 + b_{22}\tilde{x}_2) \end{bmatrix}, \quad \tilde{y}_2 = \begin{bmatrix} (a_{11}b_{11} + a_{12}b_{21})\tilde{x}_1 + (a_{11}b_{12} + a_{12}b_{22})\tilde{x}_2 \\ (a_{21}b_{11} + a_{22}b_{21})\tilde{x}_1 + (a_{21}b_{12} + a_{22}b_{22})\tilde{x}_2 \end{bmatrix}$$

It can be clearly seen that \tilde{y}_2 is sharper than \tilde{y}_1 due to the considered dependency of \tilde{x}_1 and \tilde{x}_2 throughout the rows of \tilde{y}_2 . Therefore special care should be given to the order of interval operations to obtain sharp bounds on the solution.

4. Interval Linear System of Equations

The interval linear system of equations of the form of Eq. (29) is solved using Krawczyk iteration (Krawczyk 1969) based on Brouwer's fixed point theorem (Mullen and Muhanna 1999, Muhanna and Mullen 2001, Muhanna et al. 2005). One approach of self-validating (SV) methods to find the zero of the function $f(x) = 0$, $\mathfrak{R}^n \rightarrow \mathfrak{R}^n$ is to consider a fixed point function $g(x) = x$. The transformation between $f(x)$ and $g(x)$ for a non-singular preconditioning matrix C is:

$$f(x) = 0 \Leftrightarrow g(x) = x \quad (39)$$

$$g(x) = x - C \cdot f(x) \quad (40)$$

where the function $g(x)$ is considered as a Newton operator. From Brouwer's fixed point theorem and from:

$$g(\tilde{x}) \subseteq \tilde{x} \text{ for some } \tilde{x} \in \mathfrak{R}^n \quad (41)$$

the following is true:

$$\exists x \in \tilde{x} : f(x) = 0 \quad (42)$$

This method is used to solve linear system of equations of the form of Eq. (29). The preconditioning matrix C is chosen as $C = A^{-1}$. From Eq. (40) and Eq. (41) it follows that:

$$Cb + (I - CA)\tilde{x} \subseteq \tilde{x} \quad (43)$$

The left hand side of Eq. (43) is the Krawczyk operator (Krawczyk 1969). For the iteration to provide finite solution, the preconditioning matrix needs to be proven regular (Neumaier 1990, Rump 2001). The following proves this condition.

Theorem 1. (Rump 2001) *Let $A, C \in \mathfrak{R}^{n \times n}$, $b \in \mathfrak{R}^n$, and $\tilde{x} \in \mathfrak{R}^n$ be given. If*

$$Cb + (I - CA)\tilde{x} \subseteq \text{int}(\tilde{x}) \quad (44)$$

then C and A are regular and the unique solution of $Ax = b$ satisfies $A^{-1}b \in \tilde{x}$.

$\text{int}(\tilde{x})$ refers to the interior of \tilde{x} . However, all terms in Eq. (29) can be interval terms, thus the following is a proof for the guarantee of the solution for the equation of this form.

Theorem 2. (Rump 2001) *Let $\tilde{A} \in \mathfrak{R}^{n \times n}$, $C \in \mathfrak{R}^{n \times n}$, $\tilde{b} \in \mathfrak{R}^n$, and $\tilde{x} \in \mathfrak{R}^n$ be given. If*

$$C\tilde{b} + (I - C\tilde{A})\tilde{x} \subseteq \text{int}(\tilde{x}) \quad (45)$$

then C and every matrix $A \in \tilde{A}$ is regular and

$$\sum(\tilde{A}, \tilde{b}) = \{x \in \mathfrak{R}^n \mid \exists A \in \tilde{A} \exists b \in \tilde{b} : Ax = b\} \subseteq \tilde{x} \quad (46)$$

Eq. (46) guarantees the solution to the interval linear system of equations of the form of Eq. (29). The residual form of Eq. (46) is (Neumaier 1990):

$$C\tilde{b} - C\tilde{A}x_0 + (I - C\tilde{A})\tilde{\delta} \subseteq \text{int}(\tilde{\delta}) \quad (47)$$

where $\tilde{x} = x_0 + \tilde{\delta}$. A good initial guess is $x_0 = C\hat{b}$, where $C = \hat{A}^{-1}$, \hat{A} is the midpoint matrix of A , and \hat{b} is the midpoint vector of b . The following sections describe the treatment of point-wise discretization error via interval methods.

5. Discretization Error Bounds for Boundary Element Method

The discretization error in the solutions to integral equations results from considering a finite number of collocation points for which these solutions are computed. In general, the true solutions to integral equations are functions, not discrete values, and therefore the space of the

approximate solutions does not cover the space of the true solutions. The boundary integral equations can be obtained by the use of collocation methods resulting in equation of the form of Eq. (24). The boundary integral equations are satisfied exactly only if all the locations of the source point ξ on the boundary are considered. However, to obtain a linear system of equations, a finite number of source points are considered. Moreover, the location of the source points is unique and the solution is considered as a polynomial interpolation between discrete values, whose location corresponds to the location of the source point. This allows for the solution of the linear system of equations to be unique and thus the system can be solved for the unknown boundary values. It should be noted that if all non countable source points are considered, the boundary values at all points can be computed, resulting in the true solution. The boundary integral equation can also be evaluated over n sub-domains as expressed by Eq. (27). The unique location of the source point and its correspondence to the point at which the approximate solution is computed must be satisfied for all sub-domains. Eq. (27) is satisfied exactly only if all the locations of the source point are considered. Thus the discretization error is introduced in the same manner as in Eq. (24).

In the analysis of the discretization error, all the locations of the source point, $\tilde{\xi}$, in the continuous boundary integral equation:

$$\frac{1}{2} u_i(\xi) + \int_{\Gamma} t_{ij}^*(x, \xi) u_j(x) d\Gamma = \int_{\Gamma} u_{ij}^*(x, \xi) t_j(x) d\Gamma, \quad \xi \in \Gamma \quad (48)$$

are treated via interval approach. Considering interval bounds $\tilde{\xi}$ on all the possible locations of the source points ξ allows obtaining an interval solution which bounds the true solution. From the interval bounds on the boundary values, the bounds on the true solution for any point in the domain can be computed. Eq. (48) is bounded by an interval boundary integral equation in which the terms $u_{ij}^*(x, \xi)$ and $t_{ij}^*(x, \xi)$ are known interval-valued functions. The unknown functions $u_j(x)$ and $t_j(x)$ in Eq. (48) are then bounded by interval values enclosing the true solution.

The integral over the domain can be expressed as the sum of the integrals over the elements and thus the boundary integral equation must be bounded on each element for all the locations of the source points. Hence, for the boundary Γ subdivided into n boundary elements, for each element k the interval values \tilde{u} and \tilde{t} that bound the functions $u(x)$ and $t(x)$ are found (Figure 1).

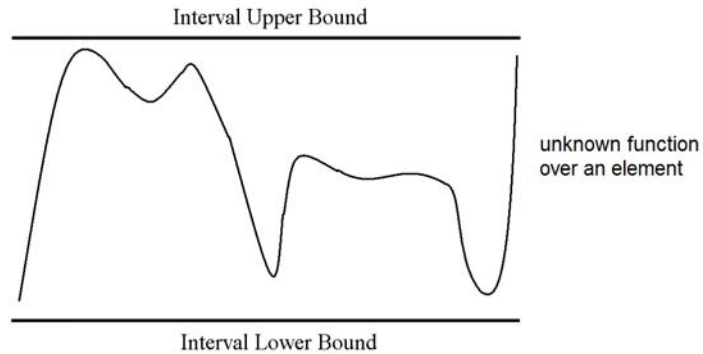


Figure 1. Constant interval bounds on a function.

For higher order elements the interval valued function, of the order of the polynomial approximation, encloses the true solution. The bounding of the function using linear elements is shown (Figure 2).

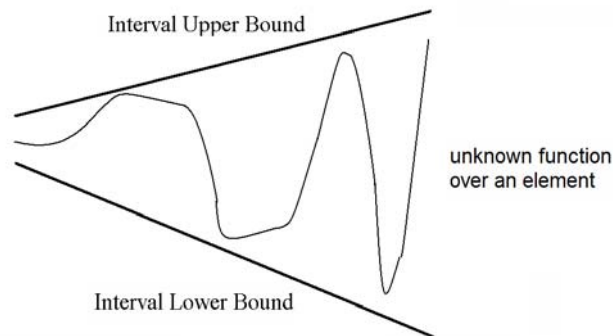


Figure 2. Linear interval bounds on a function.

It is assumed that on all other elements, except for the element in consideration, the bounds on all boundary values are known. Also either the bounds on the Dirichlet or the Neumann boundary condition bounds are known for the element in consideration. Then the remaining boundary value for the single element in consideration is bounded. The process is repeated for the second element with the assumed bounds for all the other elements, a computed bound for the previously considered element, and either the Dirichlet or the Neumann boundary condition bounds for the second element in consideration. This procedure, known as the interval Gauss-Seidel iteration (Neumaier 1990), is performed for all elements until the true solution is enclosed. Mathematically the above statement can be expressed as:

$\forall k \in \{1, 2, \dots, n\}$ Assume $\underline{u}_m \leq u_m \leq \overline{u}_m$, $\underline{t}_m \leq t_m \leq \overline{t}_m$ is known $\forall m \neq k$.

Also known $\underline{t}_k \leq t_k \leq \overline{t}_k$. Find $\underline{u}_k \leq u_k \leq \overline{u}_k$

$$\forall \xi_k \left| \begin{aligned} & \frac{1}{2} u_{ik}(\xi_k) + \int_{\Gamma_k} t_{ij}^*(x, \xi_k) u_{jk}(x) d\Gamma_k = \\ & \sum_{m=1}^n \int_{\Gamma_m} u_{ij}^*(x, \xi_k) t_{jm}(x) d\Gamma_m + \int_{\Gamma_k} u_{ij}^*(x, \xi_k) t_{jk}(x) d\Gamma_k - \sum_{m=1}^n \int_{\Gamma_m} t_{ij}^*(x, \xi_k) u_{jm}(x) d\Gamma_m \end{aligned} \right.$$

Or

$\forall k \in \{1, 2, \dots, n\}$ Assume $\underline{u}_m \leq u_m \leq \overline{u}_m$, $\underline{t}_m \leq t_m \leq \overline{t}_m$ is known $\forall m \neq k$.

Also known $\underline{u}_k \leq u_k \leq \overline{u}_k$. Find $\underline{t}_k \leq t_k \leq \overline{t}_k$

$$\forall \xi_k \left| \begin{aligned} & \int_{\Gamma_k} u_{ij}^*(x, \xi_k) t_{jk}(x) d\Gamma_k = \frac{1}{2} u_{ik}(\xi_k) + \int_{\Gamma_k} t_{ij}^*(x, \xi_k) u_{jk}(x) d\Gamma_k + \\ & \sum_{m=1}^n \int_{\Gamma_m} t_{ij}^*(x, \xi_k) u_{jm}(x) d\Gamma_m - \sum_{m=1}^n \int_{\Gamma_m} u_{ij}^*(x, \xi_k) t_{jm}(x) d\Gamma_m \end{aligned} \right. \quad (49)$$

Each term of the summation in Eq. (49) is represented graphically (Figure 3).

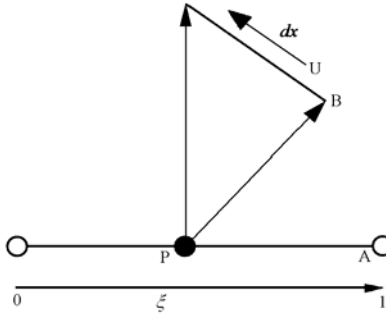


Figure 3. Integration from element B from point P on element A.

If u or q are specified boundary conditions, the interval integration can be performed explicitly as described in section 3, Eq. (35). In this work, for computational efficiency purposes, the underlying system of interval equations is solved using Krawczyk iteration (Krawczyk 1969), rather than using the interval Gauss-Seidel iteration (Neumaier 1990). This substitution of the method for bounding the unknown boundary values can be made since both of these methods are iterative methods for solving interval linear systems of equations and both obtain guaranteed bounds for the solution. Hence, the interval boundary element method (IBEM) formulation is performed such that the resulting interval linear system of equations is of the form of Eq. (29).

6. Interval Kernel Splitting Technique

The analysis of the discretization error requires that the boundary integral equations for each element be bounded for all the locations of the source point ξ . The integral equation in the boundary element formulation has the form of the Fredholm equation of the first kind. Kernel splitting techniques have been used to bound the interval Fredholm equation of the first kind in which the right side is deterministic (Dobner 2002) as:

$$\int_{\Gamma} \tilde{a}(x, \xi) u(x) d\Gamma = b(\xi) \quad (50)$$

However, the interval boundary integral equations considered herein have an interval right side, due to the interval valued location of the source point $\tilde{\xi}$, therefore a new Interval Kernel Splitting Technique (IKST) is developed. The integral of the product of two functions is bounded considering interval bounds on the unknown value as:

$$\int_{\Gamma} a(x, \tilde{\xi}) \tilde{u} d\Gamma \supseteq \int_{\Gamma} a(x, \tilde{\xi}) u(x) d\Gamma = b(\tilde{\xi}) \quad (51)$$

To separate the kernels such that the unknown \tilde{u} can be taken out of the integral on Γ , the left side integral from Eq. (51) is expressed as a sum of the integrals:

$$\int_{\Gamma} a(x, \tilde{\xi}) \tilde{u} d\Gamma = \int_{\Gamma_1} a(x, \tilde{\xi}) \tilde{u} d\Gamma_1 + \int_{\Gamma_2} a(x, \tilde{\xi}) \tilde{u} d\Gamma_2 \quad (52)$$

where $\Gamma_1 \cup \Gamma_2 = \Gamma$, $\Gamma_1 \cap \Gamma_2 = 0$ and:

$$a(x, \tilde{\xi}) > 0 \text{ or } a(x, \tilde{\xi}) < 0 \text{ on } \Gamma_1 \quad (53)$$

$$a(x, \tilde{\xi}) \in 0 \text{ on } \Gamma_2 \quad (54)$$

The interval kernel is of the same sign on Γ_1 , thus \tilde{u} can be directly taken out of the integral on Γ_1 as:

$$\int_{\Gamma_1} a(x, \tilde{\xi}) \tilde{u} d\Gamma_1 = \int_{\Gamma_1} a(x, \tilde{\xi}) d\Gamma_1 \tilde{u} \quad (55)$$

Due to the subdistributive property of interval numbers, Eq. (36), \tilde{u} cannot be taken out of the integral on Γ_2 . The direct application of the subdistributive property may result in inner bounds on the interval integral as:

$$\int_{\Gamma_2} a(x, \tilde{\xi}) d\Gamma_2 \tilde{u} \subseteq \int_{\Gamma_2} a(x, \tilde{\xi}) \tilde{u} d\Gamma_2 \quad (56)$$

Hence the interval kernel is bounded by its limits on Γ_2 :

$$\int_{\Gamma_2} \tilde{a} \tilde{u} d\Gamma_2 \supseteq \int_{\Gamma_2} a(x, \tilde{\xi}) \tilde{u} d\Gamma_2 \quad (57)$$

where \tilde{a} is defined as:

$$\tilde{a} = [\min\{a(x + \tilde{\varepsilon}, \tilde{\xi})\}, \max\{a(x + \tilde{\varepsilon}, \tilde{\xi})\}] \quad (58)$$

$$\tilde{\varepsilon} = [-\varepsilon, \varepsilon] \quad (59)$$

ε is the tolerance level of the nonlinear solver used to find the zero location of $a(x, \tilde{\xi})$. To show that by bounding the kernel on Γ_2 allows \tilde{u} to be taken out from the integral on Γ_2 , the integral on Γ_2 is expressed as an infinite sum:

$$\int_{\Gamma_2} \tilde{a} \tilde{u} d\Gamma_2 = \lim_{\Delta \rightarrow 0} \sum_{i=1}^n (\Delta \tilde{a} \tilde{u}) \Big|_{\Gamma_2} = \lim_{\Delta \rightarrow 0} (n \Delta \tilde{a} \tilde{u}) \Big|_{\Gamma_2} = \lim_{\Delta \rightarrow 0} (n \Delta \tilde{a}) \tilde{u} \Big|_{\Gamma_2} = \lim_{\Delta \rightarrow 0} \sum_{i=1}^n (\Delta \tilde{a}) \Big|_{\Gamma_2} \tilde{u} = \int_{\Gamma_2} \tilde{a} d\Gamma_2 \tilde{u} \quad (60)$$

where Δ is a small part of Γ_2 . Thus \tilde{u} can be taken out of both integrals on Γ_1 and on Γ_2 and the split interval boundary integral equation becomes:

$$\int_{\Gamma_1} a(x, \tilde{\xi}) d\Gamma_1 \tilde{u} + \int_{\Gamma_2} \tilde{a} d\Gamma_2 \tilde{u} \supseteq \int_{\Gamma} a(x, \tilde{\xi}) \tilde{u} d\Gamma \supseteq \int_{\Gamma} a(x, \tilde{\xi}) u(x) d\Gamma = b(\tilde{\xi}) \quad (61)$$

The kernels are bounded for all the elements resulting in interval linear system of equations:

$$\tilde{A}_1 \tilde{u} + \tilde{A}_2 \tilde{u} \supseteq \tilde{b} \quad (62)$$

IKST bounds the continuous boundary integral equation for all the locations of the source point ξ and Eq. (48) is guaranteed to be satisfied for all the weighting functions. The solution to Eq. (62) is described in the following sections.

7. Iterative Solver for the Interval Linear System of Equations

The bounding of the original boundary integral equation using IKST results in the interval linear system of equations different from that of Eq. (29). Hence, the algorithm to solve the interval linear system of equations, Eq. (62), must be developed. This section describes the transformation of Eq. (62) to obtain it in the form of Eq. (29). Then, Krawczyk iteration (Krawczyk 1969) is performed to obtain the guaranteed bounds on the solution. Considering the linear system of equations:

$$\tilde{A}_{1e}\tilde{x}_e + \tilde{A}_{2e}\tilde{x}_e = \tilde{b}_e \quad (63)$$

where $\tilde{A}_{1e} \in \tilde{A}_1$, $\tilde{A}_{2e} \in \tilde{A}_2$, $\tilde{b}_e \in \tilde{b}$, $\tilde{x}_e \in \tilde{x}$ and A_{1e} is regular $\forall A_{1e} | A_{1e} \in \tilde{A}_{1e}$. Eq. (63) is pre-multiplied by \tilde{A}_{1e}^{-1} as:

$$\tilde{A}_{1e}^{-1}\tilde{A}_{1e}\tilde{x}_e + \tilde{A}_{1e}^{-1}\tilde{A}_{2e}\tilde{x}_e = \tilde{A}_{1e}^{-1}\tilde{b}_e \quad (64)$$

By substituting $\tilde{A}_{1e}^{-1}\tilde{A}_{1e} = I$, $\tilde{A}_{1e}^{-1}\tilde{A}_{2e} = \tilde{A}_{3e}$ and $\tilde{A}_{1e}^{-1}\tilde{b}_e = \tilde{b}_{1e}$, Eq. (64) can be rewritten as:

$$\tilde{x}_e + \tilde{A}_{3e}\tilde{x}_e = \tilde{b}_{1e} \quad (65)$$

Since the first term in Eq. (65) is a deterministic identity matrix pre-multiplying \tilde{x}_e , the following substitution can be made directly. Letting $I + \tilde{A}_{3e} = \tilde{A}_e$ results in:

$$\tilde{A}_e\tilde{x}_e = \tilde{b}_{1e} \quad (66)$$

The transformed system of equations is subjected to Krawczyk iteration (Krawczyk 1969) as described in the previous section.

8. Discretization Error in Interval Boundary Element Method

In the preceding formulation, the bounds on the unknown boundary values are found using iterative techniques. The obtained bounds, however, are greatly overestimated since the

dependency of interval values was not considered. One reason for this overestimation is that the interval kernels are bounded such that the source point ξ is allowed to vary along the entire element. Thus, for two adjacent elements, two source points are allowed to be connecting point between the elements and have the same location, resulting in the reduction of the rank of the system of equations. The unique location of a single source point is also not considered throughout the rows of H and G matrices, which are in $R^{n \times n}$. Thus, the parameterization of the interval location of the source point, $\tilde{\xi}$, in the \tilde{H} and \tilde{G} matrices must be considered in the solver to obtain n independent interval equations and to reduce the overestimation which results from a non-unique location of the source point on any individual element. For convenience, the system is parameterized such that $\tilde{\xi} = [0, 1]$ is the location scaled by a length of an element. In performing interval matrix products, the value of $\tilde{\xi}$ is decomposed into sub-intervals such that:

$$\bigcup_{i=1}^n \tilde{\xi}_i = \tilde{\xi} \quad \text{and} \quad \bigcap_{i=1}^n \tilde{\xi}_i = 0 \quad (67)$$

The parameterized boundary integral equation is bounded by IKST for each subinterval $\tilde{\xi}_i$, resulting in the linear system of equations:

$$H_1(\tilde{\xi}_i)\tilde{u} + H_2(\tilde{\xi}_i)\tilde{u} = G_1(\tilde{\xi}_i)\tilde{t} + G_2(\tilde{\xi}_i)\tilde{t} \quad (68)$$

where the kernel is of the same sign for $H_1(\tilde{\xi}_i)$ and $G_1(\tilde{\xi}_i)$ and contains zero for $H_2(\tilde{\xi}_i)$ and $G_2(\tilde{\xi}_i)$. The system of equations is rearranged according to the boundary conditions as:

$$A_1(\tilde{\xi}_i)\tilde{x} + A_2(\tilde{\xi}_i)\tilde{x} = b(\tilde{\xi}_i) \quad (69)$$

Steps described in the previous section lead to the equation of the form:

$$A(\tilde{\xi}_i)\tilde{x} = b_1(\tilde{\xi}_i) \quad (70)$$

The initial interval guess is then considered as:

$$\tilde{x}_0 = A^{-1} \bigcup_{i=1}^n b_1(\tilde{\xi}_i) \quad (71)$$

where A is computed for $\xi = 1/2$. The difference between I and the preconditioning matrix A^{-1} post-multiplied by the interval matrix $A(\tilde{\xi}_i)$ is computed as:

$$\tilde{I}_d = I - \bigcup_{i=1}^n A^{-1} \tilde{A}(\tilde{\xi}_i) \quad (72)$$

The difference between the solution and the initial guess is computed for each $\tilde{\xi}_i$ pre-multiplied by the preconditioning matrix I , which numerically gave the sharpest results:

$$\tilde{\delta} = \bigcup_{i=1}^n (\tilde{b}(\tilde{\xi}_i) - \tilde{A}_1(\tilde{\xi}_i)\tilde{x}_0 - \tilde{A}_2(\tilde{\xi}_i)\tilde{x}_0) \quad (73)$$

Also:

$$\tilde{\delta}_1 = \tilde{\delta} \quad (74)$$

The iteration is performed as:

$$\tilde{del} = \tilde{\delta}_1 \quad (75)$$

$$\tilde{\delta}_1 = \tilde{\delta} + \tilde{I}_d \tilde{del} \quad (76)$$

$$\text{If } \tilde{del} \supset \tilde{\delta}_1 \quad (77)$$

$$\tilde{x} = \tilde{x}_0 + \tilde{\delta}_1 \quad (78)$$

For any point n on element k the bounds on the discretization error are found as:

$$\tilde{E}_{nk}^{discretization} = \tilde{x}_k - x_n \quad (79)$$

where \tilde{x}_k are the solution bounds over an element k and x_n is the solution from a conventional boundary element analysis for point n .

9. Examples

The first example demonstrates the IBEM considering discretization error for the elasticity problem. A unit square domain of the problem as well as the boundary element mesh is shown (Figure 4). The body has a unit elastic modulus and a zero Poisson ratio. The left and right sides have a zero traction boundary condition; the bottom boundary has a zero displacement boundary

condition, while the top boundary has a zero traction condition in the x direction and a unit displacement in the y direction.

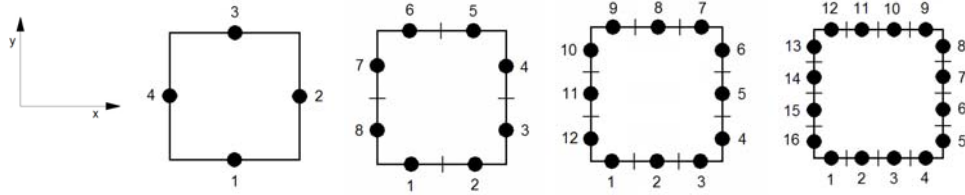


Figure 4. Boundary discretization using constant boundary elements.

The behavior of the y displacement bounds such as solution width, effectivity index, and solution bounds is depicted (Figure 5-7) for nodes 2, 3, 4, and 5 on the four respective meshes. The interval bounds, depicted by a solid line enclosing the dashed true solution, for the right edge displacement in the y direction are shown (Figure 8). The effect of the parameterization for the traction in the x direction for element 1 for the 4 and 8 element meshes is also shown (Figure 9, Figure 10).

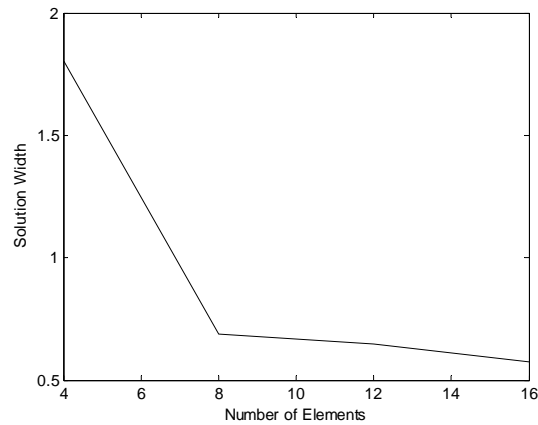


Figure 5. Behavior of the width of the interval solution with problem size.

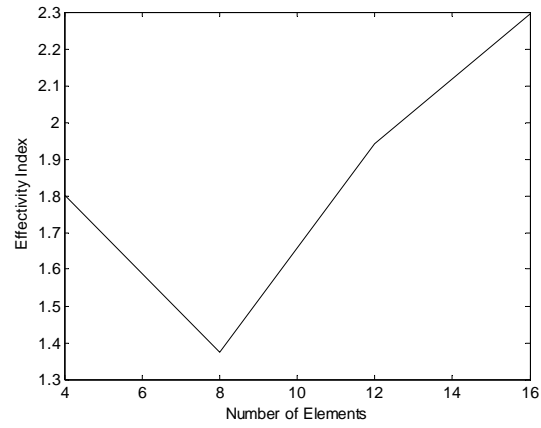


Figure 6. Behavior of the effectivity index with problem size.

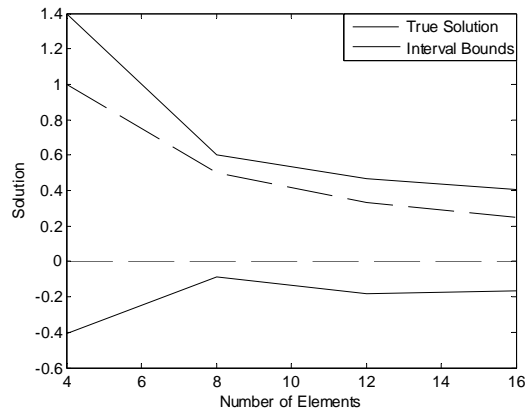


Figure 7. Behavior of the interval bounds with problem size.

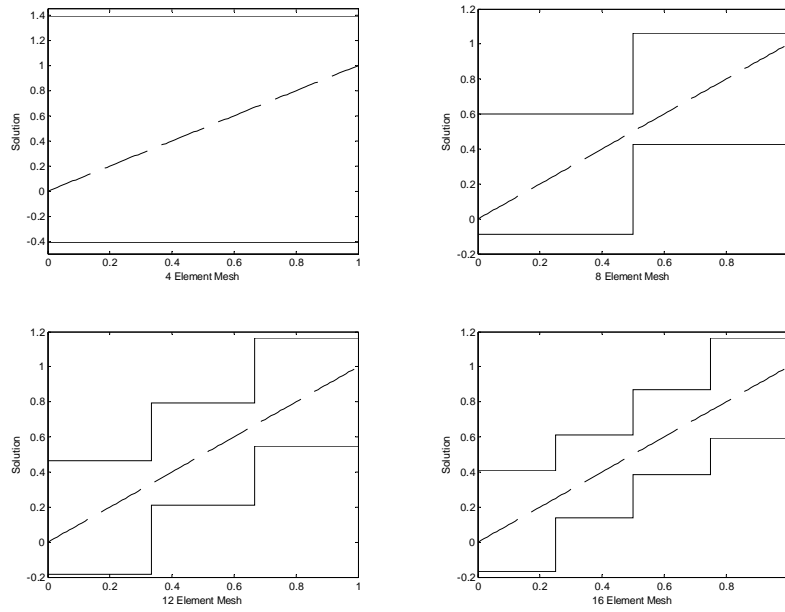


Figure 8. Behavior of the interval bounds for the different meshes.

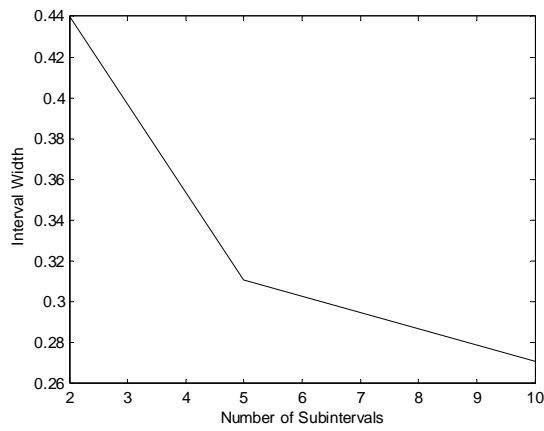


Figure 9. Behavior of the width of the interval solution with parameterization for a 4 element mesh.

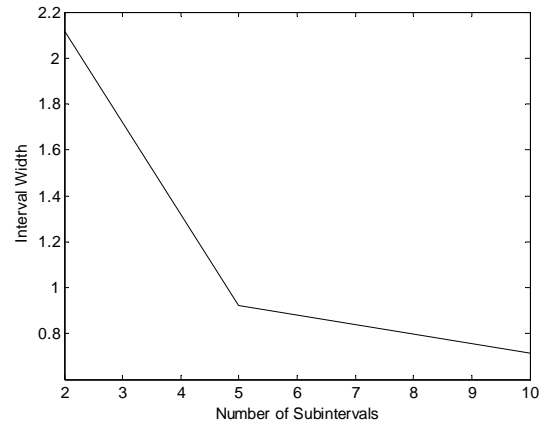


Figure 10. Behavior of the width of the interval solution with parameterization for an 8 element mesh.

The second example obtains bounds on the solution, considering the discretization error, to a hexagonal plate subjected to a unit displacement in the y direction at the top and a unit displacement in the $-y$ direction on the bottom (Figure 11). The body has a unit elastic modulus and a zero Poisson ratio.

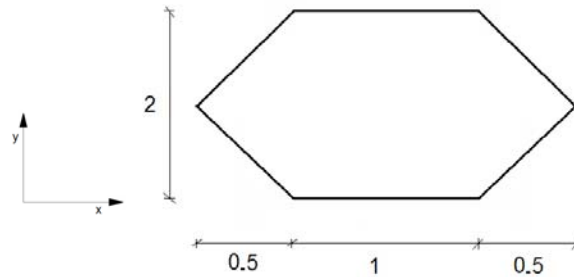


Figure 11. Hexagonal plate subjected to a unit displacement.

A symmetry model is considered, to decrease the computational time, with a unit displacement at the top and is uniformly discretized using constant boundary elements (Figure 12, Figure 13).

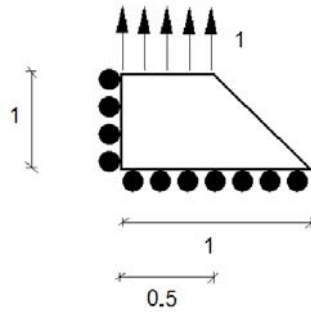


Figure 12. Symmetry model.

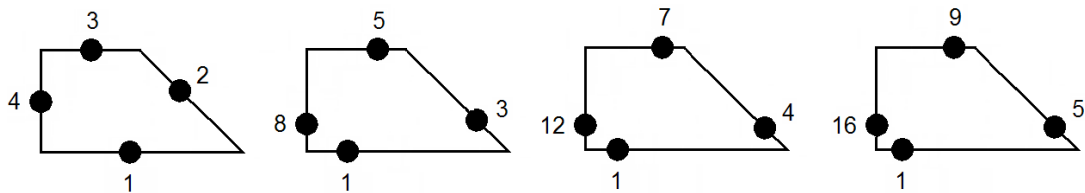


Figure 13. Uniform boundary discretization using constant boundary elements.

The behavior of the solution width, effectivity index, and solution bounds is depicted (Figure 14-16) for the displacement in the y direction for nodes 4, 8, 12, and 16 on the four respective meshes shown above. The interval bounds, depicted by a solid line enclosing the dashed true solution, for the left edge displacement in the y direction are shown (Figure 17).

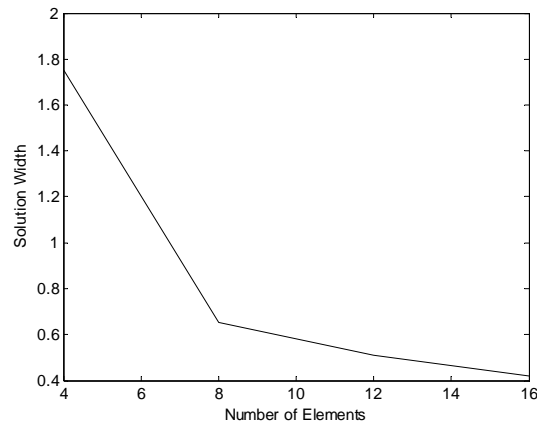


Figure 14. Behavior of the width of the interval solution with problem size.

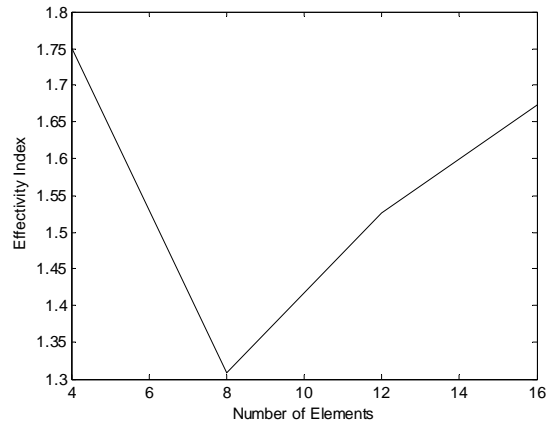


Figure 15. Behavior of the effectivity index with problem size.

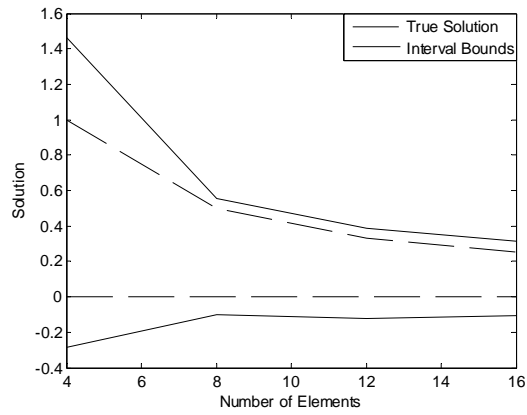


Figure 16. Behavior of the interval bounds with problem size.

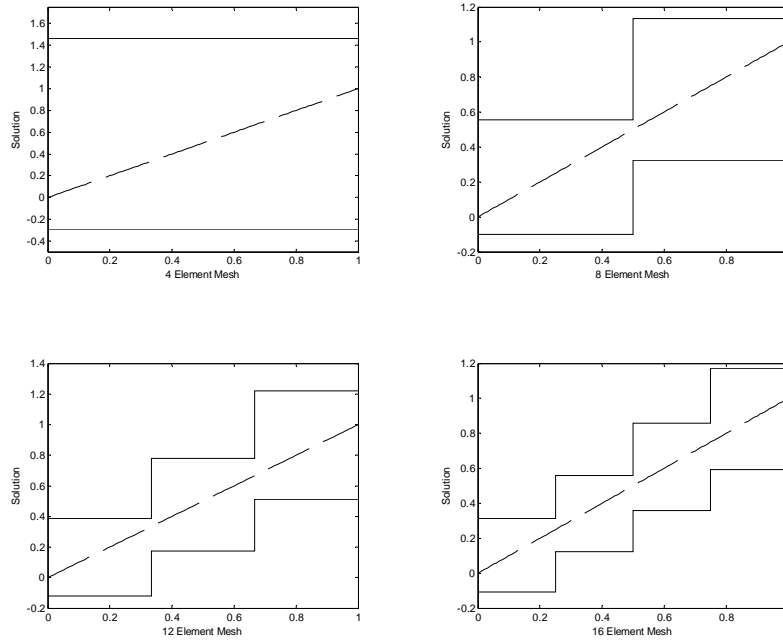


Figure 17. Behavior of the interval bounds for the different meshes.

10. Conclusion

In this work the discretization error for the elasticity problem is bounded using interval boundary element method. The interval bounds on the true solution are shown to converge for the meshes considered despite the increase in the effectivity index. The increase in the effectivity index is attributed to the slower convergence of the interval bounds than the true solution. The overestimation in the interval bounds is due to the overestimation of the terms in the interval boundary integral equation using IKST, imperfect parameterization of the location of the source point throughout the rows of the matrices H and G , and the overestimation in the iterative interval solver. There are two sources of overestimation in the iterative scheme solving the interval system of linear equations. The first one is due to the inherent overestimation when Krawczyk iteration is used to solve interval linear system of equations. This source of overestimation occurs due to the orthogonal multidimensional interval bounds enclosing a true solution which may not be, and in most cases is not, orthogonal and/or oriented in the same direction as the interval bounds (Figure 18).

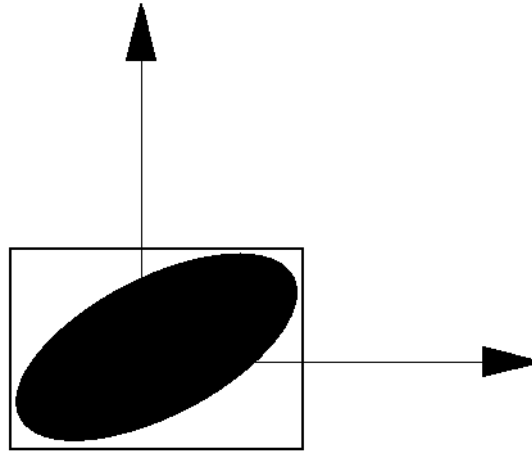


Figure 18. Interval bounds on the solution.

The second source of overestimation on the interval solver comes from incomplete consideration of the interval parameterization in Eq. (76). Each term in Eq. (76) is parameterized; however, each of these terms must be dealt with in its entirety when operated with. The solution of the linear system of equations must be satisfied for the entire system and thus the residual has to be calculated for the entire interval width, not for the length of the subinterval. If the residual is computed for the portion of the interval, for instance an interval width corresponding to a subinterval such that a complete interval parameterization can be utilized in Eq. (76), the enclosure is no longer guaranteed.

References

- Alefeld, G. and J. Herzberger. *Introduction to Interval Computations*, Academic Press, New York, NY, 1983.
- Brebbia, C.A. and J. Dominguez. *Boundary Elements: An Introductory Course*, Computational Mechanics, New York, McGraw-Hill, 1992.
- Dobner, H. Kernel-Splitting Technique for Enclosing the Solution of Fredholm Equations of the First Kind, *Reliable Computing*, Vol. 8, pp. 469-179, 2002.
- Friedman, A. *Partial differential equations*, R. E. Krieger Pub. Co., Huntington, N.Y., 1976.
- Gay, D. M. Solving Interval Linear Equations, *SIAM Journal on Numerical Analysis*, Vol. 19, 4, pp. 858-870, 1982.
- Hansen, E. *Interval arithmetic in matrix computation*, J. S. I. A. M., series B, Numerical Analysis, part I, 2, 308-320, 1965.
- Hartmann, F. *Introduction to boundary elements : theory and applications*, New York, Springer-Verlag, 1989.
- Jansson, C. Interval Linear System with Symmetric Matrices, Skew-Symmetric Matrices, and Dependencies in the Right Hand Side, *Computing*, Vol. 46, pp. 265-274, 1991.

- Krawczyk, R. Newton-Algorithmen zur Bestimmung von Nullstellen mit Fehlerschranken, *Computing*, Vol. 4, pp. 187-201, 1969.
- MATLAB 6.5.1. *Using Matlab Version 6 OEM Manual the Language of Technical Computing*, Mathworks, 2002.
- Moore, R. E. *Interval Analysis*, Prentice-Hall, Inc., Englewood Cliffs, N. J, 1966.
- Muhanna, R. L. and R. L. Mullen. Uncertainty in Mechanics Problems - Interval-Based Approach, *Journal of Engineering Mechanics, ASCE*, Vol. 127, No. 6, pp. 557-566, 2001.
- Muhanna et al. Penalty-Based Solution for the Interval Finite-Element Methods. *Journal of Engineering Mechanics, ASCE*, Vol. 131, 10, pp.1102-1111, 2005.
- Mullen, R. L. and R. L. Muhanna. Interval-Based Finite Element Methods. *Reliable Computing* 5 pp. 97-100, 1999.
- Mullen, R. L. and R. L. Muhanna. Bounds of Structural Response for All Possible Loadings, " *Journal of Structural Engineering, ASCE*, Vol. 125, No. 1, pp 98-106, 1999.
- Neumaier, A. Overestimation in Linear Interval Equations, *SIAM Journal on Numerical Analysis*, Vol. 24, 1, pp. 207-214, 1987.
- Neumaier, A. Rigorous Sensitivity Analysis for Parameter-Dependent Systems of Equations, *Journal of Mathematical Analysis and Applications*, Vol. 144, pp. 16-25, 1989.
- Neumaier, A. *Interval methods for systems of equations*, Cambridge University Press, 1990.
- Pilkey, W. D. and W. Wunderlich. *Mechanics of Structures, Variational and Computational Methods*, CRS Press, London, 1994.
- Rencis, J. J. and K-Y. Jong. An Error Estimator for Boundary Element Computations, *ASCE Journal of Engineering Mechanics*, Vol. 115 (9), pp. 1993-2010, 1989.
- Rump, S. M. *Kleine Fehlerschranken bei Matrixproblem*, Dissertation, Universitat Karlsruhe, 1980.
- Rump, S. M. Rigorous Sensitivity Analysis for Systems of Linear and Nonlinear Equations, *Mathematics of Computations*, Vol. 54, 190, pp. 721-736, 1990.
- Rump, S. M. Self-validating Methods, *Linear Algebra and its Applications*, Vol. 324 pp. 3-13, 2001.
- Sunaga, T. Theory of interval algebra and its application to numerical analysis, *RAAG Memoirs* 3, pp 29-46, 1958.
- Taylor B. *Methodus incrementorum directa & inversa*, Londini, typis Pearsnianis, 1715.
- Zalewski B. F. et al. Interval Boundary Element Method in the Presence of Uncertain Boundary Conditions, Integration Errors, and Truncation Errors, *Engineering Analysis with Boundary Elements*, 2007 (in review).
- Zalewski B. F. and R. L. Mullen. Local Discretization Errors in Boundary Element Analysis, *Journal of Computational and Applied Mathematics*, 2007 (in review).